Some Inequalities for the Dispersion of a Random Variable whose PDF is Defined on a Finite Interval

SOME INEQUALITIES FOR THE DISPERSION OF A RANDOM VARIABLE WHOSE PDF IS DEFINED ON A FINITE INTERVAL

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Abstract. Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval and applications are given.

1. Introduction

In this note we obtain some inequalities for the dispersion of a continuous random variable X having the probability density function (p.d.f.) f defined on a finite interval $[a, b]$.

Tools used include: Korkine's identity, which plays a central role in the proof of Chebychev's integral inequality for synchronous mappings [24], Hölder's weighted inequality for double integrals and an integral identity connecting the variance $\sigma^2(X)$ and the expectation $E(X)$. Perturbed results are also obtained by using Grüss, Chebyshev and Lupaş inequalities. In Section 4, results from an identity involving a double integral are obtained for a variety of norms.

2. Some Inequalities for Dispersion

Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ be the p.d.f. of the random variable X and

$$E(X) := \int_a^b t f(t)\, dt$$

its *expectation* and

$$\begin{aligned} \sigma(X) &= \left[\int_a^b (t - E(X))^2 f(t)\, dt \right]^{\frac{1}{2}} \\ &= \left[\int_a^b t^2 f(t)\, dt - [E(X)]^2 \right]^{\frac{1}{2}} \end{aligned}$$

its *dispersion* or *standard deviation.*

The following theorem holds.

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Theorem 1. *With the above assumptions, we have*

$$0 \le \sigma(X) \le \begin{cases} \frac{\sqrt{3}(b-a)^2}{6}\|f\|_\infty & \text{provided } f \in L_\infty[a,b]; \\ \frac{\sqrt{2}(b-a)^{1+\frac{1}{q}}}{2[(q+1)(2q+1)]^{\frac{2}{q}}}\|f\|_p & \text{provided } f \in L_p[a,b] \\ & \text{and } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\sqrt{2}(b-a)}{2}. \end{cases} \tag{2.1}$$

Proof. Korkine's identity [24], is

$$\begin{aligned} &\int_a^b p(t)\,dt \int_a^b p(t)\,g(t)\,h(t)\,dt - \int_a^b p(t)\,g(t)\,dt \cdot \int_a^b p(t)\,h(t)\,dt \\ &= \frac{1}{2}\int_a^b \int_a^b p(t)\,p(s)\,(g(t)-g(s))\,(h(t)-h(s))\,dtds, \end{aligned} \tag{2.2}$$

which holds for the measurable mappings $p, g, h : [a,b] \to \mathbb{R}$ for which the integrals involved in (2.2) exist and are finite. Choose in (2.2) $p(t) = f(t)$, $g(t) = h(t) = t - E(X)$, $t \in [a,b]$ to get

$$\sigma^2(X) = \frac{1}{2}\int_a^b \int_a^b f(t)\,f(s)\,(t-s)^2\,dtds. \tag{2.3}$$

It is obvious that

$$\begin{aligned} &\int_a^b \int_a^b f(t)\,f(s)\,(t-s)^2\,dtds \\ &\le \sup_{(t,s)\in[a,b]^2} |f(t)\,f(s)| \int_a^b \int_a^b (t-s)^2\,dtds \\ &= \frac{(b-a)^4}{6}\|f\|_\infty^2 \end{aligned} \tag{2.4}$$

and then, by (2.3), we obtain the first part of (2.1).

For the second part, we apply Hölder's integral inequality for double integrals to obtain

$$\begin{aligned} &\int_a^b \int_a^b f(t)\,f(s)\,(t-s)^2\,dtds \\ &\le \left(\int_a^b \int_a^b f^p(t)\,f^p(s)\,dtds\right)^{\frac{1}{p}} \left(\int_a^b \int_a^b (t-s)^{2q}\,dtds\right)^{\frac{1}{q}} \\ &= \|f\|_p^2 \left[\frac{(b-a)^{2q+2}}{(q+1)(2q+1)}\right]^{\frac{1}{q}}, \end{aligned}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and the second inequality in (2.1) is proved.

For the last part, observe that

$$\begin{aligned} \int_a^b \int_a^b f(t)\,f(s)\,(t-s)^2\,dtds &\le \sup_{(t,s)\in[a,b]^2} |(t-s)|^2 \int_a^b \int_a^b f(t)\,f(s)\,dtds \\ &= (b-a)^2 \end{aligned}$$

as

$$\int_a^b \int_a^b f(t) f(s) \, dt ds = \int_a^b f(t) \, dt \int_a^b f(s) \, ds = 1.$$

∎

Using a finer argument, the last inequality in (2.1) can be improved as follows.

Theorem 2. *Under the above assumptions, we have*

$$0 \le \sigma(X) \le \frac{1}{2}(b-a). \tag{2.5}$$

Proof. We use the following Grüss type inequality:

$$0 \le \frac{\int_a^b p(t) g^2(t) \, dt}{\int_a^b p(t) \, dt} - \left(\frac{\int_a^b p(t) g(t) \, dt}{\int_a^b p(t) \, dt} \right)^2 \le \frac{1}{4}(M-m)^2, \tag{2.6}$$

provided that p, g are measurable on $[a,b]$ and all the integrals in (2.6) exist and are finite, $\int_a^b p(t) \, dt > 0$ and $m \le g \le M$ a.e. on $[a,b]$. For a proof of this inequality see [19].

Choose in (2.6), $p(t) = f(t)$, $g(t) = t - E(X)$, $t \in [a,b]$. Observe that in this case $m = a - E(X)$, $M = b - E(X)$ and then, by (2.6) we deduce (2.5). ∎

Remark 1. *The same conclusion can be obtained for the choice $p(t) = f(t)$ and $g(t) = t$, $t \in [a,b]$.*

The following result holds.

Theorem 3. *Let X be a random variable having the p.d.f. given by $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$. Then for any $x \in [a,b]$ we have the inequality:*

$$\sigma^2(X) + (x - E(X))^2 \le \begin{cases} (b-a)\left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2\right] \|f\|_\infty & \text{provided } f \in L_\infty[a,b]; \\ \left[\frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{q}} \|f\|_p & \text{provided } f \in L_p[a,b], \ p > 1, \text{ and } \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^2. \end{cases} \tag{2.7}$$

Proof. We observe that

$$\begin{aligned} \int_a^b (x-t)^2 f(t) \, dt &= \int_a^b \left(x^2 - 2xt + t^2\right) f(t) \, dt \\ &= x^2 - 2xE(X) + \int_a^b t^2 f(t) \, dt \end{aligned} \tag{2.8}$$

and as

$$\sigma^2(X) = \int_a^b t^2 f(t) \, dt - [E(X)]^2, \tag{2.9}$$

we get, by (2.8) and (2.9),

$$[x - E(X)]^2 + \sigma^2(X) = \int_a^b (x-t)^2 f(t) \, dt, \tag{2.10}$$

which is of interest in itself too.

We observe that

$$\begin{aligned}\int_a^b (x-t)^2 f(t)\,dt &\leq ess \sup_{t\in[a,b]} |f(t)| \int_a^b (x-t)^2\,dt \\ &= \|f\|_\infty \frac{(b-x)^3+(x-a)^3}{3} \\ &= (b-a)\|f\|_\infty \left[\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right]\end{aligned}$$

and the first inequality in (2.7) is proved.

For the second inequality, observe that by Hölder's integral inequality,

$$\begin{aligned}\int_a^b (x-t)^2 f(t)\,dt &\leq \left(\int_a^b f^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_a^b (x-t)^{2q}\,dt\right)^{\frac{1}{q}} \\ &= \|f\|_p \left[\frac{(b-x)^{2q+1}+(x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{q}},\end{aligned}$$

and the second inequality in (2.7) is established.

Finally, observe that,

$$\begin{aligned}\int_a^b (x-t)^2 f(t)\,dt &\leq \sup_{t\in[a,b]} (x-t)^2 \int_a^b f(t)\,dt \\ &= \max\left\{(x-a)^2,(b-x)^2\right\} \\ &= (\max\{x-a,b-x\})^2 \\ &= \left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^2,\end{aligned}$$

and the theorem is proved. ∎

The following corollaries are easily deduced.

Corollary 1. *With the above assumptions, we have*

$$0 \leq \sigma(X) \leq \begin{cases} (b-a)^{\frac{1}{2}}\left[\frac{(b-a)^2}{12}+\left(E(X)-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}\|f\|_\infty^{\frac{1}{2}} \\ \textit{provided}\ \ f\in L_\infty[a,b]; \\ \left[\frac{(b-E(X))^{2q+1}+(E(X)-a)^{2q+1}}{2q+1}\right]^{\frac{1}{2q}}\|f\|_p^{\frac{1}{2}} \\ \textit{if}\ \ f\in L_p[a,b],\ p>1\ \ \textit{and}\ \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{b-a}{2}+\left|E(X)-\frac{a+b}{2}\right|. \end{cases} \tag{2.11}$$

Remark 2. *The last inequality in (2.11) is worse than the inequality (2.5), obtained by a technique based on Grüss' inequality.*

The best inequality we can get from (2.7) is that one for which $x = \frac{a+b}{2}$, and this applies for all the bounds as

$$\min_{x\in[a,b]} \left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2} \right)^2 \right] = \frac{(b-a)^2}{12},$$

$$\min_{x\in[a,b]} \frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} = \frac{(b-a)^{2q+1}}{2^{2q}(2q+1)},$$

and

$$\min_{x\in[a,b]} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] = \frac{b-a}{2}.$$

Consequently, we can state the following corollary as well.

Corollary 2. *With the above assumptions, we have the inequality:*

$$0 \le \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 \le \begin{cases} \frac{(b-a)^3}{12} \|f\|_\infty & \text{provided } f \in L_\infty[a,b]; \\ \frac{(b-a)^{2q+1}}{4(2q+1)^{\frac{1}{q}}} \|f\|_p & \text{if } f \in L_p[a,b],\ p>1, \\ & \text{and } \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{12}. \end{cases} \tag{2.12}$$

Remark 3. *from the last inequality in (2.12), we obtain*

$$0 \le \sigma^2(X) \le (b - E(X))(E(X) - a) \le \frac{1}{4}(b-a)^2, \tag{2.13}$$

which is an improvement on (2.5).

3. Perturbed Results Using Grüss Type inequalities

In 1935, G. Grüss (see for example [26]) proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of the integrals.

Theorem 4. *Let $h, g : [a,b] \to \mathbb{R}$ be two integrable mappings such that $\phi \le h(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a,b]$, where $\phi, \Phi, \gamma, \Gamma$ are real numbers. Then,*

$$|T(h,g)| \le \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma), \tag{3.1}$$

where

$$T(h,g) = \frac{1}{b-a}\int_a^b h(x)g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx \tag{3.2}$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

For a simple proof of this as well as for extensions, generalisations, discrete variants and other associated material, see [25], and [1]-[21] where further references are given.

A 'premature' Grüss inequality is embodied in the following theorem which was proved in [23]. It provides a sharper bound than the above Grüss inequality.

Theorem 5. *Let h, g be integrable functions defined on $[a,b]$ and let $d \leq g(t) \leq D$. Then*

$$|T(h,g)| \leq \frac{D-d}{2} |T(h,h)|^{\frac{1}{2}}, \tag{3.3}$$

where $T(h,g)$ is as defined in (3.2).

Theorem 5 will now be used to provide a perturbed rule involving the variance and mean of a p.d.f.

3.1. **Perturbed Results Using 'Premature' Inequalities.** In this subsection we develop some perturbed results.

Theorem 6. *Let X be a random variable having the p.d.f. given by $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$. Then for any $x \in [a,b]$ and $m \leq f(x) \leq M$ we have the inequality*

$$\begin{aligned} (3.4) \quad |P_V(x)| &: = \left| \sigma^2(X) + (x - E(X))^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ &\leq \frac{M-m}{2} \cdot \frac{(b-a)^2}{\sqrt{45}} \left[\left(\frac{b-a}{2}\right)^2 + 15\left(x - \frac{a+b}{2}\right) \right]^{\frac{1}{2}} \\ &\leq (M-m) \frac{(b-a)^3}{\sqrt{45}}. \end{aligned}$$

Proof. Applying the 'premature' Grüss result (3.3) by associating $g(t)$ with $f(t)$ and $h(t) = (x-t)^2$, gives, from (3.1)-(3.3)

$$\begin{aligned} (3.5) \quad & \left| \int_a^b (x-t)^2 f(t)\,dt - \frac{1}{b-a} \int_a^b (x-t)^2\,dt \cdot \int_a^b f(t)\,dt \right| \\ &\leq (b-a) \frac{M-m}{2} [T(h,h)]^{\frac{1}{2}}, \end{aligned}$$

where from (3.2)

$$T(h,h) = \frac{1}{b-a} \int_a^b (x-t)^4\,dt - \left[\frac{1}{b-a} \int_a^b (x-t)^2\,dt \right]^2. \tag{3.6}$$

Now,

$$\begin{aligned} (3.7) \quad \frac{1}{b-a} \int_a^b (x-t)^2\,dt &= \frac{(x-a)^3 + (b-x)^3}{3(b-a)} \\ &= \frac{1}{3}\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 \end{aligned}$$

and

$$\frac{1}{b-a} \int_a^b (x-t)^4\,dt = \frac{(x-a)^5 + (b-x)^5}{5(b-a)}$$

giving, from (3.6),

$$45T(h,h) = 9\left[\frac{(x-a)^5 + (b-x)^5}{b-a} \right] - 5\left[\frac{(x-a)^3 + (b-x)^3}{b-a} \right]^2. \tag{3.8}$$

Let $A = x - a$ and $B = b - x$ in (3.8) to give

$$\begin{aligned}
45T(h,h) &= 9\left(\frac{A^5+B^5}{A+B}\right) - 5\left(\frac{A^3+B^3}{A+B}\right)^2 \\
&= 9\left[A^4 - A^3B + A^2B^2 - AB^3 + B^4\right] - 5\left[A^2 - AB + B^2\right]^2 \\
&= \left(4A^2 - 7AB + 4B^2\right)(A+B)^2 \\
&= \left[\left(\frac{A+B}{2}\right)^2 + 15\left(\frac{A-B}{2}\right)^2\right](A+B)^2 .
\end{aligned}$$

Using the facts that $A + B = b - a$ and $A - B = 2x - (a+b)$ gives

$$T(h,h) = \frac{(b-a)^2}{45}\left[\left(\frac{b-a}{2}\right)^2 + 15\left(x - \frac{a+b}{2}\right)^2\right] \tag{3.9}$$

and from (3.7)

$$\begin{aligned}
\frac{1}{b-a}\int_a^b (x-t)^2\,dt &= \frac{A^3+B^3}{3(A+B)} = \frac{1}{3}\left[A^2 - AB + B^2\right] \\
&= \frac{1}{3}\left[\left(\frac{A+B}{2}\right)^2 + 3\left(\frac{A-B}{2}\right)^2\right],
\end{aligned}$$

giving

$$\frac{1}{b-a}\int_a^b (x-t)^2\,dt = \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 . \tag{3.10}$$

Hence, from (3.5), (3.9) (3.10) and (2.10), the first inequality in (3.4) results. The coarsest uniform bound is obtained by taking x at either end point. Thus the theorem is completely proved. ∎

Remark 4. *The best inequality otainable from (3.4) is at $x = \frac{a+b}{2}$ giving*

$$\left|\sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 - \frac{(b-a)^2}{12}\right| \leq \frac{M-m}{12}\frac{(b-a)^3}{\sqrt{5}}. \tag{3.11}$$

The result (3.11) is a tighter bound than that obtained in the first inequality of (2.12) since $0 < M - m < 2\|f\|_\infty$.

For a symmetric p.d.f. $E(X) = \frac{a+b}{2}$ and so the above results would give bounds on the variance.

The following results hold if the p.d.f $f(x)$ is differentiable, that is, for $f(x)$ absolutely continuous.

Theorem 7. *Let the conditions on Theorem 4 be satisfied. Further, suppose that f is differentiable and is such that*

$$\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty.$$

Then

$$|P_V(x)| \leq \frac{b-a}{\sqrt{12}}\|f'\|_\infty \cdot I(x), \tag{3.12}$$

where $P_V(x)$ is given by the left hand side of (3.4) and,

$$I(x)=\frac{(b-a)^2}{\sqrt{45}}\left[\left(\frac{b-a}{2}\right)^2+15\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}. \tag{3.13}$$

Proof. Let $h,g:[a,b]\to\mathbb{R}$ be absolutely continuous and h',g' be bounded. Then Chebychev's inequality holds (see [23])

$$T(h,g)\le\frac{(b-a)^2}{\sqrt{12}}\sup_{t\in[a,b]}|h'(t)|\cdot\sup_{t\in[a,b]}|g'(t)|.$$

Matić, Pečarić and Ujević [23] using a '*premature*' Grüss type argument proved that

$$T(h,g)\le\frac{(b-a)}{\sqrt{12}}\sup_{t\in[a,b]}|g'(t)|\sqrt{T(h,h)}. \tag{3.14}$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x-\cdot)^2$ with $h(\cdot)$ in (3.13) gives, from (3.5) and (3.9), $I(x)=(b-a)[T(h,h)]^{\frac{1}{2}}$, which simplifies to (3.13) and the theorem is proved. ∎

Theorem 8. *Let the conditions of Theorem 6 be satisfied. Further, suppose that f is locally absolutely continuous on (a,b) and let $f'\in L_2(a,b)$. Then*

$$|P_V(x)|\le\frac{b-a}{\pi}\|f'\|_2\cdot I(x), \tag{3.15}$$

where $P_V(x)$ is the left hand side of (3.4) and $I(x)$ is as given in (3.13).

Proof. The following result was obtained by Lupaş (see [23]). For $h,g:(a,b)\to\mathbb{R}$ locally absolutely continuous on (a,b) and $h',g'\in L_2(a,b)$, then

$$|T(h,g)|\le\frac{(b-a)^2}{\pi^2}\|h'\|_2\|g'\|_2,$$

where

$$\|k\|_2:=\left(\frac{1}{b-a}\int_a^b|k(t)|^2\right)^{\frac{1}{2}}\quad\text{for } k\in L_2(a,b).$$

Matić, Pečarić and Ujević [23] further show that

$$|T(h,g)|\le\frac{b-a}{\pi}\|g'\|_2\sqrt{T(h,h)}. \tag{3.16}$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x-\cdot)^2$ with h in (3.16) gives (3.15), where $I(x)$ is as found in (3.13), since from (3.5) and (3.9), $I(x)=(b-a)[T(h,h)]^{\frac{1}{2}}$. ∎

3.2. Alternate Grüss Type Results for Inequalities Involving the Variance.

Let

$$S(h(x))=h(x)-\mathcal{M}(h) \tag{3.17}$$

where

$$\mathcal{M}(h)=\frac{1}{b-a}\int_a^b h(u)\,du. \tag{3.18}$$

Then from (3.2),

$$\mathcal{T}(h,g)=\mathcal{M}(hg)-\mathcal{M}(h)\mathcal{M}(g). \tag{3.19}$$

Dragomir and McAndrew [19] have shown, that

$$\mathcal{T}(h,g)=\mathcal{T}(S(h),S(g)) \tag{3.20}$$

and proceeded to obtain bounds for a trapezoidal rule. Identity (3.20) is now applied to obtain bounds for the variance.

Theorem 9. *Let X be a random variable having the p.d.f. $f:[a,b]\subset\mathbb{R}\to\mathbb{R}_+$. Then for any $x\in[a,b]$ the following inequality holds, namely,*

$$|P_V(x)|\le\frac{8}{3}\nu^3(x)\left\|f(\cdot)-\frac{1}{b-a}\right\|_\infty \quad \text{if } f\in L_\infty[a,b], \tag{3.21}$$

where $P_V(x)$ is as defined by the left hand side of (3.4), and $\nu=\nu(x)=\frac{1}{3}\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2$.

Proof. Using identity (3.20), associate with $h(\cdot)$, $(x-\cdot)^2$ and $f(\cdot)$ with $g(\cdot)$. Then

$$\begin{aligned}&\int_a^b (x-t)^2 f(t)\,dt-\mathcal{M}\left((x-\cdot)^2\right) \\ &= \int_a^b\left[(x-t)^2-\mathcal{M}\left((x-\cdot)^2\right)\right]\left[f(t)-\frac{1}{b-a}\right]dt,\end{aligned} \tag{3.22}$$

where from (3.18),

$$\mathcal{M}\left((x-\cdot)^2\right)=\frac{1}{b-a}\int_a^b(x-t)^2\,dt=\frac{1}{3(b-a)}\left[(x-a)^3+(b-x)^3\right]$$

and so

$$3\mathcal{M}\left((x-\cdot)^2\right)=\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2. \tag{3.23}$$

Further, from (3.17),

$$S\left((x-\cdot)^2\right)=(x-t)^2-\mathcal{M}\left((x-\cdot)^2\right)$$

and so, on using (3.23)

$$S\left((x-\cdot)^2\right)=(x-t)^2-\frac{1}{3}\left(\frac{b-a}{2}\right)^2-\left(x-\frac{a+b}{2}\right)^2. \tag{3.24}$$

Now, from (3.22) and using (2.10), (3.23) and (3.24), the following identity is obtained

$$\begin{aligned}&\sigma^2(X)+[x-E(X)]^2-\frac{1}{3}\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right] \\ &= \int_a^b S\left((x-t)^2\right)\left(f(t)-\frac{1}{b-a}\right)dt,\end{aligned} \tag{3.25}$$

where $S(\cdot)$ is as given by (3.24). Taking the modulus of (3.25) gives

$$|P_V(x)|=\left|\int_a^b S\left((x-t)^2\right)\left(f(t)-\frac{1}{b-a}\right)dt\right|. \tag{3.26}$$

Observe that under different assumptions with regard to the norms of the p.d.f. $f(x)$ we may obtain a variety of bounds.

For $f \in L_\infty [a,b]$ then

$$|P_V(x)| \leq \left\| f(\cdot) - \frac{1}{b-a} \right\|_\infty \int_a^b \left| S\left((x-t)^2\right) \right| dt. \tag{3.27}$$

Now, let

$$S\left((x-t)^2\right) = (t-x)^2 - \nu^2 = (t - X_-)(t - X_+), \tag{3.28}$$

where

$$\begin{aligned} \nu^2 &= \mathcal{M}\left((x-\cdot)^2\right) = \frac{(x-a)^3 + (b-x)^3}{3(b-a)} \\ &= \frac{1}{3}\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2, \end{aligned} \tag{3.29}$$

and

$$X_- = x - \nu, \quad X_+ = x + \nu. \tag{3.30}$$

Then,

$$\begin{aligned} H(t) &= \int S\left((x-t)^2\right) dt = \int \left[(t-x)^2 - \nu^2\right] dt \\ &= \frac{(t-x)^3}{3} - \nu^2 t + k \end{aligned} \tag{3.31}$$

and so from (3.31) and using (3.28) - (3.29) gives,

$$\begin{aligned} &\int_a^b \left| S\left((x-t)^2\right) \right| dt \\ &= H(X_-) - H(a) - [H(X_+) - H(X_-)] + [H(b) - H(X_+)] \\ &= 2[H(X_-) - H(X_+)] + H(b) - H(a) \\ &= 2\left\{-\frac{\nu^3}{3} - \nu^2 X_- - \frac{\nu^3}{3} + \nu^2 X_+\right\} + \frac{(b-x)^3}{3} - \nu^2 b + \frac{(x-a)^3}{3} + \nu^2 a \\ &= 2\left[2\nu^3 - \frac{2}{3}\nu^3\right] + \frac{(b-x)^3 + (x-a)^3}{3} - \nu^2(b-a) \\ &= \frac{8}{3}\nu^3. \end{aligned} \tag{3.32}$$

Thus, substituting into (3.27), (3.26) and using (3.29) readily produces the result (3.21) and the theorem is proved. ∎

Remark 5. *Other bounds may be obtained for $f \in L_p [a,b]$, $p \geq 1$ however obtaining explicit expressions for these bounds is somewhat intricate and will not be considered further here. They involve the calculation of*

$$\sup_{t \in [a,b]} \left| (t-x)^2 - \nu^2 \right| = \max\left\{ \left|(x-a)^2 - \nu^2\right|, \nu^2, \left|(b-x)^2 - \nu^2\right| \right\}$$

for $f \in L_1 [a,b]$ and

$$\left(\int_a^b \left| (t-x)^2 - \nu^2 \right|^q dt \right)^{\frac{1}{q}}$$

for $f \in L_p [a,b]$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, where ν^2 is given by (3.29).

4. Some Inequalities for Absolutely Continuous P.D.F.s

We start with the following lemma which is interesting in itself.

Lemma 1. *Let X be a random variable whose probability density function $f : [a,b] \to \mathbb{R}_+$ is absolutely continuous on $[a,b]$. Then we have the identity*

$$\begin{aligned} & \sigma^2(X) + [E(X) - x]^2 \\ = & \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 + \frac{1}{b-a}\int_a^b \int_a^b (t-x)^2 p(t,s) f'(s)\, ds dt, \end{aligned} \tag{4.1}$$

where the kernel $p : [a,b]^2 \to \mathbb{R}$ is given by

$$p(t,s) := \begin{cases} s-a & \text{if } a \le s \le t \le b \\ s-b & \text{if } a \le t < s \le b \end{cases},$$

for all $x \in [a,b]$.

Proof. We use the identity (see (2.10))

$$\sigma^2(X) + [E(X) - x]^2 = \int_a^b (x-t)^2 f(t)\, dt \tag{4.2}$$

for all $x \in [a,b]$.

On the other hand, we know that (see for example [22] for a simple proof using integration by parts)

$$f(t) = \frac{1}{b-a}\int_a^b f(s)\, ds + \frac{1}{b-a}\int_a^b p(t,s) f'(s)\, ds \tag{4.3}$$

for all $t \in [a,b]$.

Substituting (4.3) in (4.2) we obtain

$$\begin{aligned} & \sigma^2(X) + [E(X) - x]^2 \\ = & \int_a^b (t-x)^2 \left[\frac{1}{b-a}\int_a^b f(s)\, ds + \frac{1}{b-a}\int_a^b p(t,s) f'(s)\, ds\right] dt \\ = & \frac{1}{b-a} \cdot \frac{1}{3}\left[(x-a)^3 + (b-x)^3\right] + \frac{1}{b-a}\int_a^b \int_a^b (t-x)^2 p(t,s) f'(s)\, ds dt. \end{aligned} \tag{4.4}$$

Taking into account the fact that

$$\frac{1}{3}\left[(x-a)^3 + (b-x)^3\right] = \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2, \quad x \in [a,b],$$

then, by (4.4) we deduce the desired result (4.1). ∎

The following inequality for P.D.F.s which are absolutely continuous and have the derivatives essentially bounded holds.

Theorem 10. *If $f : [a,b] \to \mathbb{R}_+$ is absolutely continuous on $[a,b]$ and $f' \in L_\infty [a,b]$, i.e., $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)| < \infty$, then we have the inequality:*

$$\left|\sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2\right| \leq \frac{(b-a)^2}{3}\left[\frac{(b-a)^2}{10} + \left(x - \frac{a+b}{2}\right)^2\right]\|f'\|_\infty \tag{4.5}$$

for all $x \in [a,b]$.

Proof. Using Lemma 1, we have

$$\begin{aligned}
&\left|\sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2\right| \\
&= \frac{1}{b-a}\left|\int_a^b\int_a^b (t-x)^2 p(t,s) f'(s)\, dsdt\right| \\
&\leq \frac{1}{b-a}\int_a^b\int_a^b (t-x)^2 |p(t,s)| |f'(s)|\, dsdt \\
&\leq \frac{\|f'\|_\infty}{b-a}\int_a^b\int_a^b (t-x)^2 |p(t,s)|\, dsdt.
\end{aligned}$$

We have

$$\begin{aligned}
I &:= \int_a^b\int_a^b (t-x)^2 |p(t,s)|\, dsdt \\
&= \int_a^b (t-x)^2\left[\int_a^t (s-a)\, ds + \int_t^b (b-s)\, ds\right] dt \\
&= \int_a^b (t-x)^2\left[\frac{(t-a)^2 + (b-t)^2}{2}\right] dt \\
&= \frac{1}{2}\left[\int_a^b (t-x)^2 (t-a)^2\, dt + \int_a^b (t-x)^2 (b-t)^2\, dt\right] \\
&= \frac{(I_a + I_b)}{2}.
\end{aligned}$$

Let $A = x - a$, $B = b - x$ then

$$\begin{aligned}
I_a &= \int_a^b (t-x)^2 (t-a)^2\, dt \\
&= \int_0^{b-a} \left(u^2 - 2Au + A^2\right) u^2 du \\
&= \frac{(b-a)^3}{3}\left[A^2 - \frac{3}{2}A(b-a) + \frac{3}{5}(b-a)^2\right]
\end{aligned}$$

and

$$
\begin{aligned}
I_b &= \int_a^b (t-x)^2 (b-t)^2 \, dt \\
&= \int_0^{b-a} \left(u^2 - 2Bu + B^2\right) u^2 du \\
&= \frac{(b-a)^3}{3} \left[B^2 - \frac{3}{2} B (b-a) + \frac{3}{5} (b-a)^2 \right]
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{I_a + I_b}{2} &= \frac{(b-a)^3}{3} \left[\frac{A^2 + B^2}{2} - \frac{3}{4} (A+B)(b-a) + \frac{3}{5} (b-a)^2 \right] \\
&= \frac{(b-a)^3}{3} \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 - 3 \frac{(b-a)^2}{20} \right] \\
& \quad \frac{(b-a)^3}{3} \left[\frac{(b-a)^2}{10} + \left(x - \frac{a+b}{2} \right)^2 \right]
\end{aligned}
$$

and the theorem is proved. ∎

The best inequality we can get from (4.5) is embodied in the following corollary.

Corollary 3. *If f is as in Theorem 10, then we have*

$$
\left| \sigma^2 (X) + \left[E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{(b-a)^4}{30} \left\| f' \right\|_\infty . \tag{4.6}
$$

We now analyze the case where f' is a Lebesgue $p-$integrable mapping with $p \in (1, \infty)$.

Remark 6. *The results of Theorem 10 may be compared with those of Theorem 7. It may be shown that both bounds are convex and symmetric about $x = \frac{a+b}{2}$. Further, the bound given by the 'premature' Chebychev approach, namely from (3.12)-(3.13) is tighter than that obtained by the current approach (4.5) which may be shown from the following. Let these bounds be described by B_p and B_c so that, neglecting the common terms*

$$
B_p = \frac{b-a}{2\sqrt{15}} \left[\left(\frac{b-a}{2} \right)^2 + 15Y \right]^{\frac{1}{2}}
$$

and

$$
B_c = \frac{(b-a)^2}{100} + Y,
$$

where

$$
Y = \left(x - \frac{a+b}{2} \right)^2 .
$$

It may be shown through some straightforward algebra that $B_c^2 - B_p^2 > 0$ for all $x \in [a, b]$ so that $B_c > B_p$.
The current development does however have the advantage that the identity (4.1) is satisfied, thus allowing bounds for $L_p [a, b]$, $p \geq 1$ rather than the infinity norm.

Theorem 11. *If $f:[a,b]\to\mathbb{R}_+$ is absolutely continuous on $[a,b]$ and $f'\in L_p$, i.e.,*

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p\,dt\right)^{\frac{1}{p}} < \infty,\ \ p\in(1,\infty)$$

then we have the inequality

$$\left|\sigma^2(X)+[E(X)-x]^2-\frac{(b-a)^2}{12}-\left(x-\frac{a+b}{2}\right)^2\right| \tag{4.7}$$
$$\le \frac{\|f'\|_p}{(b-a)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}\left[(x-a)^{3q+2}\tilde{B}\left(\frac{b-a}{x-a},2q+1,q+2\right)\right.$$
$$\left.+(b-x)^{3q+2}\tilde{B}\left(\frac{b-a}{b-x},2q+1,q+2\right)\right]$$

for all $x\in[a,b]$, when $\frac{1}{p}+\frac{1}{q}=1$ and $\tilde{B}(\cdot,\cdot,\cdot)$ is the quasi incomplete Euler's Beta mapping:

$$\tilde{B}(z;\alpha,\beta) := \int_0^z (u-1)^{\alpha-1}u^{\beta-1}du,\ \ \alpha,\beta>0,\ z\ge 1.$$

Proof. Using Lemma 1, we have, as in Theorem 10, that

$$\left|\sigma^2(X)+[E(X)-x]^2-\frac{(b-a)^2}{12}-\left(x-\frac{a+b}{2}\right)^2\right| \tag{4.8}$$
$$\le \frac{1}{b-a}\int_a^b\int_a^b (t-x)^2|p(t,s)||f'(s)|\,dsdt.$$

Using Hölder's integral inequality for double integrals, we have

$$\int_a^b\int_a^b (t-x)^2|p(t,s)||f'(s)|\,dsdt \tag{4.9}$$
$$\le \left(\int_a^b\int_a^b |f'(s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_a^b (t-x)^{2q}|p(t,s)|^q\,dsdt\right)^{\frac{1}{q}}$$
$$= (b-a)^{\frac{1}{p}}\|f'\|_p\left(\int_a^b\int_a^b (t-x)^{2q}|p(t,s)|^q\,dsdt\right)^{\frac{1}{q}},$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

We have to compute the integral

$$D := \int_a^b\int_a^b (t-x)^{2q}|p(t,s)|^q\,dsdt \tag{4.10}$$
$$= \int_a^b (t-x)^{2q}\left[\int_a^t (s-a)^q\,ds+\int_t^b (b-s)^q\,ds\right]dt$$
$$= \int_a^b (t-x)^{2q}\left[\frac{(t-a)^{q+1}+(b-t)^{q+1}}{q+1}\right]dt$$
$$= \frac{1}{q+1}\left[\int_a^b (t-x)^{2q}(t-a)^{q+1}\,dt+\int_a^b (t-x)^{2q}(b-t)^{q+1}\,dt\right].$$

Define

$$E := \int_a^b (t-x)^{2q} (t-a)^{q+1} \, dt. \tag{4.11}$$

If we consider the change of variable $t = (1-u)\, a + ux$, we have $t = a$ implies $u = 0$ and $t = b$ implies $u = \frac{b-a}{x-a}$, $dt = (x-a)\, du$ and then

$$\begin{aligned} E &= \int_0^{\frac{b-a}{x-a}} [(1-u)\, a + ux - x]^{2q} [(1-u)\, a + ux - a] (x-a)\, du \\ &= (x-a)^{3q+2} \int_0^{\frac{b-a}{x-a}} (u-1)^{2q} u^{q+1} du \\ &= (x-a)^{3q+2} \tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2\right). \end{aligned} \tag{4.12}$$

Define

$$F := \int_a^b (t-x)^{2q} (b-t)^{q+1} \, dt. \tag{4.13}$$

If we consider the change of variable $t = (1-v)\, b + vx$, we have $t = b$ implies $v = 0$, and $t = a$ implies $v = \frac{b-a}{b-x}$, $dt = (x-b)\, dv$ and then

$$\begin{aligned} F &= \int_{\frac{b-a}{b-x}}^0 [(1-v)\, b + vx - x]^{2q} [b - (1-v)\, b - vx]^{q+1} (x-b)\, dv \\ &= (b-x)^{3q+2} \int_0^{\frac{b-a}{b-x}} (v-1)^{2q} v^{q+1} dv \\ &= (b-x)^{3q+2} \tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2\right). \end{aligned} \tag{4.14}$$

Now, using the inequalities (4.8)-(4.9) and the relations (4.10)-(4.14), since $D = \frac{1}{q+1}(E+F)$, we deduce the desired estimate (4.7). ∎

The following corollary is natural to be considered.

Corollary 4. *Let f be as in Theorem 11. Then, we have the inequality:*

$$\begin{aligned} &\left| \sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 - \frac{(b-a)^2}{12} \right| \\ &\leq \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} [B(2q+1, q+1) + \Psi(2q+1, q+2)]^{\frac{1}{q}}, \end{aligned} \tag{4.15}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and $B(\cdot, \cdot)$ is Euler's Beta mapping and $\Psi(\alpha, \beta) := \int_0^1 u^{\alpha-1} (u+1)^{\beta-1} du$, $\alpha, \beta > 0$.

Proof. In (4.7) put $x = \frac{a+b}{2}$.

The left side is clear.

Now

$$\begin{aligned}\tilde{B}(2,2q+1,q+2) &= \int_0^2 (u-1)^{2q} u^{q+1} du \\ &= \int_0^1 (u-1)^{2q} u^{q+1} du + \int_1^2 (u-1)^{2q} u^{q+1} du \\ &= B(2q+1,q+2) + \Psi(2q+1,q+2).\end{aligned}$$

The right hand side of (4.7) is thus:-

$$\begin{aligned}&\frac{\|f'\|_p \left(\frac{b-a}{2}\right)^{\frac{3q+2}{q}}}{(b-a)^{\frac{1}{p}} (q+1)^{\frac{1}{q}}} \left[2B(2q+1,q+2) + 2\Psi(2q+1,q+2)\right]^{\frac{1}{q}} \\ =\ & \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} \left[B(2q+1,q+2) + \Psi(2q+1,q+2)\right]^{\frac{1}{q}}\end{aligned}$$

and the corollary is proved. ∎

Finally, as f is absolutely continuous, $f' \in L_1[a,b]$ and $\|f'\|_1 = \int_a^b |f'(t)| dt$, and we can state the following theorem.

Theorem 12. *If the p.d.f., $f : [a,b] \to \mathbb{R}_+$ is absolutely continuous on $[a,b]$, then*

$$\begin{aligned}&\left|\sigma^2(X) + [E(X)-x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2\right| \\ \leq\ & \|f'\|_1 (b-a)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2\end{aligned} \tag{4.16}$$

for all $x \in [a,b]$.

Proof. As above, we can state that

$$\begin{aligned}&\left|\sigma^2(X) + [E(X)-x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2\right| \\ \leq\ & \frac{1}{b-a}\int_a^b \int_a^b (t-x)^2 |p(t,s)| |f'(s)| dsdt \\ \leq\ & \sup_{(t,s)\in[a,b]^2} \left[(t-x)^2 |p(t,s)|\right] \frac{1}{b-a}\int_a^b \int_a^b |f'(s)| dsdt \\ =\ & \|f'\|_1 G\end{aligned}$$

where

$$\begin{aligned}G &:= \sup_{(t,s)\in[a,b]^2} \left[(t-x)^2 |p(t,s)|\right] \leq (b-a) \sup_{t\in[a,b]} (t-x)^2 \\ &= (b-a)\left[\max(x-a,b-x)\right]^2 \\ &= (b-a)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2,\end{aligned}$$

and the theorem is proved. ∎

It is clear that the best inequality we can get from (4.16) is the one when $x = \frac{a+b}{2}$, giving the following corollary.

Corollary 5. *With the assumptions of Theorem 12, we have:*

$$\left|\sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2-\frac{(b-a)^2}{12}\right|\le\frac{(b-a)^3}{4}\left\|f'\right\|_1. \tag{4.17}$$

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