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*Some Inequalities for the Dispersion of a Random Variable whose PDF is Defined on a Finite Interval*

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# SOME INEQUALITIES FOR THE DISPERSION OF A RANDOM VARIABLE WHOSE PDF IS DEFINED ON A FINITE INTERVAL

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ABSTRACT. Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval and applications are given.

## 1. INTRODUCTION

In this note we obtain some inequalities for the dispersion of a continuous random variable  $X$  having the probability density function (p.d.f.)  $f$  defined on a finite interval  $[a, b]$ .

Tools used include: Korkine's identity, which plays a central role in the proof of Chebychev's integral inequality for synchronous mappings [24], Hölder's weighted inequality for double integrals and an integral identity connecting the variance  $\sigma^2(X)$  and the expectation  $E(X)$ . Perturbed results are also obtained by using Grüss, Chebyshev and Lupaş inequalities. In Section 4, results from an identity involving a double integral are obtained for a variety of norms.

## 2. SOME INEQUALITIES FOR DISPERSION

Let  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$  be the p.d.f. of the random variable  $X$  and

$$E(X) := \int_a^b t f(t) dt$$

its *expectation* and

$$\begin{aligned} \sigma(X) &= \left[ \int_a^b (t - E(X))^2 f(t) dt \right]^{\frac{1}{2}} \\ &= \left[ \int_a^b t^2 f(t) dt - [E(X)]^2 \right]^{\frac{1}{2}} \end{aligned}$$

its *dispersion* or *standard deviation*.

The following theorem holds.

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**Theorem 1.** *With the above assumptions, we have*

$$(2.1) \quad 0 \leq \sigma(X) \leq \begin{cases} \frac{\sqrt{3}(b-a)^2}{6} \|f\|_\infty & \text{provided } f \in L_\infty[a, b]; \\ \frac{\sqrt{2}(b-a)^{1+\frac{1}{q}}}{2[(q+1)(2q+1)]^{\frac{2}{q}}} \|f\|_p & \text{provided } f \in L_p[a, b] \\ & \text{and } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\sqrt{2}(b-a)}{2}. & \end{cases}$$

*Proof.* Korkine's identity [24], is

$$(2.2) \quad \begin{aligned} & \int_a^b p(t) dt \int_a^b p(t) g(t) h(t) dt - \int_a^b p(t) g(t) dt \cdot \int_a^b p(t) h(t) dt \\ &= \frac{1}{2} \int_a^b \int_a^b p(t) p(s) (g(t) - g(s)) (h(t) - h(s)) dt ds, \end{aligned}$$

which holds for the measurable mappings  $p, g, h : [a, b] \rightarrow \mathbb{R}$  for which the integrals involved in (2.2) exist and are finite. Choose in (2.2)  $p(t) = f(t)$ ,  $g(t) = h(t) = t - E(X)$ ,  $t \in [a, b]$  to get

$$(2.3) \quad \sigma^2(X) = \frac{1}{2} \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds.$$

It is obvious that

$$(2.4) \quad \begin{aligned} & \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds \\ & \leq \sup_{(t,s) \in [a,b]^2} |f(t) f(s)| \int_a^b \int_a^b (t-s)^2 dt ds \\ & = \frac{(b-a)^4}{6} \|f\|_\infty^2 \end{aligned}$$

and then, by (2.3), we obtain the first part of (2.1).

For the second part, we apply Hölder's integral inequality for double integrals to obtain

$$\begin{aligned} & \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds \\ & \leq \left( \int_a^b \int_a^b f^p(t) f^p(s) dt ds \right)^{\frac{1}{p}} \left( \int_a^b \int_a^b (t-s)^{2q} dt ds \right)^{\frac{1}{q}} \\ & = \|f\|_p^2 \left[ \frac{(b-a)^{2q+2}}{(q+1)(2q+1)} \right]^{\frac{1}{q}}, \end{aligned}$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , and the second inequality in (2.1) is proved.

For the last part, observe that

$$\begin{aligned} \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds & \leq \sup_{(t,s) \in [a,b]^2} |(t-s)|^2 \int_a^b \int_a^b f(t) f(s) dt ds \\ & = (b-a)^2 \end{aligned}$$

as

$$\int_a^b \int_a^b f(t) f(s) dt ds = \int_a^b f(t) dt \int_a^b f(s) ds = 1.$$

■

Using a finer argument, the last inequality in (2.1) can be improved as follows.

**Theorem 2.** *Under the above assumptions, we have*

$$(2.5) \quad 0 \leq \sigma(X) \leq \frac{1}{2}(b-a).$$

*Proof.* We use the following Grüss type inequality:

$$(2.6) \quad 0 \leq \frac{\int_a^b p(t) g^2(t) dt}{\int_a^b p(t) dt} - \left( \frac{\int_a^b p(t) g(t) dt}{\int_a^b p(t) dt} \right)^2 \leq \frac{1}{4}(M-m)^2,$$

provided that  $p, g$  are measurable on  $[a, b]$  and all the integrals in (2.6) exist and are finite,  $\int_a^b p(t) dt > 0$  and  $m \leq g \leq M$  a.e. on  $[a, b]$ . For a proof of this inequality see [19].

Choose in (2.6),  $p(t) = f(t)$ ,  $g(t) = t - E(X)$ ,  $t \in [a, b]$ . Observe that in this case  $m = a - E(X)$ ,  $M = b - E(X)$  and then, by (2.6) we deduce (2.5). ■

**Remark 1.** *The same conclusion can be obtained for the choice  $p(t) = f(t)$  and  $g(t) = t$ ,  $t \in [a, b]$ .*

The following result holds.

**Theorem 3.** *Let  $X$  be a random variable having the p.d.f. given by  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$ . Then for any  $x \in [a, b]$  we have the inequality:*

$$(2.7) \quad \sigma^2(X) + (x - E(X))^2 \leq \begin{cases} (b-a) \left[ \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 \right] \|f\|_\infty & \text{provided } f \in L_\infty[a, b]; \\ \left[ \frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} \right]^{\frac{1}{q}} \|f\|_p & \text{provided } f \in L_p[a, b], p > 1, \\ & \text{and } \frac{1}{p} + \frac{1}{q} = 1; \\ \left( \frac{b-a}{2} + \left|x - \frac{a+b}{2}\right| \right)^2. & \end{cases}$$

*Proof.* We observe that

$$(2.8) \quad \begin{aligned} \int_a^b (x-t)^2 f(t) dt &= \int_a^b (x^2 - 2xt + t^2) f(t) dt \\ &= x^2 - 2xE(X) + \int_a^b t^2 f(t) dt \end{aligned}$$

and as

$$(2.9) \quad \sigma^2(X) = \int_a^b t^2 f(t) dt - [E(X)]^2,$$

we get, by (2.8) and (2.9),

$$(2.10) \quad [x - E(X)]^2 + \sigma^2(X) = \int_a^b (x-t)^2 f(t) dt,$$

which is of interest in itself too.

We observe that

$$\begin{aligned} \int_a^b (x-t)^2 f(t) dt &\leq \operatorname{ess\,sup}_{t \in [a,b]} |f(t)| \int_a^b (x-t)^2 dt \\ &= \|f\|_\infty \frac{(b-x)^3 + (x-a)^3}{3} \\ &= (b-a) \|f\|_\infty \left[ \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 \right] \end{aligned}$$

and the first inequality in (2.7) is proved.

For the second inequality, observe that by Hölder's integral inequality,

$$\begin{aligned} \int_a^b (x-t)^2 f(t) dt &\leq \left( \int_a^b f^p(t) dt \right)^{\frac{1}{p}} \left( \int_a^b (x-t)^{2q} dt \right)^{\frac{1}{q}} \\ &= \|f\|_p \left[ \frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} \right]^{\frac{1}{q}}, \end{aligned}$$

and the second inequality in (2.7) is established.

Finally, observe that,

$$\begin{aligned} \int_a^b (x-t)^2 f(t) dt &\leq \sup_{t \in [a,b]} (x-t)^2 \int_a^b f(t) dt \\ &= \max \left\{ (x-a)^2, (b-x)^2 \right\} \\ &= (\max \{x-a, b-x\})^2 \\ &= \left( \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^2, \end{aligned}$$

and the theorem is proved. ■

The following corollaries are easily deduced.

**Corollary 1.** *With the above assumptions, we have*

$$(2.11) \quad 0 \leq \sigma(X) \leq \begin{cases} (b-a)^{\frac{1}{2}} \left[ \frac{(b-a)^2}{12} + \left(E(X) - \frac{a+b}{2}\right)^2 \right]^{\frac{1}{2}} \|f\|_\infty^{\frac{1}{2}} \\ \text{provided } f \in L_\infty[a, b]; \\ \left[ \frac{(b-E(X))^{2q+1} + (E(X)-a)^{2q+1}}{2q+1} \right]^{\frac{1}{2q}} \|f\|_p^{\frac{1}{2}} \\ \text{if } f \in L_p[a, b], p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{b-a}{2} + \left| E(X) - \frac{a+b}{2} \right|. \end{cases}$$

**Remark 2.** *The last inequality in (2.11) is worse than the inequality (2.5), obtained by a technique based on Grüss' inequality.*

The best inequality we can get from (2.7) is that one for which  $x = \frac{a+b}{2}$ , and this applies for all the bounds as

$$\begin{aligned} \min_{x \in [a,b]} \left[ \frac{(b-a)^2}{12} + \left( x - \frac{a+b}{2} \right)^2 \right] &= \frac{(b-a)^2}{12}, \\ \min_{x \in [a,b]} \frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} &= \frac{(b-a)^{2q+1}}{2^{2q}(2q+1)}, \end{aligned}$$

and

$$\min_{x \in [a,b]} \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] = \frac{b-a}{2}.$$

Consequently, we can state the following corollary as well.

**Corollary 2.** *With the above assumptions, we have the inequality:*

$$(2.12) \quad 0 \leq \sigma^2(X) + \left[ E(X) - \frac{a+b}{2} \right]^2 \leq \begin{cases} \frac{(b-a)^3}{12} \|f\|_\infty & \text{provided } f \in L_\infty[a, b]; \\ \frac{(b-a)^{2q+1}}{4(2q+1)^{\frac{1}{q}}} \|f\|_p & \text{if } f \in L_p[a, b], p > 1, \\ & \text{and } \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^2}{12}. \end{cases}$$

**Remark 3.** *from the last inequality in (2.12), we obtain*

$$(2.13) \quad 0 \leq \sigma^2(X) \leq (b - E(X))(E(X) - a) \leq \frac{1}{4}(b-a)^2,$$

*which is an improvement on (2.5).*

### 3. PERTURBED RESULTS USING GRÜSS TYPE INEQUALITIES

In 1935, G. Grüss ( see for example [26]) proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of the integrals.

**Theorem 4.** *Let  $h, g : [a, b] \rightarrow \mathbb{R}$  be two integrable mappings such that  $\phi \leq h(x) \leq \Phi$  and  $\gamma \leq g(x) \leq \Gamma$  for all  $x \in [a, b]$ , where  $\phi, \Phi, \gamma, \Gamma$  are real numbers. Then,*

$$(3.1) \quad |T(h, g)| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma),$$

where

$$(3.2) \quad T(h, g) = \frac{1}{b-a} \int_a^b h(x) g(x) dx - \frac{1}{b-a} \int_a^b h(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx$$

*and the inequality is sharp, in the sense that the constant  $\frac{1}{4}$  cannot be replaced by a smaller one.*

For a simple proof of this as well as for extensions, generalisations, discrete variants and other associated material, see [25], and [1]-[21] where further references are given.

A ‘premature’ Grüss inequality is embodied in the following theorem which was proved in [23]. It provides a sharper bound than the above Grüss inequality.

**Theorem 5.** *Let  $h, g$  be integrable functions defined on  $[a, b]$  and let  $d \leq g(t) \leq D$ . Then*

$$(3.3) \quad |T(h, g)| \leq \frac{D-d}{2} |T(h, h)|^{\frac{1}{2}},$$

where  $T(h, g)$  is as defined in (3.2).

Theorem 5 will now be used to provide a perturbed rule involving the variance and mean of a p.d.f.

**3.1. Perturbed Results Using ‘Premature’ Inequalities.** In this subsection we develop some perturbed results.

**Theorem 6.** *Let  $X$  be a random variable having the p.d.f. given by  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$ . Then for any  $x \in [a, b]$  and  $m \leq f(x) \leq M$  we have the inequality*

$$(3.4) \quad |P_V(x)| \quad = \left| \sigma^2(X) + (x - E(X))^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{M-m}{2} \cdot \frac{(b-a)^2}{\sqrt{45}} \left[ \left(\frac{b-a}{2}\right)^2 + 15 \left(x - \frac{a+b}{2}\right) \right]^{\frac{1}{2}} \\ \leq (M-m) \frac{(b-a)^3}{\sqrt{45}}.$$

*Proof.* Applying the ‘premature’ Grüss result (3.3) by associating  $g(t)$  with  $f(t)$  and  $h(t) = (x-t)^2$ , gives, from (3.1)-(3.3)

$$(3.5) \quad \left| \int_a^b (x-t)^2 f(t) dt - \frac{1}{b-a} \int_a^b (x-t)^2 dt \cdot \int_a^b f(t) dt \right| \\ \leq (b-a) \frac{M-m}{2} [T(h, h)]^{\frac{1}{2}},$$

where from (3.2)

$$(3.6) \quad T(h, h) = \frac{1}{b-a} \int_a^b (x-t)^4 dt - \left[ \frac{1}{b-a} \int_a^b (x-t)^2 dt \right]^2.$$

Now,

$$(3.7) \quad \frac{1}{b-a} \int_a^b (x-t)^2 dt = \frac{(x-a)^3 + (b-x)^3}{3(b-a)} \\ = \frac{1}{3} \left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2$$

and

$$\frac{1}{b-a} \int_a^b (x-t)^4 dt = \frac{(x-a)^5 + (b-x)^5}{5(b-a)}$$

giving, from (3.6),

$$(3.8) \quad 45T(h, h) = 9 \left[ \frac{(x-a)^5 + (b-x)^5}{b-a} \right] - 5 \left[ \frac{(x-a)^3 + (b-x)^3}{b-a} \right]^2.$$

Let  $A = x - a$  and  $B = b - x$  in (3.8) to give

$$\begin{aligned} 45T(h, h) &= 9 \left( \frac{A^5 + B^5}{A + B} \right) - 5 \left( \frac{A^3 + B^3}{A + B} \right)^2 \\ &= 9 [A^4 - A^3B + A^2B^2 - AB^3 + B^4] - 5 [A^2 - AB + B^2]^2 \\ &= (4A^2 - 7AB + 4B^2) (A + B)^2 \\ &= \left[ \left( \frac{A + B}{2} \right)^2 + 15 \left( \frac{A - B}{2} \right)^2 \right] (A + B)^2. \end{aligned}$$

Using the facts that  $A + B = b - a$  and  $A - B = 2x - (a + b)$  gives

$$(3.9) \quad T(h, h) = \frac{(b-a)^2}{45} \left[ \left( \frac{b-a}{2} \right)^2 + 15 \left( x - \frac{a+b}{2} \right)^2 \right]$$

and from (3.7)

$$\begin{aligned} \frac{1}{b-a} \int_a^b (x-t)^2 dt &= \frac{A^3 + B^3}{3(A+B)} = \frac{1}{3} [A^2 - AB + B^2] \\ &= \frac{1}{3} \left[ \left( \frac{A+B}{2} \right)^2 + 3 \left( \frac{A-B}{2} \right)^2 \right], \end{aligned}$$

giving

$$(3.10) \quad \frac{1}{b-a} \int_a^b (x-t)^2 dt = \frac{(b-a)^2}{12} + \left( x - \frac{a+b}{2} \right)^2.$$

Hence, from (3.5), (3.9) (3.10) and (2.10), the first inequality in (3.4) results. The coarsest uniform bound is obtained by taking  $x$  at either end point. Thus the theorem is completely proved. ■

**Remark 4.** *The best inequality obtainable from (3.4) is at  $x = \frac{a+b}{2}$  giving*

$$(3.11) \quad \left| \sigma^2(X) + \left[ E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{M-m}{12} \frac{(b-a)^3}{\sqrt{5}}.$$

The result (3.11) is a tighter bound than that obtained in the first inequality of (2.12) since  $0 < M - m < 2 \|f\|_\infty$ .

For a symmetric p.d.f.  $E(X) = \frac{a+b}{2}$  and so the above results would give bounds on the variance.

The following results hold if the p.d.f.  $f(x)$  is differentiable, that is, for  $f(x)$  absolutely continuous.

**Theorem 7.** *Let the conditions on Theorem 4 be satisfied. Further, suppose that  $f$  is differentiable and is such that*

$$\|f'\|_\infty := \sup_{t \in [a, b]} |f'(t)| < \infty.$$

Then

$$(3.12) \quad |P_V(x)| \leq \frac{b-a}{\sqrt{12}} \|f'\|_\infty \cdot I(x),$$



where  $P_V(x)$  is given by the left hand side of (3.4) and,

$$(3.13) \quad I(x) = \frac{(b-a)^2}{\sqrt{45}} \left[ \left( \frac{b-a}{2} \right)^2 + 15 \left( x - \frac{a+b}{2} \right)^2 \right]^{\frac{1}{2}}.$$

*Proof.* Let  $h, g : [a, b] \rightarrow \mathbb{R}$  be absolutely continuous and  $h', g'$  be bounded. Then Chebychev's inequality holds (see [23])

$$T(h, g) \leq \frac{(b-a)^2}{\sqrt{12}} \sup_{t \in [a, b]} |h'(t)| \cdot \sup_{t \in [a, b]} |g'(t)|.$$

Matić, Pečarić and Ujević [23] using a ‘premature’ Grüss type argument proved that

$$(3.14) \quad T(h, g) \leq \frac{(b-a)}{\sqrt{12}} \sup_{t \in [a, b]} |g'(t)| \sqrt{T(h, h)}.$$

Associating  $f(\cdot)$  with  $g(\cdot)$  and  $(x - \cdot)^2$  with  $h(\cdot)$  in (3.13) gives, from (3.5) and (3.9),  $I(x) = (b-a) [T(h, h)]^{\frac{1}{2}}$ , which simplifies to (3.13) and the theorem is proved. ■

**Theorem 8.** *Let the conditions of Theorem 6 be satisfied. Further, suppose that  $f$  is locally absolutely continuous on  $(a, b)$  and let  $f' \in L_2(a, b)$ . Then*

$$(3.15) \quad |P_V(x)| \leq \frac{b-a}{\pi} \|f'\|_2 \cdot I(x),$$

where  $P_V(x)$  is the left hand side of (3.4) and  $I(x)$  is as given in (3.13).

*Proof.* The following result was obtained by Lupas̃ (see [23]). For  $h, g : (a, b) \rightarrow \mathbb{R}$  locally absolutely continuous on  $(a, b)$  and  $h', g' \in L_2(a, b)$ , then

$$|T(h, g)| \leq \frac{(b-a)^2}{\pi^2} \|h'\|_2 \|g'\|_2,$$

where

$$\|k\|_2 := \left( \frac{1}{b-a} \int_a^b |k(t)|^2 \right)^{\frac{1}{2}} \quad \text{for } k \in L_2(a, b).$$

Matić, Pečarić and Ujević [23] further show that

$$(3.16) \quad |T(h, g)| \leq \frac{b-a}{\pi} \|g'\|_2 \sqrt{T(h, h)}.$$

Associating  $f(\cdot)$  with  $g(\cdot)$  and  $(x - \cdot)^2$  with  $h$  in (3.16) gives (3.15), where  $I(x)$  is as found in (3.13), since from (3.5) and (3.9),  $I(x) = (b-a) [T(h, h)]^{\frac{1}{2}}$ . ■

**3.2. Alternate Grüss Type Results for Inequalities Involving the Variance.** Let

$$(3.17) \quad S(h(x)) = h(x) - \mathcal{M}(h)$$

where

$$(3.18) \quad \mathcal{M}(h) = \frac{1}{b-a} \int_a^b h(u) du.$$

Then from (3.2),

$$(3.19) \quad \mathcal{T}(h, g) = \mathcal{M}(hg) - \mathcal{M}(h) \mathcal{M}(g).$$

Dragomir and McAndrew [19] have shown, that

$$(3.20) \quad \mathcal{T}(h, g) = \mathcal{T}(S(h), S(g))$$

and proceeded to obtain bounds for a trapezoidal rule. Identity (3.20) is now applied to obtain bounds for the variance.

**Theorem 9.** *Let  $X$  be a random variable having the p.d.f.  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$ . Then for any  $x \in [a, b]$  the following inequality holds, namely,*

$$(3.21) \quad |P_V(x)| \leq \frac{8}{3} \nu^3(x) \left\| f(\cdot) - \frac{1}{b-a} \right\|_{\infty} \quad \text{if } f \in L_{\infty}[a, b],$$

where  $P_V(x)$  is as defined by the left hand side of (3.4), and  $\nu = \nu(x) = \frac{1}{3} \left( \frac{b-a}{2} \right)^2 + \left( x - \frac{a+b}{2} \right)^2$ .

*Proof.* Using identity (3.20), associate with  $h(\cdot)$ ,  $(x - \cdot)^2$  and  $f(\cdot)$  with  $g(\cdot)$ . Then

$$(3.22) \quad \begin{aligned} & \int_a^b (x-t)^2 f(t) dt - \mathcal{M}\left((x-\cdot)^2\right) \\ &= \int_a^b \left[ (x-t)^2 - \mathcal{M}\left((x-\cdot)^2\right) \right] \left[ f(t) - \frac{1}{b-a} \right] dt, \end{aligned}$$

where from (3.18),

$$\mathcal{M}\left((x-\cdot)^2\right) = \frac{1}{b-a} \int_a^b (x-t)^2 dt = \frac{1}{3(b-a)} \left[ (x-a)^3 + (b-x)^3 \right]$$

and so

$$(3.23) \quad 3\mathcal{M}\left((x-\cdot)^2\right) = \left( \frac{b-a}{2} \right)^2 + 3 \left( x - \frac{a+b}{2} \right)^2.$$

Further, from (3.17),

$$S\left((x-\cdot)^2\right) = (x-t)^2 - \mathcal{M}\left((x-\cdot)^2\right)$$

and so, on using (3.23)

$$(3.24) \quad S\left((x-\cdot)^2\right) = (x-t)^2 - \frac{1}{3} \left( \frac{b-a}{2} \right)^2 - \left( x - \frac{a+b}{2} \right)^2.$$

Now, from (3.22) and using (2.10), (3.23) and (3.24), the following identity is obtained

$$(3.25) \quad \begin{aligned} & \sigma^2(X) + [x - E(X)]^2 - \frac{1}{3} \left[ \left( \frac{b-a}{2} \right)^2 + 3 \left( x - \frac{a+b}{2} \right)^2 \right] \\ &= \int_a^b S\left((x-t)^2\right) \left( f(t) - \frac{1}{b-a} \right) dt, \end{aligned}$$

where  $S(\cdot)$  is as given by (3.24). Taking the modulus of (3.25) gives

$$(3.26) \quad |P_V(x)| = \left| \int_a^b S\left((x-t)^2\right) \left( f(t) - \frac{1}{b-a} \right) dt \right|.$$

Observe that under different assumptions with regard to the norms of the p.d.f.  $f(x)$  we may obtain a variety of bounds.

For  $f \in L_\infty [a, b]$  then

$$(3.27) \quad |P_V(x)| \leq \left\| f(\cdot) - \frac{1}{b-a} \right\|_\infty \int_a^b |S((x-t)^2)| dt.$$

Now, let

$$(3.28) \quad S((x-t)^2) = (t-x)^2 - \nu^2 = (t-X_-)(t-X_+),$$

where

$$(3.29) \quad \begin{aligned} \nu^2 &= \mathcal{M}((x-\cdot)^2) = \frac{(x-a)^3 + (b-x)^3}{3(b-a)} \\ &= \frac{1}{3} \left( \frac{b-a}{2} \right)^2 + \left( x - \frac{a+b}{2} \right)^2, \end{aligned}$$

and

$$(3.30) \quad X_- = x - \nu, \quad X_+ = x + \nu.$$

Then,

$$(3.31) \quad \begin{aligned} H(t) &= \int S((x-t)^2) dt = \int [(t-x)^2 - \nu^2] dt \\ &= \frac{(t-x)^3}{3} - \nu^2 t + k \end{aligned}$$

and so from (3.31) and using (3.28) - (3.29) gives,

$$(3.32) \quad \begin{aligned} &\int_a^b |S((x-t)^2)| dt \\ &= H(X_-) - H(a) - [H(X_+) - H(X_-)] + [H(b) - H(X_+)] \\ &= 2[H(X_-) - H(X_+)] + H(b) - H(a) \\ &= 2 \left\{ -\frac{\nu^3}{3} - \nu^2 X_- - \frac{\nu^3}{3} + \nu^2 X_+ \right\} + \frac{(b-x)^3}{3} - \nu^2 b + \frac{(x-a)^3}{3} + \nu^2 a \\ &= 2 \left[ 2\nu^3 - \frac{2}{3}\nu^3 \right] + \frac{(b-x)^3 + (x-a)^3}{3} - \nu^2(b-a) \\ &= \frac{8}{3}\nu^3. \end{aligned}$$

Thus, substituting into (3.27), (3.26) and using (3.29) readily produces the result (3.21) and the theorem is proved. ■

**Remark 5.** Other bounds may be obtained for  $f \in L_p [a, b]$ ,  $p \geq 1$  however obtaining explicit expressions for these bounds is somewhat intricate and will not be considered further here. They involve the calculation of

$$\sup_{t \in [a, b]} |(t-x)^2 - \nu^2| = \max \left\{ |(x-a)^2 - \nu^2|, \nu^2, |(b-x)^2 - \nu^2| \right\}$$

for  $f \in L_1 [a, b]$  and

$$\left( \int_a^b |(t-x)^2 - \nu^2|^q dt \right)^{\frac{1}{q}}$$

for  $f \in L_p [a, b]$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $p > 1$ , where  $\nu^2$  is given by (3.29).

## 4. SOME INEQUALITIES FOR ABSOLUTELY CONTINUOUS P.D.F.s

We start with the following lemma which is interesting in itself.

**Lemma 1.** *Let  $X$  be a random variable whose probability density function  $f : [a, b] \rightarrow \mathbb{R}_+$  is absolutely continuous on  $[a, b]$ . Then we have the identity*

$$(4.1) \quad \begin{aligned} & \sigma^2(X) + [E(X) - x]^2 \\ &= \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 + \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 p(t,s) f'(s) ds dt, \end{aligned}$$

where the kernel  $p : [a, b]^2 \rightarrow \mathbb{R}$  is given by

$$p(t, s) := \begin{cases} s - a & \text{if } a \leq s \leq t \leq b \\ s - b & \text{if } a \leq t < s \leq b \end{cases},$$

for all  $x \in [a, b]$ .

*Proof.* We use the identity (see (2.10))

$$(4.2) \quad \sigma^2(X) + [E(X) - x]^2 = \int_a^b (x-t)^2 f(t) dt$$

for all  $x \in [a, b]$ .

On the other hand, we know that (see for example [22] for a simple proof using integration by parts)

$$(4.3) \quad f(t) = \frac{1}{b-a} \int_a^b f(s) ds + \frac{1}{b-a} \int_a^b p(t,s) f'(s) ds$$

for all  $t \in [a, b]$ .

Substituting (4.3) in (4.2) we obtain

$$(4.4) \quad \begin{aligned} & \sigma^2(X) + [E(X) - x]^2 \\ &= \int_a^b (t-x)^2 \left[ \frac{1}{b-a} \int_a^b f(s) ds + \frac{1}{b-a} \int_a^b p(t,s) f'(s) ds \right] dt \\ &= \frac{1}{b-a} \cdot \frac{1}{3} [(x-a)^3 + (b-x)^3] + \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 p(t,s) f'(s) ds dt. \end{aligned}$$

Taking into account the fact that

$$\frac{1}{3} [(x-a)^3 + (b-x)^3] = \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2, \quad x \in [a, b],$$

then, by (4.4) we deduce the desired result (4.1). ■

The following inequality for P.D.F.s which are absolutely continuous and have the derivatives essentially bounded holds.

**Theorem 10.** *If  $f : [a, b] \rightarrow \mathbb{R}_+$  is absolutely continuous on  $[a, b]$  and  $f' \in L_\infty [a, b]$ , i.e.,  $\|f'\|_\infty := \text{ess sup}_{t \in [a, b]} |f'(t)| < \infty$ , then we have the inequality:*

$$(4.5) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{(b-a)^2}{3} \left[ \frac{(b-a)^2}{10} + \left(x - \frac{a+b}{2}\right)^2 \right] \|f'\|_\infty$$

for all  $x \in [a, b]$ .

*Proof.* Using Lemma 1, we have

$$\left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ = \frac{1}{b-a} \left| \int_a^b \int_a^b (t-x)^2 p(t,s) f'(s) ds dt \right| \\ \leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t,s)| |f'(s)| ds dt \\ \leq \frac{\|f'\|_\infty}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t,s)| ds dt.$$

We have

$$I : = \int_a^b \int_a^b (t-x)^2 |p(t,s)| ds dt \\ = \int_a^b (t-x)^2 \left[ \int_a^t (s-a) ds + \int_t^b (b-s) ds \right] dt \\ = \int_a^b (t-x)^2 \left[ \frac{(t-a)^2}{2} + \frac{(b-t)^2}{2} \right] dt \\ = \frac{1}{2} \left[ \int_a^b (t-x)^2 (t-a)^2 dt + \int_a^b (t-x)^2 (b-t)^2 dt \right] \\ = \frac{(I_a + I_b)}{2}.$$

Let  $A = x - a$ ,  $B = b - x$  then

$$I_a = \int_a^b (t-x)^2 (t-a)^2 dt \\ = \int_0^{b-a} (u^2 - 2Au + A^2) u^2 du \\ = \frac{(b-a)^3}{3} \left[ A^2 - \frac{3}{2}A(b-a) + \frac{3}{5}(b-a)^2 \right]$$

and

$$\begin{aligned}
I_b &= \int_a^b (t-x)^2 (b-t)^2 dt \\
&= \int_0^{b-a} (u^2 - 2Bu + B^2) u^2 du \\
&= \frac{(b-a)^3}{3} \left[ B^2 - \frac{3}{2}B(b-a) + \frac{3}{5}(b-a)^2 \right]
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{I_a + I_b}{2} &= \frac{(b-a)^3}{3} \left[ \frac{A^2 + B^2}{2} - \frac{3}{4}(A+B)(b-a) + \frac{3}{5}(b-a)^2 \right] \\
&= \frac{(b-a)^3}{3} \left[ \left( \frac{b-a}{2} \right)^2 + \left( x - \frac{a+b}{2} \right)^2 - 3 \frac{(b-a)^2}{20} \right] \\
&= \frac{(b-a)^3}{3} \left[ \frac{(b-a)^2}{10} + \left( x - \frac{a+b}{2} \right)^2 \right]
\end{aligned}$$

and the theorem is proved. ■

The best inequality we can get from (4.5) is embodied in the following corollary.

**Corollary 3.** *If  $f$  is as in Theorem 10, then we have*

$$(4.6) \quad \left| \sigma^2(X) + \left[ E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{(b-a)^4}{30} \|f'\|_\infty.$$

We now analyze the case where  $f'$  is a Lebesgue  $p$ -integrable mapping with  $p \in (1, \infty)$ .

**Remark 6.** *The results of Theorem 10 may be compared with those of Theorem 7. It may be shown that both bounds are convex and symmetric about  $x = \frac{a+b}{2}$ . Further, the bound given by the 'premature' Chebychev approach, namely from (3.12)-(3.13) is tighter than that obtained by the current approach (4.5) which may be shown from the following. Let these bounds be described by  $B_p$  and  $B_c$  so that, neglecting the common terms*

$$B_p = \frac{b-a}{2\sqrt{15}} \left[ \left( \frac{b-a}{2} \right)^2 + 15Y \right]^{\frac{1}{2}}$$

and

$$B_c = \frac{(b-a)^2}{100} + Y,$$

where

$$Y = \left( x - \frac{a+b}{2} \right)^2.$$

*It may be shown through some straightforward algebra that  $B_c^2 - B_p^2 > 0$  for all  $x \in [a, b]$  so that  $B_c > B_p$ .*

*The current development does however have the advantage that the identity (4.1) is satisfied, thus allowing bounds for  $L_p[a, b]$ ,  $p \geq 1$  rather than the infinity norm.*

**Theorem 11.** *If  $f : [a, b] \rightarrow \mathbb{R}_+$  is absolutely continuous on  $[a, b]$  and  $f' \in L_p$ , i.e.,*

$$\|f'\|_p := \left( \int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} < \infty, \quad p \in (1, \infty)$$

then we have the inequality

$$(4.7) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{\|f'\|_p}{(b-a)^{\frac{1}{p}} (q+1)^{\frac{1}{q}}} \left[ (x-a)^{3q+2} \tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2\right) \right. \\ \left. + (b-x)^{3q+2} \tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2\right) \right]$$

for all  $x \in [a, b]$ , when  $\frac{1}{p} + \frac{1}{q} = 1$  and  $\tilde{B}(\cdot, \cdot, \cdot)$  is the quasi incomplete Euler's Beta mapping:

$$\tilde{B}(z; \alpha, \beta) := \int_0^z (u-1)^{\alpha-1} u^{\beta-1} du, \quad \alpha, \beta > 0, \quad z \geq 1.$$

*Proof.* Using Lemma 1, we have, as in Theorem 10, that

$$(4.8) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t, s)| |f'(s)| ds dt.$$

Using Hölder's integral inequality for double integrals, we have

$$(4.9) \quad \int_a^b \int_a^b (t-x)^2 |p(t, s)| |f'(s)| ds dt \\ \leq \left( \int_a^b \int_a^b |f'(s)|^p ds dt \right)^{\frac{1}{p}} \left( \int_a^b \int_a^b (t-x)^{2q} |p(t, s)|^q ds dt \right)^{\frac{1}{q}} \\ = (b-a)^{\frac{1}{p}} \|f'\|_p \left( \int_a^b \int_a^b (t-x)^{2q} |p(t, s)|^q ds dt \right)^{\frac{1}{q}},$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

We have to compute the integral

$$(4.10) \quad D : = \int_a^b \int_a^b (t-x)^{2q} |p(t, s)|^q ds dt \\ = \int_a^b (t-x)^{2q} \left[ \int_a^t (s-a)^q ds + \int_t^b (b-s)^q ds \right] dt \\ = \int_a^b (t-x)^{2q} \left[ \frac{(t-a)^{q+1} + (b-t)^{q+1}}{q+1} \right] dt \\ = \frac{1}{q+1} \left[ \int_a^b (t-x)^{2q} (t-a)^{q+1} dt + \int_a^b (t-x)^{2q} (b-t)^{q+1} dt \right].$$

Define

$$(4.11) \quad E := \int_a^b (t-x)^{2q} (t-a)^{q+1} dt.$$

If we consider the change of variable  $t = (1-u)a + ux$ , we have  $t = a$  implies  $u = 0$  and  $t = b$  implies  $u = \frac{b-a}{x-a}$ ,  $dt = (x-a) du$  and then

$$(4.12) \quad \begin{aligned} E &= \int_0^{\frac{b-a}{x-a}} [(1-u)a + ux - x]^{2q} [(1-u)a + ux - a] (x-a) du \\ &= (x-a)^{3q+2} \int_0^{\frac{b-a}{x-a}} (u-1)^{2q} u^{q+1} du \\ &= (x-a)^{3q+2} \tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2\right). \end{aligned}$$

Define

$$(4.13) \quad F := \int_a^b (t-x)^{2q} (b-t)^{q+1} dt.$$

If we consider the change of variable  $t = (1-v)b + vx$ , we have  $t = b$  implies  $v = 0$ , and  $t = a$  implies  $v = \frac{b-a}{b-x}$ ,  $dt = (x-b) dv$  and then

$$(4.14) \quad \begin{aligned} F &= \int_{\frac{b-a}{b-x}}^0 [(1-v)b + vx - x]^{2q} [b - (1-v)b - vx]^{q+1} (x-b) dv \\ &= (b-x)^{3q+2} \int_0^{\frac{b-a}{b-x}} (v-1)^{2q} v^{q+1} dv \\ &= (b-x)^{3q+2} \tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2\right). \end{aligned}$$

Now, using the inequalities (4.8)-(4.9) and the relations (4.10)-(4.14), since  $D = \frac{1}{q+1} (E + F)$ , we deduce the desired estimate (4.7). ■

The following corollary is natural to be considered.

**Corollary 4.** *Let  $f$  be as in Theorem 11. Then, we have the inequality:*

$$(4.15) \quad \begin{aligned} &\left| \sigma^2(X) + \left[ E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \\ &\leq \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} [B(2q+1, q+1) + \Psi(2q+1, q+2)]^{\frac{1}{q}}, \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $p > 1$  and  $B(\cdot, \cdot)$  is Euler's Beta mapping and  $\Psi(\alpha, \beta) := \int_0^1 u^{\alpha-1} (u+1)^{\beta-1} du$ ,  $\alpha, \beta > 0$ .

*Proof.* In (4.7) put  $x = \frac{a+b}{2}$ .

The left side is clear.



Now

$$\begin{aligned}\tilde{B}(2, 2q + 1, q + 2) &= \int_0^2 (u - 1)^{2q} u^{q+1} du \\ &= \int_0^1 (u - 1)^{2q} u^{q+1} du + \int_1^2 (u - 1)^{2q} u^{q+1} du \\ &= B(2q + 1, q + 2) + \Psi(2q + 1, q + 2).\end{aligned}$$

The right hand side of (4.7) is thus:-

$$\begin{aligned}& \frac{\|f'\|_p \left(\frac{b-a}{2}\right)^{\frac{3q+2}{q}}}{(b-a)^{\frac{1}{p}} (q+1)^{\frac{1}{q}}} [2B(2q+1, q+2) + 2\Psi(2q+1, q+2)]^{\frac{1}{q}} \\ &= \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} [B(2q+1, q+2) + \Psi(2q+1, q+2)]^{\frac{1}{q}}\end{aligned}$$

and the corollary is proved. ■

Finally, as  $f$  is absolutely continuous,  $f' \in L_1[a, b]$  and  $\|f'\|_1 = \int_a^b |f'(t)| dt$ , and we can state the following theorem.

**Theorem 12.** *If the p.d.f.,  $f : [a, b] \rightarrow \mathbb{R}_+$  is absolutely continuous on  $[a, b]$ , then*

$$(4.16) \quad \begin{aligned}& \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ & \leq \|f'\|_1 (b-a) \left[ \frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right| \right]^2\end{aligned}$$

for all  $x \in [a, b]$ .

*Proof.* As above, we can state that

$$\begin{aligned}& \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ & \leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t, s)| |f'(s)| ds dt \\ & \leq \sup_{(t,s) \in [a,b]^2} [(t-x)^2 |p(t, s)|] \frac{1}{b-a} \int_a^b \int_a^b |f'(s)| ds dt \\ & = \|f'\|_1 G\end{aligned}$$

where

$$\begin{aligned}G &: = \sup_{(t,s) \in [a,b]^2} [(t-x)^2 |p(t, s)|] \leq (b-a) \sup_{t \in [a,b]} (t-x)^2 \\ &= (b-a) [\max(x-a, b-x)]^2 \\ &= (b-a) \left[ \frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right| \right]^2,\end{aligned}$$

and the theorem is proved. ■

It is clear that the best inequality we can get from (4.16) is the one when  $x = \frac{a+b}{2}$ , giving the following corollary.

**Corollary 5.** *With the assumptions of Theorem 12, we have:*

$$(4.17) \quad \left| \sigma^2(X) + \left[ E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{(b-a)^3}{4} \|f'\|_1.$$

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