A Grüss Type Discrete Inequality in Inner Product Spaces and Applications

A GRÜSS TYPE DISCRETE INEQUALITY IN INNER PRODUCT SPACES AND APPLICATIONS

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Abstract. A Grüss type inequality in inner product spaces and applications for the discrete Fourier transform, Mellin transform of sequences, polynomials with coefficients in Hilbert spaces and Lipschitzian vector valued mappings are given.

1. Introduction

In 1935, G. Grüss proved the following integral inequality [9]

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi \text{ and } \gamma\leq g(x)\leq\Gamma \text{ for all } x\in[a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is the *best possible* and is achieved for

$$f(x)=g(x)=sgn\left(x-\frac{a+b}{2}\right).$$

The discrete version of (1.1) states that:

If $a\leq a_i\leq A$, $b\leq b_i\leq B$ $(i=1,...,n)$ where a,A,a_i,b,B,b_i are real numbers, then

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i\right|\leq\frac{1}{4}(A-a)(B-b), \tag{1.3}$$

where the constant $\frac{1}{4}$ is the best possible.

For an entire chapter devoted to the history of this inequality see the book [11] where further references are given.

New results in the domain can be found in the papers [1]-[7] and [10].

In the recent paper [2], the author proved the following generalization in inner product spaces.

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Theorem 1. *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}, \mathbb{K} = \mathbb{C}, \mathbb{R}$ and $e \in X, \|e\| = 1$. If $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$ and $x, y \in X$ such that*

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0, \tag{1.4}$$

holds, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|. \tag{1.5}$$

The constant $\frac{1}{4}$ is the best possible.

It has been shown in [1] that the above theorem, for real cases, contains the usual integral and discrete Grüss inequality and also some Grüss type inequalities for mappings defined on infinite intervals.

Namely, if $\rho : (-\infty, \infty) \to (-\infty, \infty)$ is a probabilistic density function, i.e., $\int_{-\infty}^{\infty} \rho(t) \, dt = 1$, then $\rho^{\frac{1}{2}} \in L^2(-\infty, \infty)$ and obviously $\left\| \rho^{\frac{1}{2}} \right\|_2 = 1$. Consequently, if we assume that $f, g \in L^2(-\infty, \infty)$ and

$$\alpha \cdot \rho^{\frac{1}{2}} \leq f \leq \psi \cdot \rho^{\frac{1}{2}}, \; \beta \cdot \rho^{\frac{1}{2}} \leq g \leq \theta \cdot \rho^{\frac{1}{2}} \text{ a.e. on } (-\infty, \infty), \tag{1.6}$$

then we have the inequality

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} f(t) g(t) \, dt - \int_{-\infty}^{\infty} f(t) \rho^{\frac{1}{2}}(t) \, dt \cdot \int_{-\infty}^{\infty} g(t) \rho^{\frac{1}{2}}(t) \, dt \right| \\ \leq \; & \frac{1}{4} (\psi - \alpha)(\theta - \beta). \end{aligned} \tag{1.7}$$

Similarly, if $l = (l_i)_{i \in \mathbb{N}} \in l^2(\mathbb{R})$ with $\sum_{i \in \mathbb{N}} |l_i|^2 = 1$ and $x = (x_i)_{i \in \mathbb{N}}, y = (y_i)_{i \in \mathbb{N}} \in l^2(\mathbb{R})$ are such that

$$\alpha \cdot l_i \leq x_i \leq \psi \cdot l_i, \beta \cdot l_i \leq y_i \leq \theta \cdot l_i \tag{1.8}$$

for all $i \in \mathbb{N}$, then we have

$$\left| \sum_{i \in \mathbb{N}} x_i y_i - \sum_{i \in \mathbb{N}} x_i l_i \cdot \sum_{i \in \mathbb{N}} y_i l_i \right| \leq \frac{1}{4} (\psi - \alpha)(\theta - \beta). \tag{1.9}$$

In this paper we point out some other Grüss type discrete inequalities in inner product spaces. Applications for Fourier transform, Mellin transform, polynomials with coefficients in Hilbert spaces and Lipschitzian mappings with values in normed spaces are also given.

2. Preliminary Results

The following lemma is of interest in itself.

Lemma 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i \in H$ and $p_i \geq 0$ $(i = 1, ..., n)$ such that $\sum_{i=1}^{n} p_i = 1$ $(n \geq 2)$. If $x, X \in H$ are such that*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \text{ for all } i \in \{1, ..., n\}, \tag{2.1}$$

then we have the inequality

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq \frac{1}{4} \|X - x\|^2. \tag{2.2}$$

The constant $\frac{1}{4}$ is sharp.

Proof. Define

$$I_1 := \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle$$

and

$$I_2 := \sum_{i=1}^{n} p_i \langle X - x_i, x_i - x \rangle .$$

Then

$$I_1 = \sum_{i=1}^{n} p_i \langle X, x_i \rangle - \langle X, x \rangle - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 + \sum_{i=1}^{n} p_i \langle x_i, x \rangle$$

and

$$I_2 = \sum_{i=1}^{n} p_i \langle X, x_i \rangle - \langle X, x \rangle - \sum_{i=1}^{n} p_i \|x_i\|^2 + \sum_{i=1}^{n} p_i \langle x_i, x \rangle .$$

Consequently,

$$I_1 - I_2 = \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 . \tag{2.3}$$

Taking the real value in (2.3), we can state that

$$\begin{aligned} &\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \\ &= \operatorname{Re} \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle - \sum_{i=1}^{n} p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle , \end{aligned} \tag{2.4}$$

which is also an identity of interest in itself.
Using the assumption (2.1), we can conclude, by (2.4), that

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \le \operatorname{Re} \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle . \tag{2.5}$$

It is known that if $y, z \in H$, then

$$4 \operatorname{Re} \langle z, y \rangle \le \|z + y\|^2 , \tag{2.6}$$

with equality iff $z = y$.

Now, by (2.6) we can state that

$$\begin{aligned} \operatorname{Re} \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle &\le \frac{1}{4} \left\| X - \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i x_i - x \right\|^2 \\ &= \frac{1}{4} \|X - x\|^2 . \end{aligned}$$

Using (2.5), we obtain (2.2).
To prove the sharpness of the constant $\frac{1}{4}$, let us assume that the inequality (2.2)

holds with a constant $c > 0$, i.e.,

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq c \|X - x\|^2 \tag{2.7}$$

for all p_i, x_i and n as in the hypothesis of Lemma 1.
Assume that $n = 2, p_1 = p_2 = \frac{1}{2}, x_1 = x$ and $x_2 = X$ with $x, X \in H$ and $x \neq X$. Then, obviously,

$$\langle X - x_1, x_1 - x \rangle = \langle X - x_2, x_2 - x \rangle = 0,$$

which shows that the condition (2.1) holds.
If we replace n, p_1, p_2, x_1, x_2 in (2.7), we obtain

$$\begin{aligned} \sum_{i=1}^{2} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{2} p_i x_i \right\|^2 &= \frac{1}{2} \left(\|x\|^2 + \|X\|^2 \right) - \left\| \frac{x + X}{2} \right\|^2 \\ &= \frac{1}{4} \|x - X\|^2 \\ &\leq c \|x - X\|^2, \end{aligned}$$

from where we deduce that $c \geq \frac{1}{4}$, which proves the sharpness of the constant $\frac{1}{4}$. ∎

Remark 1. *The assumption* (2.1) *can be replaced by the more general condition*

$$\sum_{i=1}^{n} p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0, \tag{2.8}$$

and the conclusion (2.2) *will still remain valid.*

The following corollary is natural.

Corollary 1. *Let $a_i \in \mathbb{K}$, $p_i \geq 0, (i = 1, ..., n)$ $(n \geq 2)$ with $\sum_{i=1}^{n} p_i = 1$. If $a, A \in \mathbb{K}$ are such that*

$$\operatorname{Re} \left[(A - a_i) (\bar{a}_i - \bar{a}) \right] \geq 0 \text{ for all } i \in \{1, ..., n\}, \tag{2.9}$$

then we have the inequality

$$0 \leq \sum_{i=1}^{n} p_i |a_i|^2 - \left| \sum_{i=1}^{n} p_i a_i \right|^2 \leq \frac{1}{4} |A - a|^2. \tag{2.10}$$

The constant $\frac{1}{4}$ is sharp.

The proof follows by the above lemma by choosing $H = \mathbb{K}$, $\langle x, y \rangle := x\bar{y}, x_i = a_i, x = a$ and $X = A$. We omit the details.

Remark 2. *The condition* (2.9) *can be replaced by the more general assumption*

$$\sum_{i=1}^{n} p_i \operatorname{Re} \left[(A - a_i) (\bar{a}_i - \bar{a}) \right] \geq 0. \tag{2.11}$$

Remark 3. *If we assume that $\mathbb{K} = \mathbb{R}$, then* (2.8) *is equivalent to*

$$a \leq a_i \leq A \text{ for all } i \in \{1, ..., n\}, \tag{2.12}$$

and then, with the assumption (2.12), *we obtain the discrete Grüss type inequality*

$$0 \le \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i\right)^2 \le \frac{1}{4}(A-a)^2, \tag{2.13}$$

where the constant $\frac{1}{4}$ *is sharp.*

3. A Discrete Inequality of Grüss' Type

The following Grüss type inequality holds.

Theorem 2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}, \mathbb{K} = \mathbb{R}, \mathbb{C}$, $x_i \in H$, $a_i \in \mathbb{K}$, $p_i \ge 0$ $(i = 1, ..., n)$ $(n \ge 2)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A \in \mathbb{K}$ *and* $x, X \in H$ *are such that*

$$\operatorname{Re}\left[(A-a_i)(\bar{a}_i - \bar{a})\right] \ge 0, \operatorname{Re}\langle X - x_i, x_i - x\rangle \ge 0 \text{ for all } i \in \{1, ..., n\}; \tag{3.1}$$

then we have the inequality

$$0 \le \left\|\sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i x_i\right\| \le \frac{1}{4}|A-a|\,\|X-x\|. \tag{3.2}$$

The constant $\frac{1}{4}$ *is sharp.*

Proof. A simple computation shows that

$$\sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i x_i = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j (a_i - a_j)(x_i - x_j). \tag{3.3}$$

Taking the norm in both parts of (3.3) and using the generalized triangle inequality, we obtain

$$\left\|\sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i x_i\right\| \le \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j |a_i - a_j|\,\|x_i - x_j\|. \tag{3.4}$$

By the Cauchy-Buniakowsky-Schwartz discrete inequality for double sums, we obtain

$$\left(\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j |a_i - a_j|\,\|x_i - x_j\|\right)^2 \le \left(\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j |a_i - a_j|^2\right)\left(\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2\right). \tag{3.5}$$

As a simple calculation reveals that

$$\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j |a_i - a_j|^2 = \sum_{i=1}^{n} p_i |a_i|^2 - \left|\sum_{i=1}^{n} p_i a_i\right|^2$$

and

$$\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2 = \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2,$$

then, by (3.4) and (3.5), we conclude that

$$\left\|\sum_{i=1}^{n}p_ia_ix_i-\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ix_i\right\| \tag{3.6}$$
$$\leq\left(\sum_{i=1}^{n}p_i\left|a_i\right|^2-\left|\sum_{i=1}^{n}p_ia_i\right|^2\right)^{\frac{1}{2}}\times\left(\sum_{i=1}^{n}p_i\left\|x_i\right\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2\right)^{\frac{1}{2}}.$$

However, from Lemma 1 and Corollary 1, we know that

$$\left(\sum_{i=1}^{n}p_i\left\|x_i\right\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2\right)^{\frac{1}{2}}\leq\frac{1}{2}\left\|X-x\right\| \tag{3.7}$$

and

$$\left(\sum_{i=1}^{n}p_i\left|a_i\right|^2-\left|\sum_{i=1}^{n}p_ia_i\right|^2\right)^{\frac{1}{2}}\leq\frac{1}{2}\left|A-a\right|. \tag{3.8}$$

Consequently, by using (3.6) − (3.8), we deduce the desired estimate (3.2).
To prove the sharpness of the constant $\frac{1}{4}$, assume that (3.2) holds with a constant $c>0$, i.e.,

$$\left\|\sum_{i=1}^{n}p_ia_ix_i-\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ix_i\right\|\leq c\left|A-a\right|\left\|X-x\right\| \tag{3.9}$$

for all p_i,a_i,x_i,a,A,x,X and n as in the hypothesis of Theorem 2.
If we choose $n=2,a_1=a,a_2=A,x_1=x,x_2=X$ $(a\neq A,x\neq X)$ and $p_1=p_2=\frac{1}{2}$, then

$$\begin{aligned}\sum_{i=1}^{2}p_ia_ix_i-\sum_{i=1}^{2}p_ia_i\sum_{i=1}^{2}p_ix_i &= \frac{1}{2}\sum_{i,j=1}^{2}p_ip_j\left(a_i-a_j\right)\left(x_i-x_j\right)\\ &= \frac{1}{4}\left(a-A\right)\left(x-X\right).\end{aligned}$$

Consequently, from (3.9), we deduce

$$\frac{1}{4}\left|a-A\right|\left\|X-x\right\|\leq c\left|A-a\right|\left\|X-x\right\|,$$

which implies that $c\geq\frac{1}{4}$, and the theorem is completely proved. ∎

Remark 4. *The condition* (3.1) *can be replaced by the more general assumption*

$$\sum_{i=1}^{n}p_i\operatorname{Re}\left[\left(A-a_i\right)\left(\bar{a}_i-\bar{a}\right)\right]\geq 0,\ \sum_{i=1}^{n}p_i\operatorname{Re}\left\langle X-x_i,x_i-x\right\rangle\geq 0 \tag{3.10}$$

and the conclusion (3.2) *will still be valid.*

The following corollary for real or complex numbers holds.

Corollary 2. *Let* $a_i,b_i\in\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$, $p_i\geq 0$ $(i=1,...,n)$ *with* $\sum_{i=1}^{n}p_i=1$. *If* $a,A,b,B\in\mathbb{K}$ *are such that*

$$\operatorname{Re}\left[\left(A-a_i\right)\left(\bar{a}_i-\bar{a}\right)\right]\geq 0,\operatorname{Re}\left[\left(B-b_i\right)\left(\bar{b}_i-\bar{b}\right)\right]\geq 0, \tag{3.11}$$

then we have the inequality

$$0 \leq \left| \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| \leq \frac{1}{4} |A - a| |B - b| , \tag{3.12}$$

where the constant $\frac{1}{4}$ *is sharp.*

Remark 5. *If we assume that* a_i, b_i, a, A, b, B *are real numbers, then* (3.11) *is equivalent to*

$$a \leq a_i \leq A, b \leq b_i \leq B \text{ for all } i \in \{1, ..., n\} , \tag{3.13}$$

and (3.12) *becomes*

$$0 \leq \left| \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| \leq \frac{1}{4} (A - a) (B - b) , \tag{3.14}$$

which is the classical Grüss inequality for sequences of real numbers.

4. Applications for Discrete Fourier Transforms

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $\bar{x} = (x_1, ..., x_n)$ be a sequence of vectors in H.

For a given $w \in \mathbb{R}$, define the *discrete Fourier transform*

$$\mathcal{F}_w (\bar{x}) (m) := \sum_{k=1}^{n} \exp (2wimk) \times x_k, \; m = 1, ..., n. \tag{4.1}$$

The following approximation result for the Fourier transform (4.1) holds.

Theorem 3. *Let* $(H; \langle \cdot, \cdot \rangle)$ *and* $\bar{x} \in H^n$ *be as above. If there exists the vectors* $x, X \in H$ *such that*

$$\operatorname{Re} \langle X - x_k, x_k - x \rangle \geq 0 \text{ for all } k \in \{1, ..., n\} , \tag{4.2}$$

then we have the inequality

$$\begin{aligned} & \left\| \mathcal{F}_w (\bar{x}) (m) - \frac{\sin (wmn)}{\sin (wm)} \exp [(n+1) im] \times \frac{1}{n} \sum_{k=1}^{n} x_k \right\| \\ \leq \; & \frac{1}{2} \|X - x\| \left[n^2 - \frac{\sin^2 (wmn)}{\sin^2 (wm)} \right]^{\frac{1}{2}} , \end{aligned} \tag{4.3}$$

for all $m \in \{1, ..., n\}$ *and* $w \in \mathbb{R}, \; w \neq \frac{l}{m}\pi, \; l \in \mathbb{Z}$.

Proof. From the inequality (3.6) in Theorem 2, we can state that

$$\begin{aligned} & \left\| \frac{1}{n} \sum_{k=1}^{n} a_k x_k - \frac{1}{n} \sum_{k=1}^{n} a_k \cdot \frac{1}{n} \sum_{k=1}^{n} x_k \right\| \\ \leq \; & \left(\frac{1}{n} \sum_{k=1}^{n} |a_k|^2 - \left| \frac{1}{n} \sum_{k=1}^{n} a_k \right|^2 \right)^{\frac{1}{2}} \times \left(\frac{1}{n} \sum_{k=1}^{n} \|x_k\|^2 - \left\| \frac{1}{n} \sum_{k=1}^{n} x_k \right\|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{4.4}$$

for all $a_k \in \mathbb{K},\ x_k \in H\ \ (k = 1, ..., n)$.
However, the $x_k\ \ (k = 1, ..., n)$ satisfy (4.2), and therefore, by Lemma 1, we have

$$0 \leq \frac{1}{n}\sum_{k=1}^{n}\|x_k\|^2 - \left\|\frac{1}{n}\sum_{k=1}^{n}x_k\right\|^2 \leq \frac{1}{4}\|X - x\|^2. \tag{4.5}$$

Consequently, by (4.4) and (4.5), we conclude that

$$\left\|\sum_{k=1}^{n}a_k x_k - \sum_{k=1}^{n}a_k \cdot \frac{1}{n}\sum_{k=1}^{n}x_k\right\| \leq \frac{1}{2}\|X - x\|\left(n\sum_{k=1}^{n}|a_k|^2 - \left|\sum_{k=1}^{n}a_k\right|^2\right)^{\frac{1}{2}} \tag{4.6}$$

for all $a_k \in \mathbb{K}\ \ (k = 1, ..., n)$.

We now choose in (4.6), $a_k = \exp(2wimk)$ to obtain

$$\begin{aligned} &\left\|\mathcal{F}_w(\bar{x})(m) - \sum_{k=1}^{n}\exp(2wimk) \times \frac{1}{n}\sum_{k=1}^{n}x_k\right\| \\ \leq\ & \frac{1}{2}\|X - x\|\left(n\sum_{k=1}^{n}|\exp(2wimk)|^2 - \left|\sum_{k=1}^{n}\exp(2wimk)\right|^2\right)^{\frac{1}{2}} \end{aligned} \tag{4.7}$$

for all $m \in \{1, ..., n\}$.

As a simple calculation reveals that

$$\begin{aligned} \sum_{k=1}^{n}\exp(2wimk) &= \exp(2wim) \times \left[\frac{\exp(2wimn) - 1}{\exp(2wim) - 1}\right] \\ &= \exp(2wim) \times \left[\frac{\cos(2wmn) + i\sin(2wmn) - 1}{\cos(2wm) + i\sin(2wm) - 1}\right] \\ &= \exp(2wim) \times \frac{\sin(wmn)}{\sin(wm)}\left[\frac{\cos(wmn) + i\sin(wmn)}{\cos(wm) + i\sin(wm)}\right] \\ &= \frac{\sin(wmn)}{\sin(wm)} \times \exp(2wim)\left[\frac{\exp(iwmn)}{\exp(iwm)}\right] \\ &= \frac{\sin(wmn)}{\sin(wm)} \times \exp[(n+1)im], \end{aligned}$$

$$\sum_{k=1}^{n}|\exp(2wimk)|^2 = n$$

and

$$\left|\sum_{k=1}^{n}\exp(2wimk)\right|^2 = \frac{\sin^2(wmn)}{\sin^2(wm)},\ \text{for } w \neq \frac{l}{m}\pi,\ l \in \mathbb{Z},$$

thus, from (4.7), we deduce the desired inequality (4.3). ∎

Remark 6. *The assumption* (4.2) *can be replaced by the more general condition*

$$\sum_{i=1}^{n}\operatorname{Re}\langle X - x_i, x_i - x\rangle \geq 0, \tag{4.8}$$

and the conclusion (4.3) *will still remain valid.*

The following corollary is an obvious consequence of (4.3).

Corollary 3. *Let* $a_i \in \mathbb{K}$ $(i = 1, ..., n)$. *If* $a, A \in \mathbb{K}$ *are such that*

$$\operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a})\right] \geq 0 \text{ for all } i \in \{1, ..., n\}, \tag{4.9}$$

then we have an approximation of the Fourier transform for the vector $\tilde{a} = (a_1, ..., a_n) \in \mathbb{K}^n$:

$$\left\| \mathcal{F}_w(\tilde{a})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp\left[(n+1)im\right] \times \frac{1}{n}\sum_{k=1}^{n} a_k \right\| \tag{4.10}$$

$$\leq \frac{1}{2}|A - a|\left[n^2 - \frac{\sin^2(wmn)}{\sin^2(wm)}\right]^{\frac{1}{2}}$$

for all $m \in \{1, ..., n\}$ *and* $w \in \mathbb{R}$ *so that* $w \neq \frac{l}{m}\pi,\ l \in \mathbb{Z}$.

Remark 7. *If we assume that* $\mathbb{K} = \mathbb{R}$, *then* (4.9) *is equivalent to*

$$a \leq a_i \leq A \text{ for all } i \in \{1, ..., n\}. \tag{4.11}$$

Consequently, with the assumption (4.11), *we obtain the following approximation of the Fourier transform*

$$\left\| \mathcal{F}_w(\tilde{a})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp\left[(n+1)im\right] \times \frac{1}{n}\sum_{k=1}^{n} a_k \right\| \tag{4.12}$$

$$\leq \frac{1}{2}(A - a)\left[n^2 - \frac{\sin^2(wmn)}{\sin^2(wm)}\right]^{\frac{1}{2}},$$

for all $m \in \{1, ..., n\}$ *and* $w \neq \frac{l}{m}\pi,\ l \in \mathbb{Z}$.

5. Applications for the Discrete Mellin Transform

Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product over $\mathbb{R}$ and $\bar{x} = (x_1, ..., x_n)$ be a sequence of vectors in H.

Define the *Mellin transform*:

$$\mathcal{M}(\bar{x})(m) := \sum_{k=1}^{n} k^{m-1} x_k,\ m = 1, ..., n; \tag{5.1}$$

of the sequence $\bar{x} \in H^n$.

The following approximation result holds.

Theorem 4. *Let* H *and* $\bar{x}$ *be as above. If there exist the vectors* $x, X \in H$ *such that*

$$\operatorname{Re}\langle X - x_k, x_k - x\rangle \geq 0 \text{ for all } k = 1, ..., n; \tag{5.2}$$

then we have the inequality

$$\left\| \mathcal{M}(\bar{x})(m) - S_{m-1}(n) \cdot \frac{1}{n}\sum_{k=1}^{n} x_k \right\| \tag{5.3}$$

$$\leq \frac{1}{2}\|X - x\|\left[nS_{2m-2}(n) - S_{m-1}^2(n)\right]^{\frac{1}{2}},\ m \in \{1, ..., n\},$$

where $S_p(n)$, $p\in\mathbb{R}$, $n\in\mathbb{N}$ is the $p-$powered sum of the first n natural numbers, i.e.,

$$S_p(n) := \sum_{k=1}^{n} k^p.$$

Proof. We apply inequality (4.6) to obtain

$$\left\| \sum_{k=1}^{n} k^{m-1} x_k - \sum_{k=1}^{n} k^{m-1} \cdot \frac{1}{n} \sum_{k=1}^{n} x_k \right\| \tag{5.4}$$
$$\leq \frac{1}{2} \|X - x\| \left[n \sum_{k=1}^{n} k^{2(m-1)} - \left(\sum_{k=1}^{n} k^{m-1} \right)^2 \right]^{\frac{1}{2}}$$
$$= \frac{1}{2} \|X - x\| \left[n S_{2m-2}(n) - S_{m-1}^2(n) \right]^{\frac{1}{2}},$$

and the inequality (5.3) is proved. ∎

Consider the following particular values of Mellin Transform

$$\mu_1(\bar{x}) := \sum_{k=1}^{n} k x_k$$

and

$$\mu_2(\bar{x}) := \sum_{k=1}^{n} k^2 x_k.$$

The following corollary holds.

Corollary 4. *Let H and $\bar{x}$ be as in Theorem 4. Then we have the inequalities:*

$$\left\| \mu_1(\bar{x}) - \frac{n+1}{2} \cdot \sum_{k=1}^{n} x_k \right\| \leq \frac{1}{2} \|X - x\| n \left[\frac{n(n+1)}{2} \right]^{\frac{1}{2}} \tag{5.5}$$

and

$$\left\| \mu_2(\bar{x}) - \frac{(n+1)(2n+1)}{6} \cdot \sum_{k=1}^{n} x_k \right\| \tag{5.6}$$
$$\leq \frac{1}{12\sqrt{5}} \|X - x\| n \sqrt{(n-1)(n+1)(2n+1)(8n+1)}.$$

Remark 8. *If we assume that $p = (p_1, ..., p_n)$ is a probability distribution, i.e., $p_k \geq 0$ $(k = 1, ..., n)$ and $\sum_{k=1}^{n} p_k = 1$ and $p \leq p_k \leq P$ $(k = 1, ..., n)$, then by (5.5) and (5.6), we get the inequalities*

$$\left| \sum_{k=1}^{n} k p_k - \frac{n+1}{2} \right| \leq \frac{1}{2} (P - p) n \left[\frac{n(n+1)}{2} \right]^{\frac{1}{2}} \tag{5.7}$$

and

$$\left| \sum_{k=1}^{n} k^2 p_k - \frac{(n+1)(2n+1)}{6} \right| \tag{5.8}$$
$$\leq \frac{1}{12\sqrt{5}} (P - p) n \sqrt{(n-1)(n+1)(2n+1)(8n+1)},$$

which have been obtained in [8] *and applied to the estimation of the* 1 *and* 2*-moments of a guessing mapping.*

6. Applications for Polynomials

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $\bar{c}=(c_0,...,c_n)$ be a sequence of vectors in H.

Define the polynomial $P:\mathbb{C}\to H$ with the coefficients $\bar{c}=(c_0,...,c_n)$ by

$$P(z)=c_0+zc_1+...+z^nc_n,\ z\in\mathbb{C},\ c_n\neq 0.$$

The following approximation result for the polynomial P holds.

Theorem 5. *Let $H,\bar{c}$ and P be as above. If there exist the vectors $c,C\in H$ such that*

$$\operatorname{Re}\langle C-c_k,c_k-c\rangle\geq 0 \text{ for all } k\in\{0,...,n\}, \tag{6.1}$$

then we have the inequality

$$\begin{aligned} &\left\|P(z)-\frac{z^{n+1}-1}{z-1}\times\frac{c_0+c_1+...+c_n}{n+1}\right\| \\ \leq\ & \frac{1}{2}\|C-c\|\left[(n+1)\frac{|z|^{2n+2}-1}{|z|^2-1}-\frac{|z|^{2n+2}-2\operatorname{Re}\left(z^{n+1}\right)+1}{|z|^2-2\operatorname{Re}(z)+1}\right]^{\frac{1}{2}} \end{aligned} \tag{6.2}$$

for all $z\in\mathbb{C}$, $|z|\neq 1$.

Proof. Using the inequality (4.6), we can state that

$$\begin{aligned} &\left\|\sum_{k=0}^n z^kc_k-\sum_{k=0}^n z^k\cdot\frac{1}{n+1}\sum_{k=0}^n c_k\right\| \\ \leq\ & \frac{1}{2}\|C-c\|\left((n+1)\sum_{k=0}^n|z|^{2k}-\left|\sum_{k=0}^n z^k\right|^2\right)^{\frac{1}{2}} \\ =\ & \frac{1}{2}\|C-c\|\left[(n+1)\frac{|z|^{2n+2}-1}{|z|^2-1}-\left|\frac{z^{n+1}-1}{z-1}\right|^2\right]^{\frac{1}{2}} \\ =\ & \frac{1}{2}\|C-c\|\left[(n+1)\frac{|z|^{2n+2}-1}{|z|^2-1}-\frac{|z|^{2n+2}-2\operatorname{Re}\left(z^{n+1}\right)+1}{|z|^2-2\operatorname{Re}(z)+1}\right]^{\frac{1}{2}} \end{aligned} \tag{6.3}$$

and the inequality (6.2) is proved. ∎

The following result for the complex roots of the unity also holds.

Theorem 6. *Let $z_k:=\cos\left(\frac{k\pi}{n+1}\right)+i\sin\left(\frac{k\pi}{n+1}\right)$, $k\in\{0,...,n\}$ be the complex $(n+1)-$roots of the unity. Then we have the inequality*

$$\|P(z_k)\|\leq\frac{1}{2}(n+1)\|C-c\|,\ k\in\{1,...,n\}; \tag{6.4}$$

where the coefficients $\bar{c}=(c_0,...,c_n)\in H^{n+1}$ satisfy the assumption (6.1).

Proof. From the inequality (6.3), we can state that

$$\left\| P(z_k) - \frac{z^{n+1}-1}{z-1} \cdot \frac{1}{n+1} \sum_{k=0}^{n} c_k \right\| \leq \frac{1}{2} \|C - c\| \left[(n+1) \sum_{k=0}^{n} |z|^{2k} - \left| \frac{z^{n+1}-1}{z-1} \right|^2 \right]^{\frac{1}{2}} \tag{6.5}$$

for all $z \in \mathbb{C},\ z \neq 1$.
Putting $z = z_k,\ k \in \{1, ..., n\}$ and taking into account that $z_k^{n+1} = 1,\ |z_k| = 1$, we get the desired result (6.4). ∎

The following corollary is a natural consequence of Theorem 6.

Corollary 5. *Let $P(z) := \sum_{k=0}^{n} a_k z^k$ be a polynomial with real coefficients and z_k the $(n+1)$-roots of the unity as defined above. If $a \leq a_k \leq A,\ k = 0, ..., n,$ then we have the inequality:*

$$|P(z_k)| \leq \frac{1}{2}(n+1)(A-a). \tag{6.6}$$

7. Applications for Lipschitzian Mappings

Let $(H; \langle \cdot, \cdot \rangle)$ be as above and $F : H \to B$ a mapping defined on the inner product space H with values in the normed linear space B which satisfy the *Lipschitzian condition:*

$$|F(x) - F(y)| \leq L \|x - y\|, \text{ for all } x, y \in H, \tag{7.1}$$

where $|\cdot|$ denotes the norm on B and $\|\cdot\|$ is the Euclidean norm on H.
The following theorem holds.

Theorem 7. *Let $F : H \to B$ be as above and $x_i \in H,\ p_i \geq 0\ (i = 1, ..., n)$ with $P_n := \sum_{i=1}^{n} p_i > 0$. If there exists two vectors $x, X \in H$ such that*

$$\operatorname{Re}\langle X - x_i, x_i - x \rangle \geq 0 \text{ for all } i \in \{1, ..., n\}, \tag{7.2}$$

then we have the inequality

$$\left| \frac{1}{P_n} \sum_{i=1}^{n} p_i F(x_i) - F\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) \right| \leq \frac{1}{2} \|X - x\|. \tag{7.3}$$

Proof. As F is Lipschitzian, we have (7.1) for all $x, y \in H$. Choose $x = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$ and $y = x_j\ \ (j = 1, ..., n)$, to get

$$\left| F\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) - F(x_j) \right| \leq L \left\| \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i - x_j \right\|, \tag{7.4}$$

for all $j \in \{1, ..., n\}$.
If we multiply (7.4) by $p_j \geq 0$ and sum over j from 1 to n, we obtain

$$\sum_{j=1}^{n} p_j \left| F\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) - F(x_j) \right| \leq L \sum_{j=1}^{n} p_j \left\| \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i - x_j \right\|. \tag{7.5}$$

Using the generalized triangle inequality, we have

$$\sum_{j=1}^{n} p_j \left| F\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| \geq \left| P_n F\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \sum_{j=1}^{n} p_j F(x_j)\right|. \tag{7.6}$$

By the Cauchy-Buniakowsky-Schwartz inequality, we also have

$$\begin{aligned}
&\sum_{j=1}^{n} p_j \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - x_j \right\| \qquad (7.7)\\
\leq\ & \left[\sum_{j=1}^{n} p_j \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - x_j \right\|^2\right]^{\frac{1}{2}} P_n^{\frac{1}{2}} \\
=\ & P_n^{\frac{1}{2}} \left[\sum_{j=1}^{n} p_j \left[\left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right\|^2 - 2\operatorname{Re}\left\langle \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, x_j\right\rangle + \|x_j\|^2\right]\right]^{\frac{1}{2}} \\
=\ & P_n^{\frac{1}{2}} \left[P_n \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right\|^2 - 2\operatorname{Re}\left\langle \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j\right\rangle + \sum_{j=1}^{n} p_j \|x_j\|^2\right]^{\frac{1}{2}} \\
=\ & P_n \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right\|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Combining the above inequalities (7.5) – (7.7) we deduce, by dividing with $P_n > 0$, that

(7.8)

$$\left| F\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i F(x_i)\right| \leq \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right\|^2\right]^{\frac{1}{2}}.$$

Finally, using Lemma 1, we obtain the desired result. ∎

Remark 9. *The condition* (7.2) *can be substituted by the more general condition*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\langle X - x_i, x_i - x\rangle \geq 0, \tag{7.9}$$

and the conclusion (7.3) *will still remain valid.*

The following corollary is a natural consequence of the above findings.

Corollary 6. *Let* $x_i \in H$ $(i = 1, ..., n)$ *and* $x, X \in H$ *be such that the condition* (7.2) *holds. Then we have the inequality*

$$0 \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \|x_i\| - \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{2}\|X - x\|.$$

The proof follows by Theorem 7 by choosing $F : H \to \mathbb{R}$, $F(x) = \|x\|$ which is Lipschitzian with the constant $L = 1$, as $|F(x) - F(y)| = |\|x\| - \|y\|| \leq \|x - y\|$, for all $x, y \in H$.

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