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BETTER BOUNDS IN SOME OSTROWSKI-GRÜSS TYPE INEQUALITIES

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Abstract. The main aim of this note is to point out some improvements of the recent results in [1].

1. Introduction

As in [1], let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two sequences of harmonic polynomials, that is, polynomials satisfying

$$P_n'(t) = P_{n-1}(t), \; P_0(t) = 1, \; t \in \mathbb{R}, \tag{1.1}$$

$$Q_n'(t) = Q_{n-1}(t), \; Q_0(t) = 1, \; t \in \mathbb{R}. \tag{1.2}$$

In [1], the authors proved the following result.

Lemma 1. *Let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two harmonic polynomials. Set*

$$S_n(t,x) := \begin{cases} P_n(t), & t \in [a,x] \\ Q_n(t), & t \in (x,b]. \end{cases} \tag{1.3}$$

Then we have the equality

$$\begin{aligned} &\int_a^b f(t)\,dt \\ &= \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \\ &\quad - P_k(a) f^{(k-1)}(a) \Big] + (-1)^n \int_a^b S_n(t,x) f^{(n)}(t)\,dt, \end{aligned} \tag{1.4}$$

provided that $f : [a,b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$.

Using the following "pre-Grüss" inequality

$$|T(f,g)| \leq \frac{1}{2}\sqrt{T(f,f)}\,(\Gamma - \gamma), \tag{1.5}$$

where

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$$

is the Chebychev functional and f, g are such that the previous integrals exist and $\gamma \leq g(x) \leq \Gamma$ for a.e. $x \in [a,b]$, the authors of [1] proved basically the

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following inequality for estimating the integral $\int_a^b f(t)\,dt$ in terms of the harmonic polynomials $\{P_n\}_{n\in\mathbb{N}}$, $\{Q_n\}_{n\in\mathbb{N}}$.

Theorem 1. *Assume that* $f:[a,b]\to\mathbb{R}$ *is such that* $f^{(n)}$ *is integrable and* $\gamma_n \le f^{(n)}(t) \le \Gamma_n$ *for all* $t\in[a,b]$. *Put*

$$U_n(x) := \frac{Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)}{b-a}.$$

Then for all $x\in[a,b]$, *we have the inequality*

$$\begin{aligned} (1.6)\qquad & \left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) \right. \\ & + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a) \Big] \\ & \left. - (-1)^n U_n(x) \Big[f^{(k-1)}(b) - f^{(k-1)}(a) \Big] \right| \\ \le\ & \frac{1}{2} K (\Gamma_n - \gamma_n)(b-a), \end{aligned}$$

where

$$K := \left\{ \frac{1}{b-a} \int_a^x P_n^2(t)\,dt + \int_x^b Q_n^2(t)\,dt - [U_n(x)]^2 \right\}^{\frac{1}{2}}.$$

A number of particular cases by choosing some appropriate harmonic polynomials have been obtained in [1] as well.

The main aim of this note is to point out a sharper bound in (1.6) in terms of the Euclidean norm of $f^{(n)}$ which is valid also for a larger class of mappings, i.e., for the mappings f for which $f^{(n)}$ is unbounded on (a,b) but $f^{(n)}\in L_2[a,b]$. Some particular cases as in [1], are also considered.

2. The Results

The following theorem holds.

Theorem 2. *Assume that the mapping* $f:[a,b]\to\mathbb{R}$ *is such that* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$ *and* $f^{(n)}\in L_2[a,b]$ $(n\ge 1)$. *If we denote*

$$\left[f^{(n-1)};a,b\right] := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a},$$

then we have the inequality

$$\begin{aligned} (2.1)\qquad & \left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) \right. \\ & + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a) \Big] \\ & \left. - (-1)^n \left[Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a) \right] \Big[f^{(n-1)};a,b \Big] \right| \\ \le\ & K(b-a) \left[\frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left(\left[f^{(n)};a,b \right] \right)^2 \right]^{\frac{1}{2}} \\ & \left(\le \frac{1}{2} K (b-a)(\Gamma_n - \gamma_n) \quad \textit{if} \ \ f^{(n)} \in L_\infty(a,b) \right) \end{aligned}$$

for all $x \in [a,b]$*, where* K *is defined in Theorem 1 (and* γ_n*,* Γ_n *are as in the Introduction, i.e.,* $\gamma_n \leq f^{(n)}(t) \leq \Gamma_n$ *for all* $t \in [a,b]$*.*

Proof. Recall Korkine's identity

$$T(h,g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (h(t) - h(s))(g(t) - g(s))\, dtds, \tag{2.2}$$

where $T(\cdot,\cdot)$ is the Chebychev functional defined in the Introduction. Using (2.2) and the identity (1.4), we may write (see also [1])

$$\begin{aligned}
&\int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) \\
&\quad + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a) \Big] \\
&\quad - (-1)^n [Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)] \Big[f^{(n-1)}; a, b \Big] \\
&= (b-a) T\left(S_n(\cdot,x), f^{(n)}\right) \\
&= \frac{1}{2(b-a)} \int_a^b \int_a^b (S_n(t,x) - S_n(s,x)) \left(f^{(n)}(t) - f^{(n)}(s)\right) dtds,
\end{aligned} \tag{2.3}$$

which is an identity that is interesting in itself as well.

Using the Cauchy-Buniakowski-Schwartz integral inequality for double integrals, we may write

$$\begin{aligned}
&\left| \int_a^b \int_a^b (S_n(t,x) - S_n(s,x)) \left(f^{(n)}(t) - f^{(n)}(s)\right) dtds \right| \\
&\leq \left(\int_a^b \int_a^b (S_n(t,x) - S_n(s,x))^2 \, dtds \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_a^b \int_a^b \left(f^{(n)}(t) - f^{(n)}(s)\right)^2 dtds \right)^{\frac{1}{2}} \\
&= \left[2(b-a)^2 T(S_n(\cdot,x), S_n(\cdot,x)) \right]^{\frac{1}{2}} \left[2(b-a)^2 T\left(f^{(n)}, f^{(n)}\right) \right]^{\frac{1}{2}} \\
&= 2(b-a)^2 K \left[\frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left(\left[f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}}.
\end{aligned} \tag{2.4}$$

Now, taking the modulus in (2.3) and using the estimate (2.4), we may deduce the first inequality in (2.1).

If we assume that $f^{(n)} \in L_\infty[a,b]$ ($\subset L_2[a,b]$ and the inclusion is strict), then applying the Grüss inequality

$$\begin{aligned}
0 &\leq \frac{1}{b-a} \int_a^b \left| f^{(n)}(t) \right|^2 dt - \left(\frac{1}{b-a} \int_a^b f^{(n)}(t)\,dt \right)^2 \\
&\leq \frac{1}{4} (\Gamma_n - \gamma_n)^2,
\end{aligned}$$

we deduce the last part in (2.1). ∎

We are now able to improve the Corollaries 1-3 and Theorem 2 from [1] as follows.

Corollary 1. *Under the assumptions of Theorem 2, we have*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left[(b-B)^k f^{(k-1)}(b) \right.\right. \\ &+ \left. \left((x-A)^k - (x-B)^k \right) f^{(k-1)}(x) - (a-A)^k f^{(k-1)}(a) \right] \\ &- \frac{(-1)^n}{(n+1)!} \left[(b-B)^{n+1} - (x-b)^{n+1} \right. \\ &+ \left.\left. (x-A)^{n+1} - (a-A)^{n+1} \right] \left[f^{(n-1)}; a, b \right] \right| \\ \leq\ & (b-a) K_1 \left[\frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left(\left[f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}}, \end{aligned} \tag{2.5}$$

where K_1 *is, as defined in* [1]

$$\begin{aligned} K_1 \ :\ &= \frac{1}{n!} \left[\frac{(x-A)^{2n+1} - (a-A)^{2n+1} + (b-B)^{2n+1} - (x-B)^{2n+1}}{(2n+1)(b-a)} \right. \\ &- \left. \left(\frac{(b-B)^{n+1} - (x-B)^{n+1} + (x-a)^{n+1} - (a-A)^{n+1}}{(n+1)(b-a)} \right)^2 \right]^{\frac{1}{2}} \end{aligned}$$

and $x \in [a,b]$, $A,\ B \in \mathbb{R}$.

The proof follows from Theorem 2 with the polynomial choices of $P_n(t) = \frac{(t-A)^n}{n!}$ and $Q_n(t) = \frac{(t-B)^n}{n!}$ (see also [1, Corollary 1]).

Corollary 2. *Under the assumptions of Theorem 2, we have*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^{k+1}(b-a)^k}{k!\,(p+q)^k} \left[q^k \left(f^{(k-1)}(b) - (-1)^k f^{(k-1)}(a) \right) \right.\right. \\ &+ \left. \left(\frac{p-q}{2} \right)^k \left[1 - (-1)^k \right] f^{(k-1)}\left(\frac{a+b}{2} \right) \right] \\ &- \left. \frac{(-1)^n (b-a)^{n+1} (1+(-1)^n)}{(n+1)!\,(p+q)^{n+1}} \left[2^{n+1} + \left(\frac{p-q}{2} \right)^{n+1} \right] \left[f^{(n-1)}; a, b \right] \right| \\ \leq\ & (b-a) K_2 \left[\frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left(\left[f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}}, \end{aligned} \tag{2.6}$$

for $p, q \in \mathbb{R}$ $(p, q > 0)$, *where*

$$K_2 := \frac{(b-a)^n}{n!\,(p+q)^n} \left[\frac{2\left(q^{2n+1} + \left(\frac{p-q}{2} \right)^{2n+1} \right)}{(p+q)(2n+1)} - 2\left[1 + (-1)^n \right] \frac{\left(q^{n+1} + \left(\frac{p-q}{2} \right)^{n+1} \right)^2}{(n+1)^2 (p+q)^2} \right]^{\frac{1}{2}}.$$

The proof follows by Corollary 1 with $A = \frac{pa+qb}{p+q}$, $x = \frac{a+b}{2}$ and $B = \frac{qa+pb}{p+q}$ where $p, q \in \mathbb{R}$ and $p + q > 0$ (see also [1, Corollary 2]).

For $x = b$, Theorem 2 gives the following.

Theorem 3. *With the assumptions in Theorem 2, we have:*

$$\begin{aligned}
(2.7)\qquad & \left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(b)\,f^{(k-1)}(b) - P_k(a)\,f^{(k-1)}(a)\right]\right. \\
& \left. -(-1)^n\left[P_{n+1}(b) - P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right]\right| \\
\leq\ & K_3\,(b-a)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - \left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}},
\end{aligned}$$

where K_3 is given by (see [1, Theorem 2]*)*

$$K_3 := \left[\frac{1}{b-a}\int_a^x P_n^2(t)\,dt - \left(\frac{P_{n+1}(b) - P_{n+1}(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

The choice $P_n(t) = \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n$ provides the following corollary.

Corollary 3. *Under the assumptions of Theorem 2, we have:*

$$\begin{aligned}
(2.8)\qquad & \left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{2^k k!}(b-a)^k\left[f^{(k-1)}(b) - (-1)^k f^{(k-1)}(a)\right]\right. \\
& \left. -\frac{(-1)^n\left(1+(-1)^n\right)}{2^{n+1}(n+1)!}(b-a)^{n+1}\left[f^{(n-1)};a,b\right]\right| \\
\leq\ & K_4\,(b-a)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - \left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}},
\end{aligned}$$

where K_4 is given by (see [1, Corollary 3]*)*

$$K_4 := \frac{(b-a)^n}{n!2^n}\left[\frac{1}{2n+1} - \frac{\left(1+(-1)^n\right)^2}{(n+1)^2}\right]^{\frac{1}{2}}.$$

Remark 1. *All the other results from Sections 4 and 5 can be improved accordingly. For example, if we assume that the derivative $f^{(n)} \in L_2[a,b]$ $(n \in \{1,2,3,4\})$, then we have the Simpson's inequality (for $n \in \{1,2,3\}$)*

$$\begin{aligned}
(2.9)\qquad & \left|\int_a^b f(t)\,dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\
\leq\ & \tilde{c}_n\,(b-a)^n\,\sigma\left(f^{(n)};a,b\right)
\end{aligned}$$

where

$$\tilde{c}_1 = \frac{1}{6},\ \tilde{c}_2 = \frac{1}{12\sqrt{30}},\ \tilde{c}_3 = \frac{1}{48\sqrt{105}}$$

and

$$\sigma\left(f^{(n)};a,b\right) := \left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - \left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}},\ \ n \in \{1,2,3,4\}.$$

For $n=4$, we have the perturbed Simpson's inequality:

$$\left|\int_{a}^{b} f(t)\,dt-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]+\frac{(b-a)^{5}}{2880}\left[f^{(3)};a,b\right]\right| \tag{2.10}$$
$$\leq \frac{1}{2880}\sqrt{\frac{11}{14}}\,(b-a)^{4}\,\sigma\left(f^{(4)};a,b\right).$$

References

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