On Aczél's Inequality for Real Numbers

ON ACZÉL'S INEQUALITY FOR REAL NUMBERS

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Abstract. In this note, we point out some new inequalities of Aczel's type for real numbers.

I. Introduction

In 1956, J. Aczél has proved the following interesting inequality ([2, p. 57], [3, p. 117]):

Theorem A. *Let* $a = (a_1, a_2, \cdots, a_n)$ *and* $b = (b_1, b_2, \cdots, b_n)$ *be two sequences of real numbers such that*

$$0 < a_1^2 - a_2^2 - \cdots - a_n^2 \quad \text{or} \quad 0 < b_1^2 - b_2^2 - \cdots - b_n^2.$$

Then

$$\begin{aligned} &(a_1^2 - a_2^2 - \cdots - a_n^2)(b_1^2 - b_2^2 - \cdots - b_n^2) \\ &\leq (a_1 b_1 - a_2 b_2 - \cdots - a_n b_n)^2, \end{aligned} \tag{1.1}$$

with the equality if and only if the sequences a *and* b *are proportional.*

For various generalizations of Theorem A, see the recent book ([3, p. 117]) where further references are given.

Now, in this note, we give another proof than that embodied in [2, p. 57] for a weighted variant of (1.1).

Assume that

$$\sum_{i=1}^{n} p_i a_i^2 \leq a^2 \quad \text{and} \quad \sum_{i=1}^{n} p_i b_i^2 \leq b^2,$$

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where $a_i,\ b_i,\ a,\ b \in R$ and $0 \le p_i$ for $i = 1, 2, \cdots, n$. Then we have the following inequality:

$$\left(a^2 - \sum_{i=1}^{n} p_i a_i^2\right)\left(b^2 - \sum_{i=1}^{n} p_i b_i^2\right) \le \left(ab - \sum_{i=1}^{n} p_i a_i b_i\right)^2. \tag{1.2}$$

Indeed, by a simple calculation, we have

$$(a^2 - c^2)(b^2 - d^2) \le (|ab| - |cd|)^2$$

for all $a, b, c, d \in R$. Thus we have

$$\begin{aligned} &\left(a^2 - \sum_{i=1}^{n} p_i a_i^2\right)\left(b^2 - \sum_{i=1}^{n} p_i b_i^2\right) \\ &\le \left(|ab| - \left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2}\right)^2. \end{aligned} \tag{1.3}$$

By Cauchy-Buniakowski-Schwarz's inequality, we have

$$\left|\sum_{i=1}^{n} p_i a_i b_i\right| \le \left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2}$$

and so

$$\begin{aligned} 0 &\le |ab| - \left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2} \\ &\le |ab| - \left|\sum_{i=1}^{n} p_i a_i b_i\right| \\ &= \left||ab| - \left|\sum_{i=1}^{n} p_i a_i b_i\right|\right| \\ &\le \left|ab - \sum_{i=1}^{n} p_i a_i b_i\right|. \end{aligned}$$

Thus, we have

$$\left(|ab| - \left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2}\right)^2 \le \left(ab - \sum_{i=1}^{n} p_i a_i b_i\right)^2. \tag{1.4}$$

Therefore, from (1.3) and (1.4), we have the inequality (1.2). This completes the proof.

II. The Results

We will start with the following theorem which give a refinement of the following variant of Aczél's inequality:

$$\left(a^2-\sum_{i=1}^n p_i a_i^2\right)^{1/2}\left(b^2-\sum_{i=1}^n p_i b_i^2\right)^{1/2}\le\left|ab-\sum_{i=1}^n p_i a_i b_i\right|, \tag{2.1}$$

assuming that

$$\sum_{i=1}^n p_i a_i^2\le a^2 \quad\text{and}\quad \sum_{i=1}^n p_i b_i^2\le b^2$$

and a_i, b_i, a, $b\in R$, $0\le p_i$ for $i=1,2,\cdots,n$.

Theorem 2.1. *Assume that a_i, b_i, p_i, a, b are as above and $0\le q_i\le p_i$ for all $i=1,2,\cdots,n$. Then we have the following inequality:*

$$\begin{aligned}0&\le\sum_{i=1}^n q_i|a_ib_i|-\left|\sum_{i=1}^n q_ia_ib_i\right|\\&\le\left[\sum_{i=1}^n q_ia_i^2\sum_{i=1}^n q_ib_i^2\right]^{1/2}-\left|\sum_{i=1}^n q_ia_ib_i\right|\\&\le\left|ab-\sum_{i=1}^n p_ia_ib_i\right|-\left(a^2-\sum_{i=1}^n p_ia_i^2\right)^{1/2}\left(b^2-\sum_{i=1}^n p_ib_i^2\right)^{1/2}.\end{aligned}\tag{2.2}$$

Proof. From $a^2-\sum_{i=1}^n p_ia_i^2\ge 0$ and $b^2-\sum_{i=1}^n p_ia_i^2\ge 0$, it follows that

$$a^2-\sum_{i=1}^n(p_i-q_i)a_i^2\ge a^2-\sum_{i=1}^n p_ia_i^2\ge 0,$$
$$b^2-\sum_{i=1}^n(p_i-q_i)b_i^2\ge b^2-\sum_{i=1}^n p_ib_i^2\ge 0.$$

Now, for $t_i=p_i-q_i\ge 0$, by (1.2), we have

$$\left(a^2-\sum_{i=1}^n t_ia_i^2\right)\left(b^2-\sum_{i=1}^n t_ib_i^2\right)\le\left(ab-\sum_{i=1}^n t_ia_ib_i\right)^2,$$

i.e.,

$$\left[\left(a^2-\sum_{i=1}^n p_i a_i^2\right)+\sum_{i=1}^n q_i a_i^2\right]\left[\left(b^2-\sum_{i=1}^n p_i b_i^2\right)+\sum_{i=1}^n q_i b_i^2\right]$$
$$\leq\left[\left(ab-\sum_{i=1}^n p_i a_i b_i\right)+\sum_{i=1}^n q_i a_i b_i\right]^2.$$

Applying the well-known Cauchy-Buniakowski-Schwarz's inequality for real number, we have

(2.4)

$$\left[\Big(\sum_{i=1}^n q_i a_i^2)^{1/2}\left(\sum_{i=1}^n q_i b_i^2\right)^{1/2}+\left(a^2-\sum_{i=1}^n p_i a_i^2\right)^{1/2}\left(b^2-\sum_{i=1}^n p_i b_i^2\right)^{1/2}\right]^2$$
$$\leq\left(\sum_{i=1}^n q_i a_i^2+\left[\left(a^2-\sum_{i=1}^n p_i a_i^2\right)^{1/2}\right]^2\right)\left(\sum_{i=1}^n q_i b_i^2+\left[\left(b^2-\sum_{i=1}^n p_i b_i^2\right)^{1/2}\right]^2\right)$$

and, by the triangle inequality,

$$\left|\left(ab-\sum_{i=1}^n p_i a_i b_i\right)+\sum_{i=1}^n q_i a_i b_i\right|$$
$$\leq\left|ab-\sum_{i=1}^n p_i a_i b_i\right|+\left|\sum_{i=1}^n q_i a_i b_i\right|. \tag{2.5}$$

Thus, from (2.3), (2.4) and (2.5), it follows that

$$\left(\sum_{i=1}^n q_i a_i^2\right)^{1/2}\left(\sum_{i=1}^n q_i b_i^2\right)^{1/2}+\left(a^2-\sum_{i=1}^n p_i a_i^2\right)^{1/2}\left(b^2-\sum_{i=1}^n p_i b_i^2\right)^{1/2}$$
$$\leq\left|ab-\sum_{i=1}^n p_i a_i b_i\right|+\left|\sum_{i=1}^n q_i a_i b_i\right|,$$

which implies that

$$0\leq\sum_{i=1}^n q_i|a_i b_i|-\left|\sum_{i=1}^n q_i a_i b_i\right|$$
$$\leq\left(\sum_{i=1}^n q_i a_i^2\right)^{1/2}\left(\sum_{i=1}^n q_i b_i^2\right)^{1/2}-\left|\sum_{i=1}^n q_i a_i b_i\right|$$
$$\leq\left|ab-\sum_{i=1}^n p_i a_i b_i\right|-\left(a^2-\sum_{i=1}^n p_i a_i^2\right)^{1/2}\left(b^2-\sum_{i=1}^n p_i b_i^2\right)^{1/2}.$$

This completes the proof.

Corollary 2.2. *With the above assumptions for a_i, b_i, a, $b \in R$ and $0 \le p_i$ for $i = 1, 2, \cdots, n$, we have the following inequality:*

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} p_i |a_i b_i| - \Big| \sum_{i=1}^{n} p_i a_i b_i \Big| \\ &\le \Big[\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 \Big]^{1/2} - \Big| \sum_{i=1}^{n} p_i a_i b_i \Big| \\ &\le \Big| ab - \sum_{i=1}^{n} p_i a_i b_i \Big| - \Big(a^2 - \sum_{i=1}^{n} p_i a_i^2 \Big)^{1/2} \Big(b^2 - \sum_{i=1}^{n} p_i b_i^2)^{1/2}. \end{aligned}$$

Another result of Aczél's type is as follows:

Theorem 2.3. *Assume that a, b, a_i, $b_i \in R$ and $0 \le p_i$ for $i = 1, 2, \cdots, n$ are such that*

$$\sum_{i=1}^{n} p_i a_i^2 \le a^2 \quad \textit{and} \quad \sum_{i=1}^{n} p_i b_i^2 \le b^2.$$

Then we have the following inequality:

$$\begin{aligned} &\Big[|a| - \Big(\sum_{i=1}^{n} p_i a_i^2 \Big)^{1/2} \Big]^{1/2} \Big[|b| - \Big(\sum_{i=1}^{n} p_i b_i^2 \Big)^{1/2} \Big]^{1/2} \\ &\le |ab|^{1/2} - \Big| \sum_{i=1}^{n} p_i a_i b_i \Big|^{1/2}. \end{aligned}$$

Proof. We will start with the following elementary inequality:

$$\sqrt{(x-y)(z-u)} \le \sqrt{xz} - \sqrt{yu}, \tag{2.7}$$

where $x \ge y \ge 0$ and $z \ge u \ge 0$. Indeed, the inequality (2.7) is equivalent with

$$(x-y)(z-u) \le (\sqrt{xz} - \sqrt{yu})^2 = xz - 2\sqrt{xzyu} + yu,$$

i.e.,

$$xz + yu - yz - xu \le xz - 2\sqrt{xzyu} + yu,$$

which is equivalent with

$$2\sqrt{xzyu} \le yz + xu$$

for $x, y, z, u \geq 0$, which is obvious.

Now, putting

$$x = |a|, \quad y = \Big(\sum_{i=1}^{n} p_i a_i^2\Big)^{1/2}, \quad z = |b|, \quad u = \Big(\sum_{i=1}^{n} p_i b_i^2)^{1/2},$$

then, by the inequality (2.7), we have

$$\begin{aligned} &\Big[|a| - \Big(\sum_{i=1}^{n} p_i a_i^2\Big)^{1/2}\Big]^{1/2}\Big[|b| - \Big(\sum_{i=1}^{n} p_i b_i^2\Big)^{1/2}\Big]^{1/2} \\ &\leq |ab|^{1/2} - \Big(\sum_{i=1}^{n} p_i a_i^2\Big)^{1/4}\Big(\sum_{i=1}^{n} p_i b_i^2\Big)^{1/4}. \end{aligned} \tag{2.8}$$

By Cauchy-Buniakowski-Schwarz's inequality, we have

$$\Big|\sum_{i=1}^{n} p_i a_i b_i\Big|^{1/2} \leq \Big[\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2\Big]^{1/4}$$

and so, by (2.8), we have the desired inequality (2.6). This completes the proof.

Corollary 2.4. *Let a, b, a_i, $b_i \in R$ for $i = 1, 2, \cdots, n$ be such that*

$$\sum_{i=1}^{n} a_i^2 \leq a^2 \quad \textit{and} \quad \sum_{i=1}^{n} b_i^2 \leq b^2.$$

Then we have the following inequality:

$$\begin{aligned} &\Big[|a| - \Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2}\Big]^{1/2}\Big[|b| - \Big(\sum_{i=1}^{n} b_i^2\Big)^{1/2}\Big]^{1/2} \\ &\leq |ab|^{1/2} - \Big|\sum_{i=1}^{n} a_i b_i\Big|^{1/2}. \end{aligned}$$

Remark. The inequality (2.9) was proved in [1] as a particular case of an inequality holding in inner product spaces.

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