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**FURTHER INEQUALITIES FOR THE EXPECTATION AND
VARIANCE OF A RANDOM VARIABLE DEFINED ON A
FINITE INTERVAL**

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ABSTRACT. Some new elementary inequalities for the expectation and the variance of a continuous random variable defined on a finite interval are given.

1. INTRODUCTION

Let X be a continuous random variable having the probability density function $f : [a, b] \rightarrow (0, \infty)$ and the cumulative distribution function $F : [a, b] \rightarrow [0, 1]$.

In a recent paper [10], the authors pointed out a number of inequalities for the expectation, $E(X)$ and the variance, $\sigma^2(X)$ from which we cite only the following:

$$(1.1) \quad 0 \leq \sigma^2(X) \leq [b - E(X)][E(X) - a] \leq \frac{1}{4}(b - a)^2;$$

$$(1.2) \quad 0 \leq [b - E(X)][E(X) - a] - \sigma^2(X) \leq \begin{cases} \frac{(b-a)^3}{6} \|f\|_\infty \\ [B(q+1, q+1)]^{\frac{1}{q}} (b-a)^{2+\frac{1}{q}} \|f\|_p, \\ \text{provided } f \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

where $B(\cdot, \cdot)$ is Euler's Beta function, i.e., we recall it

$$B(\alpha, \beta) := \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt, \quad \alpha, \beta > -1.$$

Moreover, if $m \leq f \leq M$ a.e. on $[a, b]$, then

$$(1.3) \quad \frac{m(b-a)^3}{6} \leq [b - E(X)][E(X) - a] - \sigma^2(X) \leq \frac{M(b-a)^3}{6}$$

and

$$(1.4) \quad \left| [b - E(X)][E(X) - a] - \sigma^2(X) - \frac{(b-a)^3}{6} \right| \leq \frac{\sqrt{5}(b-a)^3(M-m)}{60}.$$

In this current paper, we point out some additional results.

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2. THE RESULTS

Lemma 1. *Let X be a continuous random variable having the cumulative distribution function $F : [a, b] \rightarrow [0, 1]$. Then,*

$$(2.1) \quad \begin{aligned} \sigma^2(X) &= (b - E(X))(E(X) - a) \\ &\quad + \frac{1}{b-a} \int_a^b \int_a^b (t - \tau)(F(\tau) - F(t)) d\tau dt. \end{aligned}$$

Proof. Using integration by parts, we have

$$(2.2) \quad \begin{aligned} \sigma^2(X) &= \int_a^b (t - E(X))^2 dF(t) \\ &= (t - E(X))^2 F(t) \Big|_a^b - 2 \int_a^b (t - E(X)) F(t) dt \\ &= (b - E(X))^2 - 2 \int_a^b (t - E(X)) F(t) dt. \end{aligned}$$

Further, using Korkine's identity,

$$\begin{aligned} \frac{1}{b-a} \int_a^b h(t) g(t) dt &= \frac{1}{b-a} \int_a^b h(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \\ &\quad + \frac{1}{2(b-a)^2} \int_a^b \int_a^b (h(t) - h(\tau))(g(t) - g(\tau)) d\tau dt, \end{aligned}$$

we have

$$(2.3) \quad \begin{aligned} &\int_a^b (t - E(X)) F(t) dt \\ &= \frac{1}{b-a} \int_a^b (t - E(X)) dt \int_a^b F(t) dt \\ &\quad + \frac{1}{2(b-a)} \int_a^b \int_a^b (t - \tau)(F(t) - F(\tau)) d\tau dt. \end{aligned}$$

Since,

$$\begin{aligned} \int_a^b (t - E(X)) dt &= \frac{(b - E(X))^2 - (E(X) - a)^2}{2} \\ &= (b - a) \left(\frac{b+a}{2} - E(X) \right) \end{aligned}$$

and

$$\int_a^b F(t) dt = b - E(X),$$

then, by (2.2) and (2.3),

$$\begin{aligned}
 \sigma^2(X) &= (b - E(X))^2 - 2 \left[\frac{b + a - 2E(X)}{2} \cdot (b - E(X)) \right. \\
 &\quad \left. + \frac{1}{2(b-a)} \int_a^b \int_a^b (t - \tau) (F(t) - F(\tau)) d\tau dt \right] \\
 &= (b - E(X))^2 - (b + a - 2E(X))(b - E(X)) \\
 &\quad - \frac{1}{b-a} \int_a^b \int_a^b (t - \tau) (F(t) - F(\tau)) d\tau dt \\
 &= (b - E(X))(E(X) - a) - \frac{1}{b-a} \int_a^b \int_a^b (t - \tau) (F(t) - F(\tau)) d\tau dt
 \end{aligned}$$

and the lemma is proved. ■

Remark 1. Since the mapping F is monotonic nondecreasing on $[a, b]$, then

$$(2.4) \quad (t - \tau)(F(\tau) - F(t)) \leq 0 \text{ for all } t, \tau \in [a, b];$$

which implies that

$$(2.5) \quad \sigma^2(X) \leq [b - E(X)][E(X) - a],$$

an inequality that was proved in [10] and [11] using two different methods.

The inequality (2.5) can be improved as follows.

Theorem 1. With the assumptions in Lemma 1,

$$\begin{aligned}
 (2.6) \quad &(b - E(X))(E(X) - a) - \sigma^2(X) \\
 &\geq 2 \left| \int_a^b |t| F(t) dt - \frac{1}{b-a} (b - E(X)) \int_a^b |t| dt \right| \geq 0.
 \end{aligned}$$

Proof. In [12], S.S. Dragomir proved the following refinement of Chebychev's inequality

$$(2.7) \quad T(h, g) \geq \max \{ |T(h, |g|)|, |T(|h|, g)|, |T(|h|, |g|)| \} \geq 0,$$

provided (h, g) are synchronous on $[a, b]$, i.e.,

$$(h(t) - h(\tau))(g(t) - g(\tau)) \geq 0 \text{ for all } t, \tau \in [a, b]$$

and

$$T(h, g) := \frac{1}{b-a} \int_a^b h(t)g(t) dt - \frac{1}{b-a} \int_a^b h(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt.$$

If we define $h(t) = t$, $t \in [a, b]$, then from (2.1)

$$\begin{aligned}
 T(h, F) &= \frac{1}{b-a} \int_a^b \int_a^b (t - \tau) (F(t) - F(\tau)) d\tau dt \\
 &= \frac{1}{b-a} [(b - E(X))(E(X) - a) - \sigma^2(X)].
 \end{aligned}$$

Now, from (2.7),

$$\begin{aligned} T(|h|, F) &= \frac{1}{b-a} \int_a^b \int_a^b |t| F(t) dt - \frac{1}{(b-a)^2} \int_a^b |t| dt \int_a^b F(t) dt, \\ T(h, |F|) &= T(h, F), \\ T(|h|, |F|) &= T(|h|, F). \end{aligned}$$

Using the result (2.7), we get (2.6). ■

Remark 2. If $a \leq b \leq 0$ or $0 \leq a \leq b$, then the first inequality in (2.6) becomes an identity and is of no special interest.

If $a < 0 < b$, however, then,

$$\begin{aligned} \int_a^b |t| F(t) dt &= - \int_a^0 t F(t) dt + \int_0^b t F(t) dt; \\ \frac{1}{b-a} \int_a^b |t| dt &= \frac{1}{b-a} \left[- \int_a^0 t dt + \int_0^b t dt \right] = \frac{1}{b-a} \left[\frac{a^2 + b^2}{2} \right] \end{aligned}$$

and by (2.6), we get

$$\begin{aligned} (2.8) \quad & (b - E(X))(E(X) - a) - \sigma^2(X) \\ & \geq 2 \left| \int_0^b t F(t) dt - \int_a^0 t F(t) dt - \frac{a^2 + b^2}{2(b-a)} (b - E(X)) \right| \geq 0. \end{aligned}$$

Assume that $f, f : [a, b] \rightarrow (0, \infty)$ is the p.d.f. of X , then the following theorem holds.

Theorem 2. With the assumptions in Lemma 1,

$$(2.9) \quad 0 \leq (b - E(X))(E(X) - a) - \sigma^2(X) \leq \begin{cases} \frac{(b-a)^3}{6} \|f\|_\infty & \text{if } f \in L_\infty[a, b]; \\ \frac{2q^2(b-a)^{2+\frac{1}{q}} \|f\|_p}{(2q+1)(3q+1)} & \text{if } f \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^2}{3}; \end{cases}$$

where $\|\cdot\|_p$ ($p \geq 1$) on the usual Lebesgue norm.

Proof. Using (2.1),

$$\begin{aligned} (2.10) \quad 0 &\leq (b - E(X))(E(X) - a) - \sigma^2(X) \\ &= \frac{1}{b-a} \int_a^b \int_a^b (t - \tau) \left(\int_\tau^t f(u) du \right) dt d\tau. \end{aligned}$$

By the modulus property, we have

$$\begin{aligned} (2.11) \quad 0 &\leq (b - E(X))(E(X) - a) - \sigma^2(X) \\ &= \frac{1}{b-a} \left| \int_a^b \int_a^b (t - \tau) \left(\int_\tau^t f(u) du \right) dt d\tau \right| \\ &\leq \frac{1}{b-a} \int_a^b \int_a^b |t - \tau| \left| \int_\tau^t f(u) du \right| dt d\tau =: M. \end{aligned}$$

If $f \in L_\infty [a, b]$, then we can write,

$$\left| \int_\tau^t f(u) du \right| \leq |t - \tau| \|f\|_\infty$$

for all $t, \tau \in [a, b]$, and so

$$\begin{aligned} M &\leq \frac{1}{b-a} \int_a^b \int_a^b |t - \tau| |t - \tau| \|f\|_\infty dt d\tau \\ &= \frac{\|f\|_\infty}{b-a} \int_a^b \int_a^b (t - \tau)^2 dt d\tau = \frac{\|f\|_\infty (b-a)^3}{6}. \end{aligned}$$

For the second part, we apply Hölder's integral inequality to write:

$$\begin{aligned} \left| \int_\tau^t f(u) du \right| &\leq \left| \int_\tau^t du \right|^{\frac{1}{q}} \left| \int_\tau^t f^p(u) du \right|^{\frac{1}{p}} \leq |t - \tau|^{\frac{1}{q}} \left(\int_a^b f^p(u) du \right)^{\frac{1}{p}} \\ &= |t - \tau|^{\frac{1}{q}} \|f\|_p, \end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

In addition,

$$\begin{aligned} M &\leq \frac{1}{b-a} \int_a^b \int_a^b |t - \tau| |t - \tau|^{\frac{1}{q}} \|f\|_p dt d\tau \\ &= \frac{\|f\|_p}{b-a} \int_a^b \left[\int_a^t (t - \tau)^{1+\frac{1}{q}} d\tau + \int_t^b (\tau - t)^{1+\frac{1}{q}} d\tau \right] dt \\ &= \frac{\|f\|_p}{b-a} \int_a^b \left[\frac{(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}}}{\left(2 + \frac{1}{q}\right)} \right] dt = \frac{2 \|f\|_p (b-a)^{2+\frac{1}{q}}}{\left(2 + \frac{1}{q}\right) \left(3 + \frac{1}{q}\right)} \end{aligned}$$

and the second inequality in (2.9) is proved.

Finally,

$$\begin{aligned} M &\leq \frac{1}{b-a} \int_a^b \int_a^b |t - \tau| \left(\int_a^b f(u) du \right) dt d\tau \\ &= \frac{1}{b-a} \int_a^b \left[\frac{(t-a)^2 + (b-t)^2}{2} \right] dt \\ &= \frac{1}{2(b-a)} \left[\frac{(b-a)^3}{3} + \frac{(b-a)^3}{3} \right] = \frac{(b-a)^2}{3}, \end{aligned}$$

and the theorem is completely proved. ■

Using the Cauchy-Buniakowsky-Schwartz inequality, we have the following inequality.

Theorem 3. *If X and F are as in Lemma 1, then,*

$$\begin{aligned} (2.12) \quad 0 &\leq (b - E(X))(E(X) - a) - \sigma^2(X) \\ &\leq \frac{(b-a)^2}{\sqrt{3}} \left[(b-a) \|F\|_2^2 - (b - E(X))^2 \right]^{\frac{1}{2}} \\ &\leq \frac{(b-a)^3}{2\sqrt{3}}. \end{aligned}$$

Proof. Using the Cauchy-Buniakowsky-Schwartz integral inequality for double integrals,

$$(2.13) \quad \left| \int_a^b \int_a^b (t - \tau) (F(\tau) - F(t)) dt d\tau \right| \\ \leq \left(\int_a^b \int_a^b (t - \tau)^2 dt d\tau \right)^{\frac{1}{2}} \left(\int_a^b \int_a^b (F(t) - F(\tau))^2 dt d\tau \right)^{\frac{1}{2}}.$$

However,

$$\int_a^b \int_a^b (t - \tau)^2 dt d\tau = \frac{(b - a)^4}{6}, \\ \int_a^b \int_a^b (F(\tau) - F(t))^2 dt d\tau = 2 \left[(b - a) \int_a^b F^2(t) dt - \left(\int_a^b F(t) dt \right)^2 \right] \\ = 2 \left[(b - a) \|F\|_2^2 - (b - E(X))^2 \right]$$

and, by (2.13),

$$\left| \int_a^b \int_a^b (t - \tau) (F(\tau) - F(t)) d\tau dt \right| \leq \frac{(b - a)^2}{\sqrt{3}} \left[(b - a) \|F\|_2^2 - (b - E(X))^2 \right]^{\frac{1}{2}}$$

and the first inequality in (2.12) is proved.

To prove the last part of (2.12), we use the following Grüss type inequality:

$$(2.14) \quad \frac{1}{b - a} \int_a^b g^2(t) dt - \left(\frac{1}{b - a} \int_a^b g(t) dt \right)^2 \leq \frac{1}{4} (\phi - \gamma)^2,$$

provided that $g \in L_2(a, b)$ and $\gamma \leq g(t) \leq \phi$ a.e. for $t \in (a, b)$.

From (2.14),

$$(b - a) \int_a^b F^2(t) dt - \left(\int_a^b F(t) dt \right)^2 \leq \frac{1}{4} (b - a)^2$$

since

$$\sup_{t \in [a, b]} F(t) = 1 \quad \text{and} \quad \inf_{t \in [a, b]} F(t) = 0.$$

■

If it is assumed that the mapping f is convex on $[a, b]$, then the following result can be obtained.

Theorem 4. *Assume that the p.d.f., $f : [a, b] \rightarrow (0, \infty)$ is convex. Then we have the inequality*

$$(2.15) \quad \frac{1}{b - a} \int_a^b \int_a^b (t - \tau)^2 f\left(\frac{t + \tau}{2}\right) d\tau dt \\ \leq [b - E(X)] [E(X) - a] - \sigma^2(X) \\ \leq \frac{(b - a)^2}{3} + \sigma^2(X) - (b - E(X)) (E(X) - a).$$

Proof. Using the Hermite-Hadamard inequality,

$$(2.16) \quad f\left(\frac{t+\tau}{2}\right) \leq \frac{\int_t^\tau f(u) du}{\tau-t} \leq \frac{f(t)+f(\tau)}{2}$$

for all $t, \tau \in [a, b]$, $t \neq \tau$, we have

$$(2.17) \quad (t-\tau)^2 f\left(\frac{t+\tau}{2}\right) \leq (t-\tau)(F(t)-F(\tau)) \leq \frac{f(t)+f(\tau)}{2}(t-\tau)^2$$

for all $t, \tau \in [a, b]$.

Integrating (2.17) on $[a, b]^2$ and using the representation (2.1), gives:-

$$(2.18) \quad \begin{aligned} & \frac{1}{b-a} \int_a^b \int_a^b (t-\tau)^2 f\left(\frac{t+\tau}{2}\right) dt d\tau \\ & \leq \frac{1}{b-a} \int_a^b \int_a^b (t-\tau)(F(t)-F(\tau)) dt d\tau \\ & = [b-E(X)][E(X)-a] - \sigma^2(X) \\ & \leq \frac{1}{b-a} \int_a^b \int_a^b \frac{f(t)+f(\tau)}{2} (t-\tau)^2 dt d\tau. \end{aligned}$$

Now

$$(2.19) \quad \begin{aligned} & \int_a^b \int_a^b (t-\tau)^2 \left[\frac{f(t)+f(\tau)}{2} \right] dt d\tau \\ & = \int_a^b \int_a^b (t-\tau)^2 f(t) d\tau dt = \int_a^b \left[\int_a^b (t-\tau)^2 d\tau \right] f(t) dt \\ & = \int_a^b \left[\frac{(b-t)^3 + (t-a)^3}{3} \right] f(t) dt \\ & = \frac{(b-a)}{3} \int_a^b \left[(b-t)^2 - (b-t)(t-a) + (t-a)^2 \right] f(t) dt \\ & = \frac{b-a}{3} \int_a^b \left[(b-a)^2 - 3(b-t)(t-a) \right] f(t) dt \\ & = \frac{(b-a)^3}{3} - (b-a) \int_a^b (b-t)(t-a) f(t) dt \\ & = \frac{(b-a)^3}{3} + (b-a) [\sigma^2(X) - (b-E(X))(E(X)-a)] \end{aligned}$$

on using an identity (see [10]).

Hence, from (2.19) and the above working,

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \int_a^b (t-\tau)^2 \left[\frac{f(t)+f(\tau)}{2} \right] dt d\tau \\ & = \frac{(b-a)^2}{3} + [\sigma^2(X) - (b-E(X))(E(X)-a)], \end{aligned}$$

and the second part of (2.15) is proved. ■

Remark 3. *The second inequality in (2.15) is equivalent to:-*

$$(2.20) \quad [b - E(X)][E(X) - a] \leq \sigma^2(X) + \frac{1}{6}(b - a)^2.$$

Remark 4. *For $b - a < \frac{1}{\sqrt{3}}$, then the result of Theorem 5 is better than that of Theorem 4. For $b - a > \frac{1}{\sqrt{3}}$, the opposite applies. It must be remembered that Theorem 4 relies on f being convex whereas Theorem 3 does not.*

The following representation for the absolutely continuous p.d.f., $f : [a, b] \rightarrow \mathbb{R}$ holds.

Lemma 2. *Let X be a random variable having the p.d.f., $f : [a, b] \rightarrow \mathbb{R}$ absolutely continuous on $[a, b]$. Then we have*

$$(2.21) \quad \begin{aligned} \sigma^2(X) &= (b - E(X))(E(X) - a) - \frac{(b - a)^2}{6} \\ &+ \frac{1}{2(b - a)} \int_a^b \int_a^b (t - \tau) \left(\int_\tau^t \left(u - \frac{t + \tau}{2} \right) f'(u) du \right) dt d\tau. \end{aligned}$$

Proof. We use the following identity which holds for the absolutely continuous mapping $g : [a, b] \rightarrow \mathbb{R}$

$$(2.22) \quad \int_a^b g(u) du = \frac{g(a) + g(b)}{2} (b - a) - \int_a^b \left(u - \frac{a + b}{2} \right) g'(u) du,$$

can be easily proven by using the integration by parts formula.

We know that

$$(2.23) \quad \begin{aligned} &(E(X) - a)(b - E(X)) - \sigma^2(X) \\ &= \frac{1}{b - a} \int_a^b \int_a^b (t - \tau) \int_\tau^t f(u) du dt d\tau \\ &= \frac{1}{b - a} \int_a^b \int_a^b (t - \tau) \left[\frac{f(t) + f(\tau)}{2} (t - \tau) - \int_\tau^t \left(u - \frac{t + \tau}{2} \right) f'(u) du \right] dt d\tau \\ &= \frac{1}{b - a} \int_a^b \int_a^b (t - \tau)^2 \left(\frac{f(t) + f(\tau)}{2} \right) dt d\tau \\ &\quad - \frac{1}{b - a} \int_a^b \int_a^b (t - \tau) \left(\int_\tau^t \left(u - \frac{t + \tau}{2} \right) f'(u) du \right) dt d\tau. \end{aligned}$$

However, observe that (see the proof of Theorem 4)

$$\begin{aligned} L &: = \frac{1}{b - a} \int_a^b \int_a^b (t - \tau)^2 \left(\frac{f(t) + f(\tau)}{2} \right) dt d\tau \\ &\quad \sigma^2(X) + \frac{1}{3} \left[(E(X) - b)^2 - (E(X) - a)(b - E(X)) + (E(X) - a)^2 \right]. \end{aligned}$$

Using (2.23), we have

$$\begin{aligned} &(E(X) - a)(b - E(X)) - \sigma^2(X) \\ &= \sigma^2(X) + \frac{1}{3} \left[(E(X) - b)^2 - (E(X) - a)(b - E(X)) + (E(X) - a)^2 \right] \\ &\quad - \frac{1}{b - a} \int_a^b \int_a^b (t - \tau) \left(\int_\tau^t \left(u - \frac{t + \tau}{2} \right) f'(u) du \right) dt d\tau, \end{aligned}$$

which is clearly equivalent to (2.21). ■

Using Lemma 2, we are able to obtain the following bounds.

Theorem 5. *Assume that f is as in Lemma 2. Then we have the inequality*

$$(2.24) \quad \left| \left[b - E(X) \right] \left[E(X) - a \right] - \sigma^2(X) - \frac{(b-a)^2}{6} \right| \leq \begin{cases} \frac{\|f'\|_\infty}{80} (b-a)^4 & \text{if } f' \in L_\infty[a, b]; \\ \frac{q^2 \|f'\|_p}{2(3q+1)(4q+1)(q+1)^{\frac{1}{q}}} (b-a)^{3+\frac{1}{q}} & \text{if } f' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{\|f'\|_1}{24} (b-a)^3. & \end{cases}$$

Proof. Using the equality (2.21), we may write

$$(2.25) \quad \left| \sigma^2(X) - (b - E(X))(E(X) - a) + \frac{(b-a)^2}{6} \right| \leq \frac{1}{2(b-a)} \int_a^b \int_a^b |t-\tau| \left| \int_\tau^t \left(u - \frac{t+\tau}{2} \right) f'(u) du \right| dt d\tau := N.$$

Now, it may be easily shown that,

$$\begin{aligned} \left| \int_\tau^t \left(u - \frac{t+\tau}{2} \right) f'(u) du \right| &\leq \|f'\|_\infty \left| \int_\tau^t \left| u - \frac{t+\tau}{2} \right| du \right| \\ &= \|f'\|_\infty \frac{(t-\tau)^2}{4} \end{aligned}$$

for all $t, \tau \in [a, b]$.

Also, by Hölder's integral inequality, we may write

$$\begin{aligned} \left| \int_\tau^t \left(u - \frac{t+\tau}{2} \right) f'(u) du \right| &\leq \left| \int_\tau^t |f'(u)|^p du \right|^{\frac{1}{p}} \left| \int_\tau^t \left| u - \frac{t+\tau}{2} \right|^q du \right|^{\frac{1}{q}} \\ &\leq \|f'\|_p \left[\frac{|t-\tau|^{q+1}}{2q(q+1)} \right]^{\frac{1}{q}} = \|f'\|_p \frac{|t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} \end{aligned}$$

for all $t, \tau \in [a, b]$, and further,

$$\begin{aligned} \left| \int_\tau^t \left(u - \frac{t+\tau}{2} \right) f'(u) du \right| &\leq \sup \left| u - \frac{t+\tau}{2} \right| \int_\tau^t |f'(u)| du \\ &\leq \frac{|t-\tau|}{2} \|f'\|_1. \end{aligned}$$

Consequently,

$$(2.26) \quad \left| \int_\tau^t \left(u - \frac{t+\tau}{2} \right) f'(u) du \right| \leq \begin{cases} \|f'\|_\infty \frac{(t-\tau)^2}{4} & \text{if } f' \in L_\infty[a, b]; \\ \|f'\|_p \frac{|t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} & \text{if } f' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'\|_1 \frac{|t-\tau|}{2}. & \text{if } f' \in L_1[a, b]. \end{cases}$$

Using (2.26), we may write, for f' belonging to the obvious Lebesgue space $L_p[a, b]$, $p \geq 1$,

$$(2.27) \quad N \leq \begin{cases} \frac{\|f'\|_\infty}{8(b-a)} \int_a^b \int_a^b |t - \tau|^3 dt d\tau, \\ \frac{\|f'\|_p}{4(q+1)^{\frac{1}{q}}(b-a)} \int_a^b \int_a^b |t - \tau|^{2+\frac{1}{q}} dt d\tau, \\ \frac{\|f'\|_1}{4(b-a)} \int_a^b \int_a^b (t - \tau)^2 dt d\tau. \end{cases}$$

Now, since some straight forward algebra shows that

$$\begin{aligned} \int_a^b \int_a^b |t - \tau|^3 dt d\tau &= \int_a^b \left[\int_a^t (t - \tau)^3 d\tau + \int_t^b (\tau - t)^3 d\tau \right] dt \\ &= \int_a^b \left[\frac{(t-a)^4 + (b-t)^4}{4} \right] dt = \frac{(b-a)^5}{10}, \end{aligned}$$

$$\begin{aligned} \int_a^b \int_a^b |t - \tau|^{2+\frac{1}{q}} dt d\tau &= \int_a^b \left[\int_a^t (t - \tau)^{2+\frac{1}{q}} d\tau + \int_t^b (\tau - t)^{2+\frac{1}{q}} d\tau \right] dt \\ &= \int_a^b \left[\frac{(t-a)^{3+\frac{1}{q}} + (b-t)^{3+\frac{1}{q}}}{3 + \frac{1}{q}} \right] dt \\ &= \frac{2q^2 (b-a)^{4+\frac{1}{q}}}{(3q+1)(4q+1)} \end{aligned}$$

and

$$\begin{aligned} \int_a^b \int_a^b (t - \tau)^2 dt d\tau &= \int_a^b \left[\int_a^t (t - \tau)^2 d\tau + \int_t^b (\tau - t)^2 d\tau \right] dt \\ &= \int_a^b \left[\frac{(t-a)^3 + (b-t)^3}{3} \right] dt \\ &= \frac{(b-a)^4}{6}, \end{aligned}$$

we obtain the desired inequality (2.24) from using (2.27) and (2.25). ■

The following representation for the mappings whose derivatives are absolutely continuous on $[a, b]$ also holds.

Lemma 3. *Let X be a random variable having the p.d.f. $f : [a, b] \rightarrow \mathbb{R}$ and with the property that $f' : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$. Then we have*

$$(2.28) \quad \begin{aligned} \sigma^2(X) &= (b - E(X))(E(X) - a) - \frac{(b-a)^2}{6} \\ &+ \frac{1}{4(b-a)} \int_a^b \int_a^b (t - \tau) \int_\tau^t (t - u)(u - \tau) f''(u) du dt d\tau. \end{aligned}$$

Proof. We use the following identity which holds for the mappings g whose derivatives are absolutely continuous:

$$(2.29) \quad \int_a^b g(u) du = \frac{g(a) + g(b)}{2} (b-a) - \frac{1}{2} \int_a^b (b-u)(u-a) g''(u) du$$

and can easily be proven by using the integration by parts formula twice.

We know that

$$(b - E(X))(E(X) - a) - \sigma^2(X) = \frac{1}{b-a} \int_a^b \int_a^b (t-\tau) \int_t^\tau f(u) du dt d\tau$$

and then, using the representation (2.29) written for f instead of g , and proceeding as in the proof of Lemma 2, we end up with the identity (2.28). ■

Using the representation of Lemma 3, we are able to obtain the following bounds.

Theorem 6. *Assume that f is as in Lemma 3. Then we have the inequality*

$$(2.30) \quad \left| [b - E(X)][E(X) - a] - \sigma^2(X) - \frac{(b-a)^2}{6} \right| \leq \begin{cases} \frac{\|f''\|_\infty}{360} (b-a)^5 & \text{if } f'' \in L_\infty[a, b] \\ \frac{\|f''\|_q}{2(4p+1)(5p+1)} [B(p+1, p+1)]^{\frac{1}{p}} (b-a)^{4+\frac{1}{p}} & \text{if } f'' \in L_q[a, b], \\ & \frac{1}{p} + \frac{1}{q} = 1, p > 1; \\ \frac{\|f''\|_1}{160} (b-a)^4, & \end{cases}$$

where the p -norms are taken on the interval $[a, b]$.

Proof. Using the equality (2.28), we may write

$$\begin{aligned} & \left| \sigma^2(X) - [b - E(X)][E(X) - a] - \frac{(b-a)^2}{6} \right| \\ & \leq \frac{1}{4(b-a)} \int_a^b \int_a^b |t-\tau| \left| \int_\tau^t (t-u)(u-\tau) f''(u) du \right| dt d\tau := K. \end{aligned}$$

First of all, let us observe that

$$\begin{aligned} \left| \int_\tau^t (t-u)(u-\tau) f''(u) du \right| & \leq \|f''\|_\infty \left| \int_\tau^t (t-u)(u-\tau) du \right| \\ & \leq \frac{\|f''\|_\infty}{6} |t-\tau|^3, \end{aligned}$$

for all $t, \tau \in [a, b]$.

Further, by Holder's integral inequality, we obtain

$$\begin{aligned} \left| \int_\tau^t (t-u)(u-\tau) f''(u) du \right| & \leq \|f''\|_q \left| \int_\tau^t |t-u|^p |u-\tau|^p du \right|^{\frac{1}{p}} \\ & = \|f''\|_q |t-\tau|^{2+\frac{1}{p}} [B(p+1, p+1)]^{\frac{1}{p}} \end{aligned}$$

for all $t, \tau \in [a, b]$, where B is the Beta function of Euler and $\frac{1}{p} + \frac{1}{q} = 1; p > 1$.

Also, we have

$$\begin{aligned} \left| \int_{\tau}^t (t-u)(u-\tau) f''(u) du \right| &\leq \|f''\|_1 \max_{u \in [\tau, t]} |(t-u)(u-\tau)| \\ &= \frac{|t-\tau|^2}{4} \|f''\|_1 \end{aligned}$$

for all $t, \tau \in [a, b]$.

Consequently, we may state the inequality

$$(2.31) \quad \left| \int_{\tau}^t (t-u)(u-\tau) f''(u) du \right| \leq \begin{cases} \frac{\|f''\|_{\infty}}{6} |t-\tau|^3 & \text{if } f'' \in L_{\infty}[a, b]; \\ \|f''\|_q [B(p+1, p+1)]^{\frac{1}{p}} |t-\tau|^{2+\frac{1}{p}} & \text{if } f'' \in L_q[a, b], \\ \frac{|t-\tau|^2}{4} \|f''\|_1, & \frac{1}{p} + \frac{1}{q} = 1, p > 1; \end{cases}$$

for all $t, \tau \in [a, b]$.

Using (2.31) and the definition of K above, we may write

$$(2.32) \quad K \leq \begin{cases} \frac{\|f''\|_{\infty}}{24(b-a)} \int_a^b \int_a^b (t-\tau)^4 dt d\tau & \text{if } f'' \in L_{\infty}[a, b]; \\ \frac{\|f''\|_q}{4(b-a)} [B(p+1, p+1)]^{\frac{1}{p}} \int_a^b \int_a^b |t-\tau|^{3+\frac{1}{p}} dt d\tau & \text{if } f'' \in L_q[a, b], \\ \frac{\|f''\|_1}{16(b-a)} \int_a^b \int_a^b |t-\tau|^3 dt d\tau. & \frac{1}{p} + \frac{1}{q} = 1, p > 1; \end{cases}$$

Now, since some straight forward algebra shows that

$$\begin{aligned} \int_a^b \int_a^b (t-\tau)^4 dt d\tau &= \frac{(b-a)^6}{15}. \\ \int_a^b \int_a^b |t-\tau|^{3+\frac{1}{p}} dt d\tau &= \int_a^b \left[\int_a^t (t-\tau)^{3+\frac{1}{p}} d\tau + \int_t^b (\tau-t)^{3+\frac{1}{p}} d\tau \right] dt \\ &= \int_a^b \left[\frac{(t-a)^{4+\frac{1}{p}} + (b-t)^{4+\frac{1}{p}}}{4 + \frac{1}{p}} \right] dt \\ &= 2 \frac{(b-a)^{5+\frac{1}{p}}}{\left(4 + \frac{1}{p}\right) \left(5 + \frac{1}{p}\right)} = \frac{2p^2 (b-a)^{5+\frac{1}{p}}}{(4p+1)(5p+1)} \end{aligned}$$

and

$$\begin{aligned} \int_a^b \int_a^b |t-\tau|^3 dt d\tau &= \int_a^b \left[\int_a^t (t-\tau)^3 d\tau + \int_t^b (\tau-t)^3 d\tau \right] dt \\ &= \int_a^b \left[\frac{(t-a)^4 + (b-t)^4}{4} \right] dt = \frac{(b-a)^5}{10}, \end{aligned}$$

then by (2.32), we deduce the desired inequality (2.30). \blacksquare

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