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SOME INEQUALITIES FOR THE EXPECTATION AND VARIANCE OF A RANDOM VARIABLE WHOSE PDF IS n-TIME DIFFERENTIABLE

N. S. BARNETT, P. CERONE, S.S. DRAGOMIR, AND J. ROUMELIOTIS

Abstract. Some inequalities for the expectation and variance of a random variable whose p.d.f. is n-time differentiable are given.

1. Introduction

Let $f:[a,b]\to\mathbb{R}_+$ be the p.d.f. of the random variable X and

$$E(X):=\int_a^b tf(t)\,dt$$

its *expectation* and

$$\begin{aligned}\sigma(X) &= \left[\int_a^b (t-E(X))^2 f(t)\,dt\right]^{\frac{1}{2}} \\ &= \left[\int_a^b t^2 f(t)\,dt-[E(X)]^2\right]^{\frac{1}{2}}\end{aligned}$$

its dispersion or standard deviation.

In [1], using the identity

$$[x-E(X)]^2+\sigma^2(X)=\int_a^b (x-t)^2 f(t)\,dt \tag{1.1}$$

and applying a variety of inequalities such as: Hölder's inequality, pre-Grüss, pre-Chebychev, pre-Lupaş, or Ostrowski type inequalities, a number of results concerning the expectation and variance of the random variable X were obtained.

For example,

$$\begin{aligned}&\sigma^2(X)+[x-E(X)]^2 \\ &\leq \begin{cases}(b-a)\left[\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right]\|f\|_\infty, & \text{if } f\in L_\infty[a,b]; \\ \left[\frac{(b-x)^{2q+1}+(x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{q}}\|f\|_p, & \text{if } f\in L_p[a,b], \\ & p>1,\frac{1}{p}+\frac{1}{q}=1; \\ \left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^2, & \end{cases}\end{aligned} \tag{1.2}$$

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for all $x \in [a, b]$, which imply, amongst other things, that

$$
\begin{aligned}
0 &\leq \sigma(X) \\
&\leq \begin{cases} (b-a)^{\frac{1}{2}} \left[\frac{(b-a)^2}{12} + \left[E(X) - \frac{a+b}{2}\right]^2\right]^{\frac{1}{2}} \|f\|_\infty^{\frac{1}{2}}, & \text{if } f \in L_\infty[a,b]; \\ \left\{\frac{[b-E(X)]^{2q+1} + [E(X)-a]^{2q+1}}{2q+1}\right\}^{\frac{1}{2q}} \|f\|_p^{\frac{1}{2}}, & \text{if } f \in L_p[a,b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{b-a}{2} + \left|E(X) - \frac{a+b}{2}\right|, \end{cases}
\end{aligned} \tag{1.3}
$$

and

$$0 \leq \sigma^2(X) \leq [b - E(X)][E(X) - a] \leq \frac{1}{4}(b-a)^2. \tag{1.4}$$

In this paper more accurate inequalities are obtained by assuming that the p.d.f. of X is n-time differentiable and that $f^{(n)}$ is absolutely continuous on $[a, b]$. For other recent results on the application of Ostrowski type inequalities in Probability Theory, see [2]-[4].

2. Some Preliminary Integral Identities

The following lemma, which is interesting in itself, holds.

Lemma 1. *Let X be a random variable whose probability distribution function $f : [a, b] \to \mathbb{R}_+$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a, b]$. Then*

$$
\begin{aligned}
&\sigma^2(X) + [E(X) - x]^2 \\
&= \sum_{k=0}^{n} \frac{(b-x)^{k+3} + (-1)^k (x-a)^{k+3}}{(k+3)\,k!} f^{(k)}(x) \\
&\quad + \frac{1}{n!} \int_a^b (t-x)^2 \left(\int_x^t (t-s)^n f^{(n+1)}(s)\,ds\right) dt
\end{aligned} \tag{2.1}
$$

for all $x \in [a, b]$.

Proof. Is by Taylor's formula with integral remainder. Recall that

$$f(t) = \sum_{k=0}^{n} \frac{(t-x)^k}{k!} f^{(k)}(x) + \frac{1}{n!} \int_x^t (t-s)^n f^{(n+1)}(s)\,ds \tag{2.2}$$

for all $t, x \in [a, b]$.

Together with

$$\sigma^2(X) + [E(X) - x]^2 = \int_a^b (t-x)^2 f(t)\,dt, \tag{2.3}$$

where f is the p.d.f. of the random variable X, we obtain

$$
\begin{aligned}
&\sigma^2(X) + [E(X) - x]^2 \\
&= \int_a^b (t-x)^2 \left[\sum_{k=0}^{n} \frac{(t-x)^k}{k!} f^{(k)}(x) + \frac{1}{n!} \int_x^t (t-s)^n f^{(n+1)}(s)\,ds\right] dt \\
&= \sum_{k=0}^{n} f^{(k)}(x) \int_a^b \frac{(t-x)^{k+2}}{k!} dt + \frac{1}{n!} \int_a^b (t-x)^2 \left(\int_x^t (t-s)^n f^{(n+1)}(s)\,ds\right) dt
\end{aligned} \tag{2.4}
$$

and since

$$\int_a^b \frac{(t-x)^{k+2}}{k!}dt = \frac{(b-x)^{k+3} + (-1)^k (x-a)^{k+3}}{(k+3)\,k!},$$

the identity (2.4) readily produces (2.1) ∎

Corollary 1. *Under the above assumptions, we have*

$$\begin{aligned}(2.5)\qquad & \sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 \\ = & \sum_{k=0}^{n} \frac{\left[1+(-1)^k\right](b-a)^{k+3}}{2^{k+3}(k+3)\,k!} f^{(k)}\left(\frac{a+b}{2}\right) \\ & + \frac{1}{n!}\int_a^b \left(t-\frac{a+b}{2}\right)^2 \left(\int_{\frac{a+b}{2}}^t (t-s)^n f^{(n+1)}(s)\,ds\right)dt.\end{aligned}$$

The proof follows by using (2.4) with $x = \frac{a+b}{2}$.

Corollary 2. *Under the above assumptions,*

$$\begin{aligned}(2.6)\qquad & \sigma^2(X) + \frac{1}{2}\left[(E(X)-a)^2 + (E(X)-b)^2\right] \\ = & \sum_{k=0}^{n} \frac{(b-a)^{k+3}}{(k+3)\,k!}\left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2}\right] \\ & + \frac{1}{n!}\int_a^b \int_a^b K(t,s)(t-s)^n f^{(n+1)}(s)\,dsdt,\end{aligned}$$

where

$$K(t,s) := \begin{cases} \frac{(t-a)^2}{2} & \text{if} \quad a \le s \le t \le b, \\ -\frac{(t-b)^2}{2} & \text{if} \quad a \le t < s \le b. \end{cases}$$

Proof. In (2.1), choose $x = a$ and $x = b$, giving

$$\begin{aligned}(2.7)\qquad & \sigma^2(X) + [E(X) - a]^2 \\ = & \sum_{k=0}^{n} \frac{(b-a)^{k+3}}{(k+3)\,k!} f^{(k)}(a) + \frac{1}{n!}\int_a^b (t-a)^2 \left(\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right)dt\end{aligned}$$

and

$$\begin{aligned}(2.8)\qquad & \sigma^2(X) + [E(X) - b]^2 \\ = & \sum_{k=0}^{n} \frac{(-1)^k (b-a)^{k+3}}{(k+3)\,k!} f^{(k)}(b) + \frac{1}{n!}\int_a^b (t-b)^2 \left(\int_b^t (t-s)^n f^{(n+1)}(s)\,ds\right)dt.\end{aligned}$$

Adding these and dividing by 2 gives (2.6). ∎

Taking into account that $\mu = E(X) \in [a,b]$, then we also obtain the following.

Corollary 3. *With the above assumptions,*

$$\sigma^2(X) = \sum_{k=0}^{n} \frac{(b-\mu)^{k+3} + (-1)^k (\mu-a)^{k+3}}{(k+3)\,k!} f^{(k)}(\mu) + \frac{1}{n!}\int_a^b (t-\mu)^2 \left(\int_\mu^t (t-s)^n f^{(n+1)}(s)\,ds\right) dt. \tag{2.9}$$

Proof. The proof follows from (2.1) with $x = \mu \in [a,b]$. ∎

Lemma 2. *Let the conditions of Lemma 1 relating to f hold. Then the following identity is valid.*

$$\begin{aligned} &\sigma^2(X) + [E(X)-x]^2 \\ = &\sum_{k=0}^{n} \frac{(b-x)^{k+3} + (-1)^k (x-a)^{k+3}}{k+3} \cdot \frac{f^{(k)}(x)}{k!} \\ &+ \frac{1}{n!}\int_a^b K_n(x,s) f^{(n+1)}(s)\,ds, \end{aligned} \tag{2.10}$$

where

$$K(x,s) = \begin{cases} (-1)^{n+1}\psi_n(s-a, x-s), & a \le s \le x \\ \psi_n(b-s, s-x), & x < s \le b \end{cases} \tag{2.11}$$

with

$$\psi_n(u,v) = \frac{u^{n+1}}{(n+3)(n+2)(n+1)} \cdot \left[(n+2)(n+1)u^2 + 2(n+3)(n+1)uv + (n+3)(n+2)v^2\right]. \tag{2.12}$$

Proof. From (2.1), an interchange of the order of integration gives

$$\begin{aligned} &\frac{1}{n!}\int_a^b (t-x)^2\,dt \int_x^t (t-s)^n f^{(n+1)}(s)\,ds \\ = &\frac{1}{n!}\left\{-\int_a^x \int_a^s (t-x)^2 (t-s)^n f^{(n+1)}(s)\,dtds \right. \\ &\left. + \int_x^b \int_s^b (t-x)^2 (t-s)^n f^{(n+1)}(s)\,dtds\right\} \\ = &\frac{1}{n!}\int_a^b \tilde{K}_n(x,s) f^{(n+1)}(s)\,ds, \end{aligned}$$

where

$$\tilde{K}_n(x,s) = \begin{cases} p_n(x,s) = -\int_a^s (t-x)^2 (t-s)^n\,dt, & a \le s \le x \\ q_n(x,s) = \int_s^b (t-x)^2 (t-s)^n\,dt, & x < s < b. \end{cases}$$

To prove the lemma it is sufficient to show that $K \equiv \tilde{K}$.

Now,

$$\begin{aligned}\tilde{p}_n(x,s) &= -\int_a^s (t-x)^2 (t-s)^n\,dt \\ &= (-1)^{n+1}\int_0^{s-a} (u+x-s)^2 u^n du \\ &= (-1)^{n+1}\int_0^{s-a} \left[u^2+2(x-s)u+(x-s)^2\right] u^n du \\ &= (-1)^{n+1}\psi_n(s-a,x-s),\end{aligned}$$

where $\psi(\cdot,\cdot)$ is as given by (2.12).

Further,

$$\begin{aligned}\tilde{q}_n(x,s) &= \int_s^b (t-x)^2 (t-s)^n\,dt \\ &= \int_0^{b-s} [u+(s-x)]^2 u^n du \\ &= \psi_n(b-s,s-x),\end{aligned}$$

where, again, $\psi(\cdot,\cdot)$ is as given by (2.12). Hence $K\equiv\tilde{K}$ and the lemma is proved. ∎

3. Some Inequalities

We are now able to obtain the following inequalities.

Theorem 1. *Let X be a random variable whose probability density function $f:[a,b]\to\mathbb{R}_+$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$, then*

$$\left|\sigma^2(X)+[E(X)-x]^2-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{(k+3)\,k!}f^{(k)}(x)\right| \tag{3.1}$$

$$\leq\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+4)}\left[(x-a)^{n+4}+(b-x)^{n+4}\right], & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{\left\|f^{(n+1)}\right\|_p}{n!\left(n+3+\frac{1}{q}\right)}\frac{\left[(x-a)^{n+3+\frac{1}{q}}+(b-x)^{n+3+\frac{1}{q}}\right]}{(nq+1)^{\frac{1}{q}}}, & \text{if } f^{(n+1)}\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\left\|f^{(n+1)}\right\|_1}{n!(n+3)}\left[(x-a)^{n+3}+(b-x)^{n+3}\right], & \text{if } f^{(n+1)}\in L_1[a,b]\end{cases}$$

for all $x\in[a,b]$, where $\|\cdot\|_p$ $(1\leq p\leq\infty)$ are the usual Lebesque norms on $[a,b]$, i.e.,

$$\|g\|_\infty := ess\sup_{t\in[a,b]}|g(t)| \quad \textit{and} \quad \|g\|_p := \left(\int_a^b |g(t)|^p\,dt\right)^{\frac{1}{p}},\ \ p\geq 1.$$

Proof. By Lemma 1,

$$\sigma^2(X)+[E(X)-x]^2-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{k!\,(k+3)}f^{(k)}(x) \tag{3.2}$$

$$= \frac{1}{n!}\int_a^b (t-x)^2\left(\int_x^t (t-s)^n f^{(n+1)}(s)\,ds\right)dt := M(a,b;x).$$

Clearly,

$$\begin{aligned}
|M(a,b;x)| &\leq \frac{1}{n!}\int_a^b (t-x)^2 \left|\int_x^t (t-s)^n f^{(n+1)}(s)\,ds\right| dt \\
&\leq \frac{1}{n!}\int_a^b (t-x)^2 \left[\sup_{s\in[x,t]}\left|f^{(n+1)}(s)\right| \left|\int_x^t |t-s|^n\,ds\right|\right] dt \\
&\leq \frac{\left\|f^{(n+1)}\right\|_\infty}{n!}\int_a^b \frac{(t-x)^2|t-x|^{n+1}}{n+1}dt \\
&= \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!}\int_a^b |t-x|^{n+3}\,dt \\
&= \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!}\left[\int_a^x (x-t)^{n+3}\,dt + \int_x^b (t-x)^{n+3}\,dt\right] \\
&= \frac{\left\|f^{(n+1)}\right\|_\infty\left[(x-a)^{n+4}+(b-x)^{n+4}\right]}{(n+1)!\,(n+4)}
\end{aligned}$$

and the first inequality in (3.1) is obtained.

For the second, we use Hölder's integral inequality to obtain

$$\begin{aligned}
&|M(a,b;x)| \\
&\leq \frac{1}{n!}\int_a^b (t-x)^2\left|\int_x^t |t-s|^{nq}\,ds\right|^{\frac{1}{q}}\left|\int_x^t \left|f^{(n+1)}(s)\right|^p ds\right|^{\frac{1}{p}} dt \\
&\leq \frac{1}{n!}\left(\int_a^b \left|f^{(n+1)}(s)\right|^p ds\right)^{\frac{1}{p}}\int_a^b (t-x)^2|t-x|^{\frac{nq+1}{q}}\,dt \\
&= \frac{1}{n!}\frac{\left\|f^{(n+1)}\right\|_p}{(nq+1)^{\frac{1}{q}}}\int_a^b |t-x|^{n+2+\frac{1}{q}}\,dt \\
&= \frac{1}{n!}\frac{\left\|f^{(n+1)}\right\|_p}{(nq+1)^{\frac{1}{q}}}\left[\frac{(b-x)^{n+3+\frac{1}{q}}+(x-a)^{n+3+\frac{1}{q}}}{n+3+\frac{1}{q}}\right].
\end{aligned}$$

Finally, note that

$$\begin{aligned}
|M(a,b;x)| &\leq \frac{1}{n!}\int_a^b (t-x)^2|t-x|^n\left|\int_x^t \left|f^{(n+1)}(s)\right| ds\right| dt \\
&\leq \frac{\left\|f^{(n+1)}\right\|_1}{n!}\int_a^b |t-x|^{n+2}\,dt \\
&= \frac{\left\|f^{(n+1)}\right\|_1}{n!}\left[\frac{(x-a)^{n+3}+(b-x)^{n+3}}{n+3}\right]
\end{aligned}$$

and the third part of (3.1) is obtained. ∎

It is obvious that the best inequality in (3.1) is when $x = \frac{a+b}{2}$, giving Corollary 4.

Corollary 4. *With the above assumptions on X and f,*

$$\left|\sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2-\sum_{k=0}^{n}\frac{\left[1+(-1)^k\right](b-a)^{k+3}}{2^{k+3}(k+3)k!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{3.3}$$

$$\leq\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{2^{n+3}(n+1)!(n+4)}(b-a)^{n+4}, & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{\left\|f^{(n+1)}\right\|_p}{2^{n+2+\frac{1}{q}}n!\left(n+3+\frac{1}{q}\right)}\frac{(b-a)^{n+3+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}}, & \text{if } f^{(n+1)}\in L_p[a,b],\ p>1,\\ & \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\left\|f^{(n+1)}\right\|_1}{2^{n+2}n!(n+3)}(b-a)^{n+3}, & \text{if } f^{(n+1)}\in L_1[a,b].\end{cases}$$

The following corollary is interesting as it provides the opportunity to approximate the variance when the values of $f^{(k)}(\mu)$ are known, $k=0,...,n$.

Corollary 5. *With the above assumptions and $\mu=\frac{a+b}{2}$, we have*

$$\left|\sigma^2(X)-\sum_{k=0}^{n}\frac{(b-\mu)^{k+3}+(-1)^k(\mu-a)^{k+3}}{(k+3)k!}f^{(k)}(\mu)\right| \tag{3.4}$$

$$\leq\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+4)}\left[(\mu-a)^{n+4}+(b-\mu)^{n+4}\right], & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{\left\|f^{(n+1)}\right\|_p}{n!\left(n+3+\frac{1}{q}\right)}\frac{\left[(\mu-a)^{n+3+\frac{1}{q}}+(b-\mu)^{n+3+\frac{1}{q}}\right]}{(nq+1)^{\frac{1}{q}}}, & \text{if } f^{(n+1)}\in L_p[a,b],\\ & p>1,\frac{1}{p}+\frac{1}{q}=1;\\ \frac{\left\|f^{(n+1)}\right\|_1}{n!(n+3)}\left[(\mu-a)^{n+3}+(b-\mu)^{n+3}\right], & \text{if } f^{(n+1)}\in L_1[a,b].\end{cases}$$

The following result also holds.

Theorem 2. *Let X be a random variable whose probability density function $f:[a,b]\to\mathbb{R}_+$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$, then*

$$\left|\sigma^2(X)+\frac{1}{2}\left[(E(X)-a)^2+(E(X)-b)^2\right]\right.$$
$$\left.-\sum_{k=0}^{n}\frac{(b-a)^{k+3}}{(k+3)k!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \tag{3.5}$$

$$\leq\begin{cases}\frac{1}{(n+4)(n+1)!}\left\|f^{(n+1)}\right\|_\infty(b-a)^{n+4},\\ \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{1}{n!(qn+1)^{\frac{1}{q}}[(n+2)q+2]^{\frac{1}{q}}}\left\|f^{(n+1)}\right\|_p\frac{(b-a)^{n+3+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}},\\ \text{if } f^{(n+1)}\in L_p[a,b],\ \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2n!}\left\|f^{(n+1)}\right\|_1(b-a)^{n+3},\end{cases}$$

where $\|\cdot\|_p$ $(1\leq p\leq\infty)$ are the usual Lebesgue $p-$norms.

Proof. Using Corollary 2,

$$\begin{aligned}
&\left|\sigma^2(X)+\frac{1}{2}\left[(E(X)-a)^2+(E(X)-b)^2\right]\right.\\
&\left.-\sum_{k=0}^{n}\frac{(b-a)^{k+3}}{(k+3)\,k!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right|\\
\leq\ &\frac{1}{n!}\int_a^b\int_a^b |K(t,s)|\,|t-s|^n\left|f^{(n+1)}(s)\right| dsdt =: N(a,b).
\end{aligned}$$

It is obvious that

$$\begin{aligned}
&N(a,b)\\
\leq\ &\left\|f^{(n+1)}\right\|_\infty \frac{1}{n!}\int_a^b\int_a^b |K(t,s)|\,|t-s|^n\, dsdt\\
=\ &\left\|f^{(n+1)}\right\|_\infty \frac{1}{n!}\int_a^b\left(\int_a^t |K(t,s)|\,|t-s|^n\, ds+\int_t^b |K(t,s)|\,|t-s|^n\, ds\right)dt\\
=\ &\frac{1}{n!}\left\|f^{(n+1)}\right\|_\infty\int_a^b\left[\frac{(t-a)^2}{2}\cdot\frac{(t-a)^{n+1}}{n+1}+\frac{(t-b)^2}{2}\cdot\frac{(b-t)^{n+1}}{n+1}\right]dt\\
=\ &\frac{1}{2(n+1)!}\left\|f^{(n+1)}\right\|_\infty\int_a^b\left[(t-a)^{n+3}+(b-t)^{n+3}\right]dt\\
=\ &\frac{1}{2(n+1)!}\left\|f^{(n+1)}\right\|_\infty\left[\frac{(b-a)^{n+4}}{n+4}+\frac{(b-a)^{n+4}}{n+4}\right]\\
=\ &\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+4)(n+1)!}(b-a)^{n+4}
\end{aligned}$$

so the first part of (3.5) is proved.

Using Hölder's integral inequality for double integrals,

$$\begin{aligned}
&N(a,b)\\
\leq\ &\frac{1}{n!}\left(\int_a^b\int_a^b\left|f^{(n+1)}(s)\right|^p dsdt\right)^{\frac{1}{p}}\times\left(\int_a^b\int_a^b |K(t,s)|^q\,|t-s|^{qn}\, dsdt\right)^{\frac{1}{q}}\\
=\ &\frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\times\\
&\left[\int_a^b\left(\int_a^t |K(t,s)|^q\,|t-s|^{qn}\, ds+\int_t^b |K(t,s)|^q\,|t-s|^{qn}\, ds\right)dt\right]^{\frac{1}{q}}\\
=\ &\frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\\
&\times\left[\int_a^b\left[\frac{(t-a)^{2q}}{2^q}\int_a^t |t-s|^{qn}\, ds+\frac{(t-b)^{2q}}{2^q}\int_t^b |t-s|^{qn}\, ds\right]dt\right]^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(b-a)^{\frac{1}{p}} \left\| f^{(n+1)} \right\|_p}{n!} \\
&\quad \times \left[\int_a^b \left[\frac{(t-a)^{2q} (t-a)^{qn+1}}{2^q (qn+1)} + \frac{(t-b)^{2q} (b-t)^{qn+1}}{2^q (qn+1)} \right] dt \right]^{\frac{1}{q}} \\
&= \frac{(b-a)^{\frac{1}{p}} \left\| f^{(n+1)} \right\|_p}{n!} \cdot \left[\frac{1}{2^q (qn+1)} \right]^{\frac{1}{q}} \\
&\quad \times \left[\int_a^b (t-a)^{(n+2)q+1} \, dt + \int_a^b (b-t)^{(n+2)q+1} \, dt \right]^{\frac{1}{q}} \\
&= \frac{(b-a)^{\frac{1}{p}} \left\| f^{(n+1)} \right\|_p}{n!} \cdot \left[\frac{1}{2^q (qn+1)} \right]^{\frac{1}{q}} \left[\frac{(b-a)^{(n+2)q+2}}{(n+2)\, q+2} + \frac{(b-a)^{(n+2)q+2}}{(n+2)\, q+2} \right]^{\frac{1}{q}} \\
&= \frac{2 \left\| f^{(n+1)} \right\|_p (b-a)^{n+2+\frac{1}{p}+\frac{2}{q}}}{n! 2 (qn+1)^{\frac{1}{q}} ((n+2)\, q+2)^{\frac{1}{q}}} \\
&= \frac{\left\| f^{(n+1)} \right\|_p \left[(b-a)^{n+3+\frac{1}{q}} \right]}{n! \, (qn+1)^{\frac{1}{q}} \left[(n+2)\, q+2 \right]^{\frac{1}{q}}}
\end{aligned}
$$

and the second part of (3.5) is proved.

Finally, we observe that

$$
\begin{aligned}
N(a,b) &\leq \frac{1}{n!} \sup_{(t,s)\in[a,b]^2} |K(t,s)| \, |t-s|^n \int_a^b \int_a^b \left| f^{(n+1)}(s) \right| ds dt \\
&= \frac{1}{n!} \frac{(b-a)^2}{2} \cdot (b-a)^n (b-a) \int_a^b \left| f^{(n+1)}(s) \right| ds \\
&= \frac{1}{2n!} (b-a)^{n+3} \left\| f^{(n+1)} \right\|_1 ,
\end{aligned}
$$

which is the final result of (3.5). ∎

The following particular cases can be useful in practical applications.

1. For $n = 0$, (3.1) becomes

$$
\left| \sigma^2(X) + [E(X) - x]^2 - (b-a) \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} \right] f(x) \right| \tag{3.6}
$$

$$
\leq \begin{cases}
\frac{\|f'\|_\infty}{4} \left[(x-a)^4 + (b-x)^4 \right], & \text{if } f' \in L_\infty [a,b]; \\
\frac{q\|f'\|_p}{3q+1} \left[(x-a)^{3+\frac{1}{q}} + (b-x)^{3+\frac{1}{q}} \right], & \text{if } f' \in L_p [a,b], \\
& p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\
\|f'\|_1 \left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2} \right)^2 \right], & \text{if } f' \in L_1 [a,b],
\end{cases}
$$

for all $x \in [a,b]$.

In particular, for $x = \frac{a+b}{2}$,

$$\left|\sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 - \frac{(b-a)^3}{12} f\left(\frac{a+b}{2}\right)\right| \tag{3.7}$$
$$\leq \begin{cases} \frac{\|f'\|_\infty}{32}(b-a)^4, & \text{if } f' \in L_\infty[a,b]; \\ \frac{q\|f'\|_p (b-a)^{3+\frac{1}{q}}}{2^{2+\frac{1}{q}}(3q+1)}, & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{12}(b-a)^3, \end{cases}$$

which is, in a sense, the best inequality that can be obtained from (3.6). If in (3.6) $x = \mu = E(X)$, then

$$\left|\sigma^2(X) - (b-a)\left[\left(E(X) - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12}\right] f(E(X))\right| \tag{3.8}$$
$$\leq \begin{cases} \frac{\|f'\|_\infty}{4}\left[(E(X)-a)^4 + (b-E(X))^4\right], & \text{if } f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{\left(3+\frac{1}{q}\right)}\left[(E(X)-a)^4 + (b-E(X))^4\right], & \text{if } f' \in L_p[a,b],\ p>1, \\ \|f'\|_1\left[\frac{(b-a)^2}{12} + \left(E(X) - \frac{a+b}{2}\right)^2\right], & \text{if } f' \in L_1[a,b]. \end{cases}$$

In addition, from (3.5),

$$\left|\sigma^2(X) + \frac{1}{2}\left[(E(X)-a)^2 + (E(X)-b)^2\right] - \frac{(b-a)^3}{3}\left[\frac{f(a)+f(b)}{2}\right]\right| \tag{3.9}$$
$$\leq \begin{cases} \frac{1}{4}\|f'\|_\infty (b-a)^4, & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{n!2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\|f'\|_p (b-a)^{3+\frac{1}{q}}, & \text{if } f' \in L_p[a,b],\ p>1, \\ \frac{1}{2}\|f'\|_1 (b-a)^3, \end{cases}$$

which provides an approximation for the variance in terms of the expectation and the values of f at the end points a and b.

Theorem 3. *Let X be a random variable whose p.d.f. $f:[a,b]\to\mathbb{R}_+$ is $n-$time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$. Then*

$$\left|\sigma^2(X)+(E(X)-x)^2-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{k+3}\cdot\frac{f^{(k)}(x)}{k!}\right| \tag{3.10}$$

$$\leq\begin{cases}\left[(x-a)^{n+4}+(b-x)^{n+4}\right]\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+4)},\\[2ex] C^{\frac{1}{q}}\left[(x-a)^{(n+3)q+1}+(b-x)^{(n+3)q+1}\right]^{\frac{1}{q}}\frac{\left\|f^{(n+1)}\right\|_p}{n!},\\[2ex] \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^{n+3}\cdot\frac{\left\|f^{(n+1)}\right\|_1}{n!(n+3)},\end{cases}$$

where

$$C=\int_0^1\left[\frac{u^{n+3}}{n+3}+2(1-u)\frac{u^{n+2}}{n+2}+(1-u)^2\frac{u^{n+1}}{n+1}\right]^q du. \tag{3.11}$$

Proof. From (2.10),

$$\left|\sigma^2(X)+(E(X)-x)^2-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{k+3}\cdot\frac{f^{(k)}(x)}{k!}\right| \tag{3.12}$$

$$=\left|\frac{1}{n!}\int_a^b K_n(x,s)f^{(n+1)}(s)\,ds\right|.$$

Now, on using the fact that from (2.11), (2.12), $\psi_n(u,v)\geq 0$ for $u,v\geq 0$,

$$\left|\frac{1}{n!}\int_a^b K_n(x,s)f^{(n+1)}(s)\,ds\right| \tag{3.13}$$

$$\leq\frac{\left\|f^{(n+1)}\right\|_\infty}{n!}\left\{\int_a^x\psi_n(s-a,x-s)\,ds+\int_x^b\psi_n(b-s,s-x)\,ds\right\}.$$

Further,

$$\psi_n(u,v)=\frac{u^{n+3}}{n+3}+2v\frac{u^{n+2}}{n+2}+v^2\frac{u^{n+1}}{n+1} \tag{3.14}$$

and so

$$\int_a^x\psi_n(s-a,x-s)\,ds \tag{3.15}$$

$$=\int_a^x\left[\frac{(s-a)^{n+3}}{n+3}+2(x-s)\frac{(s-a)^{n+2}}{n+2}+(x-s)^2\frac{(s-a)^{n+1}}{n+1}\right]ds$$

$$=(x-a)^{n+4}\int_0^1\left[\frac{\lambda^{n+3}}{n+3}+2(1-\lambda)\frac{\lambda^{n+2}}{n+2}+(1-\lambda)^2\frac{\lambda^{n+1}}{n+1}\right]d\lambda,$$

where we have made the substitution $\lambda=\frac{s-a}{x-a}$.

Collecting powers of λ gives

$$\lambda^{n+3}\left[\frac{1}{n+3}-\frac{2}{n+2}+\frac{1}{n+1}\right]-\frac{2\lambda^{n+2}}{(n+2)(n+1)}+\frac{\lambda^{n+1}}{n+1}$$

and so, from (3.15),

$$\begin{aligned}
&\int_a^x \psi_n(s-a,x-s)\,ds \\
&= (x-a)^{n+4}\left\{\frac{1}{n+4}\left[\frac{1}{n+3}-\frac{2}{n+2}+\frac{1}{n+1}\right]\right. \\
&\quad \left. -\frac{2}{(n+3)(n+2)(n+1)}+\frac{1}{(n+2)(n+1)}\right\} \\
&= \frac{(x-a)^{n+4}}{(n+4)(n+1)}.
\end{aligned} \tag{3.16}$$

Similarly, on using (3.14),

$$\begin{aligned}
&\int_x^b \psi_n(b-s,s-x)\,ds \\
&= \int_x^b \left[\frac{(b-s)^{n+3}}{n+3}+2(s-x)\frac{(b-s)^{n+2}}{n+2}+(s-x)^2\frac{(b-s)^{n+1}}{n+1}\right]ds
\end{aligned}$$

and making the substitution $\nu=\frac{b-s}{b-x}$ gives

$$\begin{aligned}
&\int_x^b \psi_n(b-s,s-x)\,ds \\
&= (b-x)^{n+4}\int_0^1\left[\frac{\nu^{n+3}}{n+3}+2(1-\nu)\frac{\nu^{n+2}}{n+2}+(1-\nu)^2\frac{\nu^{n+1}}{n+1}\right]d\nu \\
&= \frac{(b-x)^{n+4}}{(n+4)(n+1)},
\end{aligned} \tag{3.17}$$

where we have used (3.15) and (3.16). Combining (3.16) and (3.17) gives the first inequality in (3.10).

For the second inequality in (3.10), we use Hölder's integral inequality to obtain

$$\left|\frac{1}{n!}\int_a^b K_n(x,s)\,f^{(n+1)}(s)\,ds\right| \le \frac{\left\|f^{(n+1)}(s)\right\|_p}{n!}\left(\int_a^b |K_n(x,s)|^q\,ds\right)^{\frac{1}{q}}. \tag{3.18}$$

Now, from (2.11) and (3.14)

$$\begin{aligned}
\int_a^b |K_n(x,s)|^q\,ds &= \int_a^x \psi^q(s-a,x-s)\,ds+\int_x^b \psi^q(b-s,s-x)\,ds \\
&= C\left[(x-a)^{(n+3)q+1}+(b-x)^{(n+3)q+1}\right],
\end{aligned}$$

where C is as defined in (3.11) and we have used (3.15) and (3.16). Substitution into (3.18) gives the second inequality in (3.10).

Finally, for the third inequality in (3.10). From (3.12),

$$
\begin{aligned}
(3.19)\quad & \left|\frac{1}{n!}\int_a^b K_n(x,s)\, f^{(n+1)}(s)\,ds\right| \\
&\leq \frac{1}{n!}\left\{\int_a^x \psi_n(s-a,x-s)\left|f^{(n+1)}(s)\right|ds \right.\\
&\quad \left. +\int_x^b \psi_n(b-s,s-x)\left|f^{(n+1)}(s)\right|ds\right\} \\
&\leq \frac{1}{n!}\left\{\psi_n(x-a,0)\int_a^x \left|f^{(n+1)}(s)\right|ds+\psi_n(b-x,0)\int_x^b \left|f^{(n+1)}(s)\right|ds\right\},
\end{aligned}
$$

where, from (3.14),

$$
\psi_n(u,0)=\frac{u^{n+3}}{n+3}. \tag{3.20}
$$

Hence, from (3.19) and (3.20)

$$
\begin{aligned}
& \left|\frac{1}{n!}\int_a^b K_n(x,s)\, f^{(n+1)}(s)\,ds\right| \\
&\leq \frac{1}{n!}\max\left\{\frac{(x-a)^{n+3}}{n+3},\frac{(b-x)^{n+3}}{n+3}\right\}\left\|f^{(n+1)}(\cdot)\right\|_1 \\
&= \frac{1}{n!\,(n+3)}\left[\max\{x-a,b-x\}\right]^{n+3}\left\|f^{(n+1)}(\cdot)\right\|_1,
\end{aligned}
$$

which, on using the fact that for X, $Y\in\mathbb{R}$

$$
\max\{X,Y\}=\frac{X+Y}{2}+\left|\frac{X-Y}{2}\right|
$$

gives, from (3.12), the third inequality in (3.10). The theorem is now completely proved. ∎

Remark 1. *The results of Theorem 3 may be compared with those of Theorem 1. Theorem 3 is based on the single integral identity developed in Lemma 2, while Theorem 1 is based on the double integral identity representation for the bound. It may be noticed from (3.1) and (3.10) that the bounds are the same for $f^{(n+1)}\in L_\infty[a,b]$, that for $f^{(n+1)}\in L_1[a,b]$ the bound obtained in (3.1) is better and for $f^{(n+1)}\in L_p[a,b]$, $p>1$, the result is inconclusive.*

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