Some Inequalities for Random Variables whose Probability Density Functions are Absolutely Continuous Using a Pre-Chebychev Inequality

SOME INEQUALITIES FOR RANDOM VARIABLES WHOSE PROBABILITY DENSITY FUNCTIONS ARE ABSOLUTELY CONTINUOUS USING A PRE-CHEBYCHEV INEQUALITY

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Abstract. Using the pre-Chebychev inequality considered by Matić, Pečarić and Ujević in [2], some inequalities are obtained for random variables whose p.d.f.s are absolutely continuous and whose derivatives are in $L_\infty[a,b]$.

1. Introduction

The following inequality is well known in the literature as Chebychev's inequality (see for example [1, p. 297]).

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}$ be two absolutely continuous mappings on $[a,b]$ whose derivatives $f',g':[a,b]\to\mathbb{R}$ belong to the Lebesgue space $L_\infty[a,b]$. Then,*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty. \tag{1.1}$$

The constant $\frac{1}{12}$ is the best possible.

In [2], Matić, Pečarić and Ujević proved the following refinement of (1.1) which we call the "pre-Chebychev" inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \frac{1}{2\sqrt{3}}(b-a)\|f'\|_\infty\left[\frac{1}{b-a}\int_a^b g^2(x)\,dx-\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^2\right]^{\frac{1}{2}}, \tag{1.2}$$

provided that f is as in Theorem 1 and all the integrals in (1.2) exist and are finite.

Matić, Pečarić and Ujević observed that: if a factor is known, say $g(t)$, $t\in[a,b]$, then instead of using (1.1) to estimate the difference

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt,$$

it is better to use (1.2). They demonstrated this by improving some results of the second author in [6] related to Taylor's formula with integral remainder.

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Using the same approach here, we obtain some inequalities for the expectation, $E(X)$, and cumulative distribution function $F(x)$ of a random variable having the probability density function $f:[a,b]\to\mathbb{R}$ which is assumed to be absolutely continuous and whose derivative $f'\in L_\infty[a,b]$.

2. Some Inequalities

We start with the following result for expectation.

Theorem 2. *Let X be a random variable having the probability density function $f:[a,b]\to\mathbb{R}$. Assume that f is absolutely continuous on $[a,b]$ and $f'\in L_\infty[a,b]$. Then,*

$$\left|E(X)-\frac{a+b}{2}\right|\leq\frac{1}{12}(b-a)^2\|f'\|_\infty, \tag{2.1}$$

where $E(X)$ is the expectation of the random variable X.

Proof. If we put $g(t)=t$ in (1.2),

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b tf(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b tdt\right| \\ &\leq\frac{1}{2\sqrt{3}}(b-a)\|f'\|_\infty\left[\frac{1}{b-a}\int_a^b t^2dt-\left(\frac{1}{b-a}\int_a^b tdt\right)^2\right]^{\frac{1}{2}}.\end{aligned} \tag{2.2}$$

However,

$$\frac{1}{b-a}\int_a^b t^2dt-\left(\frac{1}{b-a}\int_a^b tdt\right)^2=\frac{(b-a)^2}{12}$$

and so (2.1) is true. ∎

Remark 1. *We could obtain the same inequality by applying Chebychev's inequality (1.1). Note, however, that for further results, the pre-Chebychev inequality provides a better estimate than would be obtained using the classical result (1.1).*

Theorem 3. *Let X and f be as above. If*

$$\sigma_\mu(X):=\left[\int_a^b(t-\mu)^2f(t)\,dt\right]^{\frac{1}{2}},\ \ \mu\in[a,b],$$

then,

$$\begin{aligned}&\left|\sigma_\mu^2(X)-\left(\mu-\frac{a+b}{2}\right)^2-\frac{1}{12}(b-a)^2\right| \\ &\leq\frac{1}{2\sqrt{3}}(b-a)^2\left[\frac{1}{3}\left(\mu-\frac{a+b}{2}\right)^2+\frac{1}{180}(b-a)^2\right]\|f'\|_\infty \\ &\leq\frac{1}{3\sqrt{15}}(b-a)^3\|f'\|_\infty,\end{aligned} \tag{2.3}$$

for all $\mu\in[a,b]$.

Proof. If $g(t) = (t-\mu)^2$ in (1.2), then,

$$\left| \frac{1}{b-a} \int_a^b (t-\mu)^2 f(t)\,dt - \frac{1}{b-a} \int_a^b f(t)\,dt \cdot \frac{1}{b-a} \int_a^b (t-\mu)^2\,dt \right| \tag{2.4}$$

$$\leq \frac{1}{2\sqrt{3}} \|f'\|_\infty \left[\frac{1}{b-a} \int_a^b (t-\mu)^4\,dt - \left(\frac{1}{b-a} \int_a^b (t-\mu)^2\,dt \right)^2 \right]^{\frac{1}{2}}.$$

However,

$$\frac{1}{b-a} \int_a^b (t-\mu)^2\,dt = \left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{12}(b-a)^2$$

and

$$\begin{aligned} & \frac{1}{b-a} \int_a^b (t-\mu)^4\,dt - \left(\frac{1}{b-a} \int_a^b (t-\mu)^2\,dt \right)^2 \\ = \; & \frac{1}{5} \cdot \frac{(b-\mu)^5 + (\mu-a)^5}{b-a} - \left[\frac{(b-\mu)^3 + (\mu-a)^3}{3(b-a)} \right]^2 \\ = \; & \frac{1}{45} \left[4 \left[(b-\mu)^2 - (\mu-a)^2 \right]^2 + 2(b-\mu)^2 (\mu-a)^2 \right. \\ & \left. + (\mu-a)(b-\mu) \left[(b-\mu)^2 + (\mu-a)^2 \right] \right] := A, \end{aligned}$$

which simplifies further to give:-

$$\begin{aligned} A &= \frac{(b-a)^2}{45} \left[15 \left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2 \right] \\ &= (b-a)^2 \left[\frac{1}{3} \left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{180}(b-a)^2 \right]. \end{aligned}$$

Using (2.4), we deduce the desired inequality (2.3). ∎

The best inequality we can obtain from (2.3) is that for which $\mu = \frac{a+b}{2}$, giving the following corollary.

Corollary 1. *With the above assumptions and denoting* $\sigma_0(X) := \sigma_{\frac{a+b}{2}}(X)$,

$$\left| \sigma_0^2(X) - \frac{(b-a)^2}{12} \right| \leq \frac{1}{12\sqrt{15}} (b-a)^3 \|f'\|_\infty . \tag{2.5}$$

The following theorem provides an inequality that connects the expectation $E(X)$ and the cumulative distribution function $F(x) := \int_a^x f(t)\,dt$ of a random variable X having the p.d.f. $f : [a,b] \to \mathbb{R}$.

Theorem 4. *Let X be a random variable whose p.d.f., $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $f' \in L_\infty[a,b]$. Then,*

$$\left| E(X) + (b-a) F(x) - x - \frac{b-a}{2} \right| \leq \frac{1}{12} (b-a)^3 \|f'\|_\infty \tag{2.6}$$

for all $x \in [a,b]$.

Proof. We use the following equality established by Barnett and Dragomir in [4]

$$(b-a)F(x)+E(X)-b=\int_a^b p(x,t)\,dF(t)=\int_a^b p(x,t)\,f(t)\,dt, \tag{2.7}$$

where

$$p(x,t):=\begin{cases} t-a & \text{if} \quad a\le t\le x\le b \\ t-b & \text{if} \quad a\le x<t\le b \end{cases}.$$

Now, if we apply the inequality (1.2) for $g(t)=p(x,t)$, we obtain

$$\left|\frac{1}{b-a}\int_a^b p(x,t)\,f(t)\,dt-\frac{1}{b-a}\int_a^b p(x,t)\,dt\cdot\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.8}$$
$$\le \frac{1}{2\sqrt{3}}(b-a)\|f'\|_\infty\left[\frac{1}{b-a}\int_a^b p^2(x,t)\,dt-\left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2\right]^{\frac{1}{2}}.$$

Observe that

$$\frac{1}{b-a}\int_a^b p(x,t)\,dt=x-\frac{a+b}{2},$$

and

$$\begin{aligned} D \quad : &= \frac{1}{b-a}\int_a^b p^2(x,t)\,dt-\left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2 \\ &= \frac{1}{b-a}\left[\frac{(b-x)^3+(x-a)^3}{3}\right]-\left(x-\frac{a+b}{2}\right)^2 \\ &= \frac{1}{12}(b-a)^2. \end{aligned}$$

Using (2.8), we deduce (2.6). ∎

Remark 2. *If in (2.6) either $x=a$ or $x=b$,*

$$\left|E(X)-\frac{a+b}{2}\right|\le\frac{1}{12}(b-a)^3\|f'\|_\infty,$$

which is inequality (2.1).

Remark 3. *If in (2.6) $x=\frac{a+b}{2}$, then*

$$\left|E(X)+(b-a)\Pr\left(X\le\frac{a+b}{2}\right)-b\right|\le\frac{1}{12}(b-a)^3\|f'\|_\infty. \tag{2.9}$$

Theorem 5. *Let X, F and f be as above. Then,*

$$\left|E(X)+\frac{b-a}{2}F(x)-\frac{x+b}{2}\right| \tag{2.10}$$
$$\le \frac{1}{4}(b-a)\|f'\|_\infty\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{12}(b-a)^2\right]$$
$$\le \frac{1}{12}(b-a)^3\|f'\|_\infty$$

for all $x\in[a,b]$.

Proof. Using the same identity of Barnett and Dragomir [4] as in Theorem 4 and applying the pre-Chebychev inequality (1.2), for $x \in [a,b]$ we get:-

$$\begin{aligned}
(2.11)\quad & \left| \frac{1}{x-a} \int_a^x (t-a) f(t)\, dt - \frac{1}{x-a} \int_a^x (t-a)\, dt \cdot \frac{1}{x-a} \int_a^x f(t)\, dt \right| \\
\leq\ & \frac{1}{2\sqrt{3}} (x-a) \|f'\|_\infty \left[\frac{1}{x-a} \int_a^x (t-a)^2 dt - \left(\frac{1}{x-a} \int_a^x (t-a)\, dt \right)^2 \right]^{\frac{1}{2}} \\
=\ & \frac{1}{12} (x-a)^2 \|f'\|_\infty
\end{aligned}$$

and, similarly,

$$\begin{aligned}
(2.12)\quad & \left| \frac{1}{b-x} \int_x^b (t-b) f(t)\, dt - \frac{1}{b-x} \int_x^b (t-b)\, dt \cdot \frac{1}{b-x} \int_x^b f(t)\, dt \right| \\
\leq\ & \frac{1}{12} (b-x)^2 \|f'\|_\infty ,
\end{aligned}$$

for all $x \in [a,b)$.

From (2.11) and (2.12) we can write

$$(2.13)\quad \left| \int_a^x (t-a) f(t)\, dt - \frac{x-a}{2} F(x) \right| \leq \frac{1}{12} (x-a)^3 \|f'\|_\infty$$

and

$$(2.14)\quad \left| \int_x^b (t-b) f(t)\, dt + \frac{b-x}{2} (1 - F(x)) \right| \leq \frac{1}{12} (b-x)^3 \|f'\|_\infty ,$$

for all $x \in [a,b]$.

Summing (2.13) and (2.14) and using the triangle inequality, we deduce

$$\begin{aligned}
& \left| \int_a^x (t-a) f(t)\, dt + \int_x^b (t-b) f(t)\, dt - \frac{b-a}{2} F(x) + \frac{b-x}{2} \right| \\
\leq\ & \frac{1}{12} \|f'\|_\infty \left[(x-a)^3 + (b-x)^3 \right] \\
=\ & \frac{1}{12} (b-a) \|f'\|_\infty \left[3 \left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4} (b-a)^2 \right] \\
=\ & \frac{1}{4} (b-a) \|f'\|_\infty \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{12} (b-a)^2 \right].
\end{aligned}$$

Using the identity (2.7), the desired result (2.10) is obtained. ∎

Remark 4. *If in (2.10) either $x = a$ or $x = b$, the inequality (2.1) is recaptured.*

Remark 5. *If in (2.10), $x = \frac{a+b}{2}$, then the best inequality that can be obtained is:-*

$$\left| E(X) + \frac{b-a}{2} \Pr \left(X \leq \frac{a+b}{2} \right) - \frac{a+3b}{4} \right| \leq \frac{1}{48} (b-a)^3 \|f'\|_\infty .$$

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