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SOME BOUNDS IN TERMS OF Δ -SEMINORMS FOR OSTROWSKI-GRÜSS TYPE INEQUALITIES

P. CERONE AND S.S. DRAGOMIR

ABSTRACT. In this paper we point out some bounds for the remainder of a generalised Ostrowski type formula by the use of Δ -seminorms.

1. INTRODUCTION

As in [1], let $\{P_n\}_{n \in \mathbb{N}}$ and $\{Q_n\}_{n \in \mathbb{N}}$ be two sequences of harmonic polynomials, that is, polynomials satisfying

$$(1.1) \quad P'_n(t) = P_{n-1}(t), \quad P_0(t) = 1, \quad t \in \mathbb{R},$$

$$(1.2) \quad Q'_n(t) = Q_{n-1}(t), \quad Q_0(t) = 1, \quad t \in \mathbb{R}.$$

In [1], the authors proved the following result.

Lemma 1. *Let $\{P_n\}_{n \in \mathbb{N}}$ and $\{Q_n\}_{n \in \mathbb{N}}$ be two harmonic polynomials. Set*

$$(1.3) \quad S_n(t, x) := \begin{cases} P_n(t), & t \in [a, x] \\ Q_n(t), & t \in (x, b] \end{cases}, \quad (t, x) \in [a, b]^2.$$

Then we have the equality

$$(1.4) \quad \begin{aligned} & \int_a^b f(t) dt \\ &= \sum_{k=1}^n (-1)^{k+1} \left[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \right. \\ & \quad \left. - P_k(a) f^{(k-1)}(a) \right] + (-1)^n \int_a^b S_n(t, x) f^{(n)}(t) dt, \end{aligned}$$

provided that $f : [a, b] \rightarrow \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$.

Using the following “pre-Grüss” inequality

$$(1.5) \quad |T(f, g)| \leq \frac{1}{2} \sqrt{T(f, f)} (\Gamma - \gamma),$$

where

$$T(f, g) := \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{(b-a)^2} \int_a^b f(x) dx \cdot \int_a^b g(x) dx$$

is the Chebychev functional and f, g are such that the previous integrals exist and $\gamma \leq g(x) \leq \Gamma$ for a.e. $x \in [a, b]$, the authors of [1] proved basically the

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following inequality for estimating the integral $\int_a^b f(t) dt$ in terms of the harmonic polynomials $\{P_n\}_{n \in \mathbb{N}}$ and $\{Q_n\}_{n \in \mathbb{N}}$.

Theorem 1. *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is such that $f^{(n)}$ is integrable and $\gamma_n \leq f^{(n)} \leq \Gamma_n$ for all $t \in [a, b]$. Put*

$$(1.6) \quad U_n(x) := \frac{1}{b-a} [Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)].$$

Then for all $x \in [a, b]$, we have the inequality

$$(1.7) \quad \left| \int_a^b f(t) dt - \sum_{k=1}^n (-1)^{k+1} [Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a)] - (-1)^n U_n(x) [f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \\ \leq \frac{1}{2} K(x) (\Gamma_n - \gamma_n) (b-a),$$

where

$$(1.8) \quad K(x) := \left\{ \frac{1}{b-a} \int_a^x P_n^2(t) dt + \int_x^b Q_n^2(t) dt - [U_n(x)]^2 \right\}^{\frac{1}{2}}.$$

A number of particular cases that were obtained by an appropriate choice of harmonic polynomials have also been presented in [1].

In the recent paper [2], Dragomir proved the following refinement of (1.7).

Theorem 2. *Assume that the mapping $f : [a, b] \rightarrow \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ and $f^{(n)} \in L_2[a, b]$ ($n \geq 1$). If we denote*

$$[f^{(n-1)}; a, b] := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a},$$

then we have the inequality

$$(1.9) \quad \left| \int_a^b f(t) dt - \sum_{k=1}^n (-1)^{k+1} [Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a)] - (-1)^n [Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)] [f^{(n-1)}; a, b] \right| \\ \leq K(x) (b-a) \left[\frac{1}{b-a} \|f^{(n)}\|_2^2 - \left([f^{(n)}; a, b] \right)^2 \right]^{\frac{1}{2}} \\ \left(\leq \frac{1}{2} K(x) (b-a) (\Gamma_n - \gamma_n) \text{ if } f^{(n)} \in L_\infty(a, b) \right),$$

for all $x \in [a, b]$ and $K(x)$ as is given in (1.8).

2. SOME PRELIMINARY RESULTS INVOLVING LEBESGUE NORMS ON $[a, b]^2$

For $f \in L_p[a, b]$ ($p \in [1, \infty)$), we can define the functional (see also [3])

$$(2.1) \quad \|f\|_p^\Delta := \left(\int_a^b \int_a^b |f(t) - f(s)|^p dt ds \right)^{\frac{1}{p}}$$

and for $f \in L_\infty [a, b]$, we can define

$$(2.2) \quad \|f\|_\infty^\Delta := \operatorname{ess\,sup}_{(t,s) \in [a,b]^2} |f(t) - f(s)|.$$

If we consider $f_\Delta : [a, b]^2 \rightarrow \mathbb{R}$,

$$(2.3) \quad f_\Delta(t, s) = f(t) - f(s),$$

then, obviously

$$(2.4) \quad \|f\|_p^\Delta = \|f_\Delta\|_p, \quad p \in [1, \infty],$$

where $\|\cdot\|_p$ are the usual Lebesgue p -norms on $[a, b]^2$.

Using the properties of the Lebesgue p -norms, we may deduce the following semi-norm properties for $\|\cdot\|_p^\Delta$:

- (i) $\|f\|_p^\Delta \geq 0$ for $f \in L_p [a, b]$ and $\|f\|_p^\Delta = 0$ implies that $f = c$ (c is a constant) a.e. in $[a, b]$;
- (ii) $\|f + g\|_p^\Delta \leq \|f\|_p^\Delta + \|g\|_p^\Delta$ if $f, g \in L_p [a, b]$;
- (iii) $\|\alpha f\|_p^\Delta = |\alpha| \|f\|_p^\Delta$.

We note that if $p = 2$, then,

$$\begin{aligned} \|f\|_2^\Delta &= \left(\int_a^b \int_a^b (f(t) - f(s))^2 dt ds \right)^{\frac{1}{2}} \\ &= \sqrt{2} \left[(b-a) \|f\|_2^2 - \left(\int_a^b f(t) dt \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$, then we can point out the following bounds for $\|f\|_p^\Delta$ in terms of $\|f'\|_p$.

Theorem 3. *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$.*

(i) *If $p \in [1, \infty)$, then we have the inequality*

$$(2.5) \quad \|f\|_p^\Delta \leq \begin{cases} \frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|f'\|_\infty & \text{if } f' \in L_\infty [a, b]; \\ \frac{(2\beta^2)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}} \|f'\|_\alpha & \text{if } f' \in L_\alpha [a, b], \\ & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ (b-a)^{\frac{2}{p}} \|f'\|_1 & \text{if } f' \in L_1 [a, b], \end{cases}$$

(ii) *If $p = \infty$, then we have the inequality*

$$(2.6) \quad \|f\|_\infty^\Delta \leq \begin{cases} (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty [a, b]; \\ (b-a)^{\frac{1}{\beta}} \|f'\|_\alpha & \text{if } f' \in L_\alpha [a, b], \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|f'\|_1. & \end{cases}$$

Proof. As $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous, then $f(t) - f(s) = \int_s^t f'(u) du$ for all $t, s \in [a, b]$, and then

$$(2.7) \quad |f(t) - f(s)| = \left| \int_s^t f'(u) du \right| \leq \begin{cases} |t-s| \|f'\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ |t-s|^{\frac{1}{\beta}} \|f'\|_\alpha & \text{if } f' \in L_\alpha[a, b], \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|f'\|_1 & \text{if } f' \in L_1[a, b] \end{cases}$$

and so for $p \in [1, \infty)$, we may write

$$\begin{aligned} & |f(t) - f(s)|^p \\ & \leq \begin{cases} |t-s|^p \|f'\|_\infty^p & \text{if } f' \in L_\infty[a, b]; \\ |t-s|^{\frac{p}{\beta}} \|f'\|_\alpha^p & \text{if } f' \in L_\alpha[a, b], \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|f'\|_1^p & \text{if } f' \in L_1[a, b], \end{cases} \end{aligned}$$

and then from (2.3), (2.4)

$$(2.8) \quad \|f\|_p^\Delta \leq \begin{cases} \|f'\|_\infty \left(\int_a^b \int_a^b |t-s|^p dt ds \right)^{\frac{1}{p}} & \text{if } f' \in L_\infty[a, b]; \\ \|f'\|_\alpha \left(\int_a^b \int_a^b |t-s|^{\frac{p}{\beta}} dt ds \right)^{\frac{1}{p}} & \text{if } f' \in L_\alpha[a, b], \\ & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|f'\|_1 \left(\int_a^b \int_a^b dt ds \right)^{\frac{1}{p}} & \text{if } f' \in L_1[a, b]. \end{cases}$$

Further, since

$$\begin{aligned} \left(\int_a^b \int_a^b |t-s|^p dt ds \right)^{\frac{1}{p}} &= \left[\int_a^b \left(\int_a^t (t-s)^p ds + \int_t^b (s-t)^p ds \right) dt \right]^{\frac{1}{p}} \\ &= \left(\int_a^b \left[\frac{(t-a)^{p+1} + (b-t)^{p+1}}{p+1} \right] dt \right)^{\frac{1}{p}} \\ &= \frac{2^{\frac{1}{p}} (b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}, \end{aligned}$$

giving

$$\left(\int_a^b \int_a^b |t-s|^{\frac{p}{\beta}} dt ds \right)^{\frac{1}{p}} = \frac{(2\beta^2)^{\frac{1}{p}} (b-a)^{\frac{1}{\beta} + \frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}},$$

and

$$\left(\int_a^b \int_a^b dt ds \right)^{\frac{1}{p}} = (b-a)^{\frac{2}{p}},$$

we obtain, from (2.8), the stated result (2.5).

Using (2.7) we have (for $p = \infty$) that

$$\|f\|_{\infty}^{\Delta} \leq \begin{cases} \|f'\|_{\infty} \operatorname{ess\,sup}_{(t,s) \in [a,b]^2} |t-s| \\ \|f'\|_{\alpha} \operatorname{ess\,sup}_{(t,s) \in [a,b]} |t-s|^{\frac{1}{\beta}} \\ \|f'\|_1 \end{cases} = \begin{cases} (b-a) \|f'\|_{\infty} \\ (b-a)^{\frac{1}{\beta}} \|f'\|_{\alpha} \\ \|f'\|_1 \end{cases}$$

and the inequality (2.6) is also proved. ■

3. SOME BOUNDS IN TERMS OF Δ–SEMINORMS

We start with the following result which obtains bounds for the left hand side of (1.9) (or equivalently (1.7)) in terms of the Δ–seminorms of the previous section.

Theorem 4. *Let $\{P_n\}_{n \in \mathbb{N}}$ and $\{Q_n\}_{n \in \mathbb{N}}$ be two harmonic polynomials. Set $S_n(\cdot, \cdot)$ as in Lemma 1 and assume that $f : [a, b] \rightarrow \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then we have the inequality:*

$$(3.1) \quad \left| \int_a^b f(t) dt - \sum_{k=1}^n (-1)^{k+1} \left[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a) \right] - (-1)^n [Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)] [f^{(n-1)}; a, b] \right| \\ \leq \frac{1}{2(b-a)} \times \begin{cases} \|S_n(\cdot, x)\|_1^{\Delta} \|f^{(n)}\|_{\infty}^{\Delta} & \text{if } f^{(n)} \in L_{\infty}[a, b]; \\ \|S_n(\cdot, x)\|_q^{\Delta} \|f^{(n)}\|_p^{\Delta} & \text{if } \frac{1}{p} + \frac{1}{q} = 1, p > 1, \\ & f^{(n)} \in L_p[a, b]; \\ \|S_n(\cdot, x)\|_{\infty}^{\Delta} \|f^{(n)}\|_1^{\Delta} & \end{cases}$$

and the Δ–seminorms $\|\cdot\|_p^{\Delta}$ ($p \in [1, \infty]$) are defined as in Section 2.

Proof. Recall Korkine’s identity

$$(3.2) \quad T(h, g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (h(t) - h(s))(g(t) - g(s)) dt ds,$$

where $T(\cdot, \cdot)$ is the Chebychev functional. That is, we recall that

$$T(h, g) = \frac{1}{b-a} \int_a^b h(x) g(x) dx - \frac{1}{(b-a)^2} \int_a^b h(x) dx \cdot \int_a^b g(x) dx,$$

provided that all the involved integrals exist.

Using (3.2) and the identity (1.4), gives (see also [1]):

$$\begin{aligned}
(3.3) \quad & \int_a^b f(t) dt - \sum_{k=1}^n (-1)^{k+1} \left[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \right. \\
& \quad \left. - P_k(a) f^{(k-1)}(a) \right] - (-1)^n [Q_{n+1}(b) - Q_{n+1}(x) \\
& \quad + P_{n+1}(x) - P_{n+1}(a)] \left[f^{(n-1)}; a, b \right] \\
& = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (S_n(t, x) - S_n(s, x)) (f^{(n)}(t) - f^{(n)}(s)) dt ds.
\end{aligned}$$

Using Hölder's integral inequality for double integrals, we may write

$$\begin{aligned}
B & := \left| \int_a^b \int_a^b (S_n(t, x) - S_n(s, x)) (f^{(n)}(t) - f^{(n)}(s)) dt ds \right| \\
& \leq \left(\int_a^b \int_a^b |S_n(t, x) - S_n(s, x)|^q dt ds \right)^{\frac{1}{q}} \left(\int_a^b \int_a^b |f^{(n)}(t) - f^{(n)}(s)|^p dt ds \right)^{\frac{1}{p}} \\
& = \|S_n(\cdot, x)\|_q^\Delta \|f^{(n)}\|_p^\Delta,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

If $q = 1$, we have

$$B \leq \|S_n(\cdot, x)\|_1^\Delta \|f^{(n)}\|_\infty^\Delta$$

and if $p = 1$, then

$$B \leq \|S_n(\cdot, x)\|_\infty^\Delta \|f^{(n)}\|_1^\Delta.$$

Further, using the identity (3.3) and the properties of modulus, we obtain (3.1). ■

Remark 1. For $p = q = 2$, we recapture Theorem 2 and so Theorem 4 represents an Ostrowski-Grüss result whose bounds are given in terms of the Δ -seminorms whose properties are given in Section 2.

The following corollary holds.

Corollary 1. Let $\{P_n\}_{n \in \mathbb{N}}, \{Q_n\}_{n \in \mathbb{N}}$ and $\{S_n\}_{n \in \mathbb{N}}$ be as in Theorem 4. If $f : [a, b] \rightarrow \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous, then we have the inequality:

$$\begin{aligned}
(3.4) \quad & \left| \int_a^b f(t) dt - \sum_{k=1}^n (-1)^{k+1} \left[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \right. \right. \\
& \quad \left. \left. - P_k(a) f^{(k-1)}(a) \right] - (-1)^n [Q_{n+1}(b) - Q_{n+1}(x) \right. \\
& \quad \left. + P_{n+1}(x) - P_{n+1}(a)] \left[f^{(n-1)}; a, b \right] \right|
\end{aligned}$$

$$\leq \left\{ \begin{array}{ll} \frac{1}{2} \|S_n(\cdot, x)\|_1^\Delta \|f^{(n+1)}\|_\infty & \text{if } f^{(n+1)} \in L_\infty[a, b]; \\ \frac{1}{2} (b-a)^{\frac{1}{\beta}-1} \|S_n(\cdot, x)\|_1^\Delta \|f^{(n+1)}\|_\alpha & \text{if } f^{(n+1)} \in L_\alpha[a, b], \\ & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \frac{1}{2(b-a)} \|S_n(\cdot, x)\|_1^\Delta \|f^{(n+1)}\|_1 & \\ \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|S_n(\cdot, x)\|_q^\Delta \|f^{(n+1)}\|_\infty & \text{if } \frac{1}{p} + \frac{1}{q} = 1, p > 1, \\ & f^{(n+1)} \in L_\alpha[a, b]; \\ \frac{2^{\frac{1}{p}-1} (\beta^2)^{\frac{1}{p}} (b-a)^{\frac{1}{\beta} + \frac{2}{p} - 1}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}} \|S_n(\cdot, x)\|_q^\Delta \|f^{(n+1)}\|_\alpha & \text{if } f^{(n+1)} \in L_\alpha[a, b], \\ & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} (b-a)^{\frac{2}{p}-1} \|S_n(\cdot, x)\|_q^\Delta \|f^{(n+1)}\|_1 & \text{if } \frac{1}{p} + \frac{1}{q} = 1, p > 1; \\ \frac{(b-a)^2}{6} \|S_n(\cdot, x)\|_\infty^\Delta \|f^{(n+1)}\|_\infty & \text{if } f^{(n+1)} \in L_\infty[a, b]; \\ \frac{\beta^2 (b-a)^{\frac{1}{\beta}+1}}{2(\beta+1)(\beta+2)} \|S_n(\cdot, x)\|_\infty^\Delta \|f^{(n+1)}\|_\alpha & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ & f^{(n+1)} \in L_\alpha[a, b]; \\ \frac{1}{2} (b-a) \|S_n(\cdot, x)\|_\infty^\Delta \|f^{(n+1)}\|_1. & \end{array} \right.$$

The proof follows by Theorem 4 and by Theorem 3 (Section 2) applied for the Δ -seminorm of the mapping $f^{(n)}$. We omit the details.

Remark 2. *If we choose*

$$(3.5) \quad S_n(t, x) = \begin{cases} P_n(t), & t \in [a, x], \\ kH_n(t-x) + P_n(t), & t \in (x, b] \end{cases}$$

with $P_n(\cdot)$ and $H_n(\cdot-x)$ harmonic polynomials satisfying (1.1) and $H_n(0) = 0$ for all $n \in \mathbb{N}$ then, $S_n(\cdot, x)$ is absolutely continuous on $[a, b]$. Hence, the bounds obtained in Theorem 3 will hold for $\|S_n(\cdot, x)\|_q$, $q \geq 1$ in terms of $\|S'_n(\cdot, x)\|_\gamma$, $\gamma \geq 1$. Here,

$$(3.6) \quad S'_n(t, x) = \begin{cases} P_{n-1}(t), & t \in [a, x], \\ kH_{n-1}(t-x) + P_{n-1}(t), & t \in (x, b], \end{cases}$$

where the differentiation is with respect to t within each of the two subintervals. With the above development, we have from Theorem 3:

(i) For $q \in [1, \infty)$,

$$(3.7) \quad \|S_n(\cdot, x)\|_q^\Delta \leq \begin{cases} 2^{\frac{1}{q}} (b-a)^{1+\frac{2}{q}} \|S'_n(\cdot, x)\|_\infty, & S'_n \in L_\infty[a, b]; \\ \frac{(2\delta^2)^{\frac{1}{q}} (b-a)^{\frac{1}{\delta} + \frac{2}{q}}}{(q+\delta)(q+2\delta)^{\frac{1}{q}}} \|S'_n(\cdot, x)\|_\gamma, & S'_n \in L_\gamma[a, b], \\ & \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ (b-a)^{\frac{2}{q}} \|S'_n(\cdot, x)\|_1, & S'_n \in L_1[a, b], \end{cases}$$

(ii)

$$(3.8) \quad \|S_n(\cdot, x)\|_\infty^\Delta \leq \begin{cases} (b-a) \|S'_n(\cdot, x)\|_\infty, & S'_n \in L_\infty[a, b]; \\ (b-a)^{\frac{1}{\delta}} \|S'_n(\cdot, x)\|_\gamma, & S'_n \in L_\gamma[a, b], \\ & \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \|S'_n(\cdot, x)\|_1, & S'_n \in L_1[a, b], \end{cases}$$

and so substitution into the right hand side of Corollary 1 would give bounds involving 27 branches.

Corollary 2. Let $\{P_n\}_{n \in \mathbb{N}}$ be as in Theorem 4. Then for $f : [a, b] \rightarrow \mathbb{R}$ and $f^{(n)}$ absolutely continuous, we have

$$(3.9) \quad \left| \int_a^b f(t) dt - \sum_{k=1}^n (-1)^{k+1} [P_k(b) f^{(k-1)}(b) - P_k(a) f^{(k-1)}(a)] \right. \\ \left. - (-1)^n [P_{n+1}(b) - P_{n+1}(a)] [f^{(n-1)}; a, b] \right| \\ \leq \begin{cases} \frac{1}{2} \|P_n\|_1^\Delta \|f^{(n+1)}\|_\infty & \text{if } f^{(n+1)} \in L_\infty[a, b]; \\ \frac{1}{2} (b-a)^{\frac{1}{\beta}-1} \|P_n\|_1^\Delta \|f^{(n+1)}\|_\alpha & \text{if } f^{(n+1)} \in L_\alpha[a, b], \\ & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \frac{1}{2(b-a)} \|P_n\|_1^\Delta \|f^{(n+1)}\|_1 & \\ \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|P_n\|_q^\Delta \|f^{(n+1)}\|_\infty & \text{if } \frac{1}{p} + \frac{1}{q} = 1, p > 1, \\ & f^{(n+1)} \in L_\alpha[a, b]; \\ \frac{2^{\frac{1}{p}-1} (\beta^2)^{\frac{1}{p}} (b-a)^{\frac{1}{\beta} + \frac{2}{p} - 1}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}} \|P_n\|_q^\Delta \|f^{(n+1)}\|_\alpha & \text{if } f^{(n+1)} \in L_\alpha[a, b], \\ & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} (b-a)^{\frac{2}{p}-1} \|P_n\|_q^\Delta \|f^{(n+1)}\|_1 & \text{if } \frac{1}{p} + \frac{1}{q} = 1, p > 1; \\ \frac{(b-a)^2}{6} \|P_n\|_\infty^\Delta \|f^{(n+1)}\|_\infty & \text{if } f^{(n+1)} \in L_\infty[a, b]; \\ \frac{\beta^2 (b-a)^{\frac{1}{\beta}+1}}{2(\beta+1)(\beta+2)} \|P_n\|_\infty^\Delta \|f^{(n+1)}\|_\alpha & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ & f^{(n+1)} \in L_\alpha[a, b]; \\ \frac{1}{2} (b-a) \|P_n\|_\infty^\Delta \|f^{(n+1)}\|_1. & \end{cases}$$

where for $q \in [1, \infty)$

$$(3.10) \quad \|P_n\|_q^\Delta \leq \begin{cases} 2^{\frac{1}{q}} (b-a)^{1+\frac{2}{q}} \|P_{n-1}\|_\infty, & P_n \in L_\infty[a, b]; \\ (2\delta^2)^{\frac{1}{q}} (b-a)^{\frac{1}{\delta} + \frac{2}{q}} \|P_{n-1}\|_\gamma, & P_n \in L_\gamma[a, b], \\ & \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ (b-a)^{\frac{2}{q}} \|P_{n-1}\|_1, & P_n \in L_1[a, b], \end{cases}$$

and

$$(3.11) \quad \|P_n\|_\infty^\Delta \leq \begin{cases} (b-a) \|P_{n-1}\|_\infty, & P_n \in L_\infty[a, b]; \\ (b-a)^{\frac{1}{\delta}} \|P_{n-1}\|_\gamma, & P_n \in L_\gamma[a, b], \\ & \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \|P_{n-1}\|_1, & P_n \in L_1[a, b]. \end{cases}$$

Proof. Taking $k = 0$ in (3.5) or equivalently $Q_n(t) \equiv P_n(t)$ in (1.3) gives from Corollary 1 the stated results where we have used (3.6) – (3.8). ■

Remark 3. As an example, take

$$(3.12) \quad \tilde{P}_n(t) = \frac{(t-\theta)^n}{n!}$$

with

$$\theta = (1-\lambda)a + \lambda b, \quad \lambda \in [0, 1],$$

then,

$$(3.13) \quad \begin{aligned} \|\tilde{P}_n\|_\infty &= \sup_{t \in [a, b]} |\tilde{P}_n(t)| = \frac{(b-a)^n}{n!} \max\{\lambda^n, (1-\lambda)^n\} \\ &= \frac{(b-a)^n}{n!} \left[\frac{1}{2} \left| \lambda - \frac{1}{2} \right| \right]^n. \end{aligned}$$

Further,

$$\begin{aligned} \|\tilde{P}_n\|_\gamma &= \left(\int_a^b |\tilde{P}_n(t)|^\gamma dt \right)^{\frac{1}{\gamma}} \quad (\gamma \in [1, \infty)) \\ &= \frac{1}{n!} \left[\int_a^\theta (\theta-t)^{n\gamma} dt + \int_\theta^b (t-\theta)^{n\gamma} dt \right]^{\frac{1}{\gamma}} \\ &= \frac{1}{n!} \left[\frac{(\theta-a)^{n\gamma+1} + (b-\theta)^{n\gamma+1}}{n\gamma+1} \right]^{\frac{1}{\gamma}}, \end{aligned}$$

giving

$$(3.14) \quad \|\tilde{P}_n\|_\gamma = \frac{(b-a)^{n+\frac{1}{\gamma}}}{n!} \left[\frac{\lambda^{n\gamma+1} + (1-\lambda)^{n\gamma+1}}{n\gamma+1} \right]^{\frac{1}{\gamma}}.$$

Thus, from (3.9) with $P_n(t)$ as given by (3.12) gives

$$(3.15) \quad \begin{aligned} \tau_\lambda &: = \left| \int_a^b f(t) dt - \sum_{k=1}^n (-1)^{k+1} \frac{(b-a)^k}{k!} \right. \\ &\quad \times \left[(1-\lambda)^k f^{(k-1)}(b) - \lambda^k f^{(k-1)}(a) \right] - (-1)^n \frac{(b-a)^{n+1}}{(n+1)!} \\ &\quad \times \left[(1-\lambda)^{n+1} f^{(n)}(b) - \lambda^{n+1} f^{(n)}(a) \right] \left[f^{(n-1)}; a, b \right] \Big| \\ &\leq B \left(\|\tilde{P}_n\|^\Delta \right), \end{aligned}$$

where $B\left(\|P_n\|^\Delta\right)$ is the right hand side of (3.9) with $\|\tilde{P}_n\|_q^\Delta$, $\|\tilde{P}_n\|_\infty^\Delta$ given by (3.10), (3.11) on using (3.13) and (3.14).

For $\lambda = \frac{1}{2}$ the left hand side of (3.15) simplifies to

$$\begin{aligned} \frac{1}{b-a} \tau_{\frac{1}{2}} &= \left| \frac{1}{b-a} \int_a^b f(t) dt - \sum_{k=1}^n \frac{(-1)^{k+1}}{k!} \left(\frac{b-a}{2}\right)^k [f^{(k-1)}; a, b] \right. \\ &\quad \left. - \frac{(-1)^n}{(n+1)!} \left(\frac{b-a}{2}\right)^{n+1} [f^{(n)}; a, b] \times [f^{(n-1)}; a, b] \right|, \end{aligned}$$

where $[g; a, b] = \frac{g(b)-g(a)}{b-a}$.

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SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO BOX 14428, MELBOURNE CITY MC, VICTORIA 8001, AUSTRALIA

E-mail address: `pc@matilda.vu.edu.au`

URL: `http://sci.vu.edu.au/~pc`

E-mail address: `sever@matilda.vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`