New Steffensen Pairs

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Abstract. In this article, using mathematical induction and analytic techniques, some new Steffensen pairs are established.

1. Introduction

Let f and g be integrable functions on $[a,b]$ such that f is decreasing and $0 \leqslant g(x) \leqslant 1$ for $x \in [a,b]$. Then

$$\int_{b-\lambda}^{b} f(x)\mathrm{d}x \leqslant \int_{a}^{b} f(x)g(x)\mathrm{d}x \leqslant \int_{a}^{a+\lambda} f(x)\mathrm{d}x, \tag{1}$$

where $\lambda = \int_a^b g(x)\mathrm{d}x$.

The inequality (1) is called Steffensen's inequality. For more information, please see [3, 4, 14].

In [1], its discrete analogue of inequality (1) was proved: Let $\{x_i\}_{i=1}^n$ be a decreasing finite sequence of nonnegative real numbers, $\{y_i\}_{i=1}^n$ be a finite sequence of real numbers such that $0 \leqslant y_i \leqslant 1$ for $1 \leqslant i \leqslant n$. Let $k_1, k_2 \in \{1, 2, \dots, n\}$ be such that $k_2 \leqslant \sum_{i=1}^{n} y_i \leqslant k_1$. Then

$$\sum_{i=n-k_2+1}^{n} x_i \leqslant \sum_{i=1}^{n} x_i y_i \leqslant \sum_{i=1}^{k_1} x_i. \tag{2}$$

As a direct consequence of inequality (2), we have: Let $\{x_i\}_{i=1}^n$ be nonnegative real numbers such that $\sum_{i=1}^{n} x_i \leqslant A$ and $\sum_{i=1}^{n} x_i^2 \geqslant B^2$, where A and B are positive real numbers. Let $k \in \{1, 2, \dots, n\}$ be such that $k \geqslant \frac{A}{B}$. Then there are k numbers among $x_1, x_2, \dots, x_n$ whose sum is bigger than or equals to B.

The so-called Steffensen pair was defined in [2] as follows:

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Definition 1. Let $\varphi : [c,\infty) \to [0,\infty)$ and $\tau : (0,\infty) \to (0,\infty)$ be two strictly increasing functions, $c \geqslant 0$, let $\{x_i\}_{i=1}^n$ be a finite sequence of real numbers such that $x_i \geqslant c$ for $1 \leqslant i \leqslant n$, A and B be positive real numbers, and $\sum\limits_{i=1}^n x_i \leqslant A$, $\sum\limits_{i=1}^n \varphi(x_i) \geqslant \varphi(B)$. If, for any $k \in \{1,2,\dots,n\}$ such that $k \geqslant \tau(\frac{A}{B})$, there are k numbers among $x_1,\dots,x_n$ whose sum is not less than B, then we call (φ,τ) a Steffensen pair on $[c,\infty)$.

In [1] and [2], the following Steffensen pairs were found respectively:

$$\begin{gathered}(x^\alpha, x^{1/(\alpha-1)}), \quad \alpha \geqslant 2, \quad x \in [0,\infty);\\ (x\exp(x^\alpha-1), (1+\ln x)^{1/\alpha}), \quad \alpha \geqslant 1, \quad x \in [1,\infty).\end{gathered}$$

Let a and b be real numbers satisfying $b > a > 1$ and $\sqrt{ab} \geqslant e$. Define

$$\begin{aligned}\varphi(x) &= \begin{cases}\dfrac{x^{1+\ln b} - x^{1+\ln a}}{\ln x}, & x > 1;\\ \ln b - \ln a, & x = 1,\end{cases}\\ \tau(x) &= x^{1/\ln\sqrt{ab}}.\end{aligned}$$

Then it was verified in [2] that (φ,τ) is a Steffensen pair on $[1,\infty)$.

In this article, we will establish some new Steffensen pairs, that is

Theorem 1. *If a and b are real numbers satisfying $b > a > 1$ or $b > a^{-1} > 1$, and $\sqrt{ab} \geqslant e$, then*

$$\left(x\int_a^b t^{\ln x - 1}\,\mathrm{d}t, x^{1/\ln\sqrt{ab}}\right) \tag{3}$$

is a Steffensen pair on $[1,+\infty)$.

If a and b are real numbers satisfying $b > a > 1$ and $\sqrt{ab} \geqslant e$, then

$$\left(x\int_a^b (\ln t)^n t^{\ln x - 1}\,\mathrm{d}t, x^{\frac{n+2}{n+1}\cdot\frac{(\ln b)^{n+1}-(\ln a)^{n+1}}{(\ln b)^{n+2}-(\ln a)^{n+2}}}\right) \tag{4}$$

are Steffensen pairs on $[1,+\infty)$ for any positive integer n.

Remark 1. This theorem generalizes the Proposition 2 in [2].

2. Lemmas

Lemma 1 ([2])**.** *Let $\psi : [c,\infty) \to [0,\infty)$ be increasing and convex, $c \geqslant 0$. Assume that ψ satisfies $\psi(xy) \geqslant \psi(x)g(y)$ for all $x \geqslant c$ and $y \geqslant 1$, where $g : [1,\infty) \to [0,\infty)$ is strictly increasing. Set $\varphi(x) = x\psi(x)$, $\tau(x) = g^{-1}(x)$, where g^{-1} is the inverse function of g. Then (φ,τ) is a Steffensen pair on $[c,\infty)$.*

Let $b > a > 1$ or $b > a^{-1} > 1$, and $\sqrt{ab} > e$. Define

$$h(x) = \begin{cases} \dfrac{b^x - a^x}{x}, & x \neq 0; \\ \ln b - \ln a, & x = 0. \end{cases} \tag{5}$$

It can be represented in integral form in [5]—[13] as follows

$$h(x) = \int_a^b t^{x-1} \mathrm{d}t, \quad x \in \mathbb{R}. \tag{6}$$

It had been verified in [10] that the function $h(x)$ is absolutely and regularly monotonic on $(-\infty, +\infty)$ for $b > a > 1$, or on $(0, +\infty)$ for $b > a^{-1} > 1$, completely and regularly monotonic on $(-\infty, +\infty)$ for $0 < a < b < 1$, or on $(-\infty, 0)$ for $1 < b < a^{-1}$. Furthermore, $h(x)$ is absolutely convex on $(-\infty, +\infty)$.

A function $f(t)$ is said to be absolutely monotonic on (c, d) if it has derivatives of all orders and $f^{(k)}(t) \geqslant 0$ for $t \in (c, d)$ and $k \in \mathbb{N}$. For information of absolutely (completely, regularly, respectively) monotonic (convex, respectively) function, please refer to [4, 6, 10].

Lemma 2. *For $x \geqslant 0$ and $n \geqslant 0$, we have*

$$h^{(n+1)}(x) \geqslant h^{(n)}(x). \tag{7}$$

Proof. It is clear that

$$h^{(n)}(x) = \int_a^b t^{x-1} (\ln t)^n \mathrm{d}t. \tag{8}$$

By the Tchebysheff's integral inequality or by Cauchy-Schwarz-Buniakowski inequality as in [5]—[13], we have

$$[h^{(n+1)}(x)]^2 \leqslant h^{(n)}(x) h^{(n+2)}(x). \tag{9}$$

Since the extended mean values $E(r, s; u, v)$ defined in [5, 9, 12] by

$$E(r, s; u, v) = \left[\frac{r}{s} \cdot \frac{u^s - v^s}{u^r - v^r}\right]^{1/(s-r)}, \qquad rs(r-s)(u-v) \neq 0; \tag{10}$$

$$E(r, 0; u, v) = \left[\frac{u^r - v^r}{\ln u - \ln v} \cdot \frac{1}{r}\right]^{1/r}, \qquad r(v-u) \neq 0; \tag{11}$$

$$E(r, r; u, v) = \mathrm{e}^{-1/r} \left(\frac{u^{u^r}}{v^{v^r}}\right)^{1/(u^r - v^r)}, \qquad r(u-v) \neq 0; \tag{12}$$

$$E(0, 0; u, v) = \sqrt{uv}, \qquad u \neq v;$$

$$E(r, s; u, u) = u, \qquad u = v;$$

are increasing with r and s for fixed positive numbers u and v, then, for every $y \geqslant 0$, the function $F(x)=\frac{h(x+y)}{h(x)}$ is increasing with x. Therefore

$$F'(x)=\frac{h'(x+y)h(x)-h(x+y)h'(x)}{[h(x)]^2}\geqslant 0,$$

hence

$$h'(x+y)h(x)-h(x+y)h'(x)\geqslant 0 \tag{13}$$

holds for all x and $y \geqslant 0$.

Taking $x=0$ in (13), for all $y \geqslant 0$, we obtain

$$h'(y)h(0)-h(y)h'(0)\geqslant 0. \tag{14}$$

Since $h'(0)=h(0)\ln\sqrt{ab}$ and $\sqrt{ab}\geqslant e$, we have $h'(0)\geqslant h(0)$, and $h'(y)\geqslant h(y)$ for $y\geqslant 0$.

Note that inequality (13) can also be obtained from Lemma 4 in [12]: The functions $\frac{h^{(2(k+i)+1)}(t)}{h^{(2k)}(t)}$ are increasing with respect to t for i and k being nonnegative integers.

By mathematical induction, assume that $h^{(n+1)}(x)\geqslant h^{(n)}(x)$ for $n>1$ and $x\geqslant 0$. Then, from inequality (9), we obtain

$$h^{(n)}(x)h^{(n+1)}(x)\leqslant [h^{(n+1)}(x)]^2\leqslant h^{(n)}(x)h^{(n+2)}(x), \tag{15}$$

therefore

$$h^{(n+1)}(x)\leqslant h^{(n+2)}(x).$$

The proof is completed. □

3. Proof of Theorem 1

Now we give a proof of Theorem 1.

Set $\psi(x)=h^{(n)}(\ln x)$ for $x\geqslant 1$ and $n\geqslant 0$. Direct computation yields that $\psi'(x)=\frac{h^{(n+1)}(\ln x)}{x}>0$ and $\psi''(x)=\frac{h^{(n+2)}(\ln x)-h^{(n+1)}(\ln x)}{x^2}\geqslant 0$. Hence $\psi(x)$ is increasing and convex.

Let $u,v,r,s\in\mathbb{R}$, let $p\not\equiv 0$ be a nonnegative and integrable function and f a positive and integrable function on the interval between x and y. Then the generalized weighted mean values $M_{p,f}(r,s;u,v)$ of the function f with weight p and two parameters r and s are defined in [6] by

$$M_{p,f}(r,s;u,v)=\left(\frac{\int_u^v p(t)f^s(t)\mathrm{d}t}{\int_u^v p(t)f^r(t)\mathrm{d}t}\right)^{1/(s-r)}, \qquad (r-s)(u-v)\neq 0; \tag{16}$$

$$M_{p,f}(r,r;u,v)=\exp\left(\frac{\int_u^v p(t)f^r(t)\ln f(t)\mathrm{d}t}{\int_u^v p(t)f^r(t)\mathrm{d}t}\right), \qquad u-v\neq 0; \tag{17}$$

$$M(r,s;u,u)=f(u).$$

¿From the Cauchy-Schwarz-Buniakowski inequality and standard argument, it was obtained in [13] that: The generalized weighted mean values $M_{p,f}(r,s;u,v)$ are increasing with both r and s for any given continuous nonnegative weight p, continuous positive function f, and fixed real numbers u and v. Then, if $b>a>1$, for $x,y\geqslant 0$ and $n\geqslant 1$, we have

$$\frac{h^{(n)}(x+y)}{h^{(n)}(x)}=\frac{\int_a^b t^{x+y-1}(\ln t)^n\mathrm{d}t}{\int_a^b t^{x-1}(\ln t)^n\mathrm{d}t}\geqslant\exp\left(y\cdot\frac{n+1}{n+2}\cdot\frac{(\ln b)^{n+2}-(\ln a)^{n+2}}{(\ln b)^{n+1}-(\ln a)^{n+1}}\right). \tag{18}$$

Therefore, for $x,y\geqslant 1$,

$$\frac{\psi(xy)}{\psi(x)}=\frac{h^{(n)}(\ln(xy))}{h^{(n)}(\ln x)}=\frac{h^{(n)}(\ln x+\ln y)}{h^{(n)}(\ln x)}\geqslant y^{\frac{n+1}{n+2}\cdot\frac{(\ln b)^{n+2}-(\ln a)^{n+2}}{(\ln b)^{n+1}-(\ln a)^{n+1}}}. \tag{19}$$

Let $g(x)=x^{\frac{n+1}{n+2}\cdot\frac{(\ln b)^{n+2}-(\ln a)^{n+2}}{(\ln b)^{n+1}-(\ln a)^{n+1}}}$ for $x\geqslant 1$, then $g^{-1}(x)=x^{\frac{n+2}{n+1}\cdot\frac{(\ln b)^{n+1}-(\ln a)^{n+1}}{(\ln b)^{n+2}-(\ln a)^{n+2}}}$, $x\in[1,+\infty)$. By Lemma 1, (φ,τ), where $\varphi(x)=x\psi(x)=xh^{(n)}(\ln x)=x\int_a^b(\ln t)^n t^{\ln x-1}\mathrm{d}t$ and $\tau(x)=x^{\frac{n+2}{n+1}\cdot\frac{(\ln b)^{n+1}-(\ln a)^{n+1}}{(\ln b)^{n+2}-(\ln a)^{n+2}}}$ for $x\geqslant 1$ and $n\geqslant 0$, are Steffensen pairs on $[1,+\infty)$ for any given $n\geqslant 0$.

If a and b are real numbers satisfying $b>a>1$ or $b>a^{-1}>1$, and $\sqrt{ab}\geqslant e$, then, for $x,y\geqslant 0$, we have

$$\frac{h(x+y)}{h(x)}\geqslant\left(\sqrt{ab}\right)^y. \tag{20}$$

Therefore, for $x,y\geqslant 1$,

$$\frac{\psi(xy)}{\psi(x)}=\frac{h(\ln(xy))}{h(\ln x)}=\frac{h(\ln x+\ln y)}{h(\ln x)}\geqslant y^{\ln\sqrt{ab}}. \tag{21}$$

Let $g(x)=x^{\ln\sqrt{ab}}$ for $x\geqslant 1$, then $g^{-1}(x)=x^{1/\ln\sqrt{ab}}$, $x\in[1,+\infty)$. By Lemma 1, (φ,τ), where $\varphi(x)=x\psi(x)=xh(\ln x)=x\int_a^b t^{\ln x-1}\mathrm{d}t$ and $\tau(x)=x^{1/\ln\sqrt{ab}}$ for $x\geqslant 1$, is a Steffensen pair on $[1,+\infty)$.

The proof is complete.

Remark 2. If considering the function $\int_x^y p(u)f^t(u)\mathrm{d}u$, then more new Steffensen pairs can be obtained. We will discuss this in a subsequent paper [8].

References

[1] J.-C. Evard and H. Gauchman, *Steffensen type inequalities over general measure spaces*, Analysis **17** (1997), 301-322.

[2] H. Gauchman, *Steffensen pairs and associated inequalities*, Journal of Inequalities and Applications **5** (2000), no. 1, 53–61

[3] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[4] D. S. Mitrinović, J. E. Pečaric and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.

[5] Josip Pečarić, Feng Qi, V. Šimić, and Sen-Lin Xu, *Refinements and extensions of an inequality*, III, Journal of Mathematical Analysis and Applications **227** (1998), no. 2, 439–448.

[6] Feng Qi, *Generalized weighted mean values with two parameters*, Proceedings of the Royal Society of London Series A—Mathematical, Physical and Engineering Sciences **454** (1998), no. 1978, 2723–2732.

[7] Feng Qi, *On a two-parameter family of nonhomogeneous mean values*, Tamkang Journal of Mathematics **29** (1998), no. 2, 155–163.

[8] Feng Qi and Jun-Xiang Cheng, *On Steffensen pairs*, RGMIA Research Report Collection **3** (2000). http://rgmia.vu.edu.au.

[9] Feng Qi and Qiu-Ming Luo, *A simple proof of monotonicity for extended mean values*, Journal of Mathematical Analysis and Applications **224** (1998), 356–359.

[10] Feng Qi and Sen-Lin Xu, *The function* $(b^x - a^x)/x$*: Inequalities and properties*, Proceedings of the American Mathematical Society **126** (1998), 3355–3359.

[11] Feng Qi and Sen-Lin Xu, *Refinements and extensions of an inequality*, II, Journal of Mathematical Analysis and Applications **211** (1997), 616–620.

[12] Feng Qi, Sen-Lin Xu, and Lokenath Debnath, *A new proof of monotonicity for extended mean values*, International Journal of Mathematics and Mathematical Sciences **22** (1999), no. 2, 415–420.

[13] Feng Qi and Shi-Qin Zhang, *Note on monotonicity of generalized weighted mean values*, Proceedings of the Royal Society of London Series A—Mathematical, Physical and Engineering Sciences **455** (1999), no. 1989, 3259–3260.

[14] J. F. Steffensen, *On certain inequalities and methods of approximation*, J. Inst. Actuaries **51** (1919), 274-297.

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