On Steffensen Pairs

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Abstract. In this article, by mathematical induction and properties of the generalized weighted mean values, some general Steffensen pairs are established.

1. Introduction

Let f and g be integrable functions on $[a,b]$ such that f is decreasing and $0 \leqslant g(x) \leqslant 1$ for $x \in [a,b]$. Then

$$\int_{b-\lambda}^{b} f(x)\mathrm{d}x \leqslant \int_{a}^{b} f(x)g(x)\mathrm{d}x \leqslant \int_{a}^{a+\lambda} f(x)\mathrm{d}x, \tag{1}$$

where $\lambda = \int_a^b g(x)\mathrm{d}x$.

The inequality (1) is called Steffensen's inequality in [3, 8].

In [1], its discrete analogue of the inequality (1) was proved: Let $\{x_i\}_{i=1}^n$ be a decreasing finite sequence of nonnegative real numbers, $\{y_i\}_{i=1}^n$ be a finite sequence of real numbers such that $0 \leqslant y_i \leqslant 1$ for $1 \leqslant i \leqslant n$. Let $k_1, k_2 \in \{1, 2, \dots, n\}$ be such that $k_2 \leqslant \sum_{i=1}^{n} y_i \leqslant k_1$. Then

$$\sum_{i=n-k_2+1}^{n} x_i \leqslant \sum_{i=1}^{n} x_i y_i \leqslant \sum_{i=1}^{k_1} x_i. \tag{2}$$

As a direct consequence of inequality (2), we obtain: Let $\{x_i\}_{i=1}^n$ be a finite sequence of non-negative real numbers such that $\sum_{i=1}^{n} x_i \leqslant A$ and $\sum_{i=1}^{n} x_i^2 \geqslant B^2$, where A and B are positive real numbers. Let $k \in \{1, 2, \dots, n\}$ be such that $k \geqslant \frac{A}{B}$. Then there are k numbers among $\{x_i\}_{i=1}^n$ whose sum is not less than B.

From above results, the so-called Steffensen pair was defined in [2] by Dr. H. Gauchman as follows:

Date: July 13, 2000.

1991 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Steffensen inequality, Steffensen pair, generalized weighted mean values, mathematical induction, absolutely monotonic function.

The authors were supported in part by NSF of Henan Province (no. 004051800), SF for Pure Research of the Education Committee of Henan Province (no. 1999110004), and Doctor Fund of Jiaozuo Institute of Technology, The People's Republic of China.

Definition 1. Let $\varphi : [c, \infty) \to [0, \infty)$ and $\tau : (0, \infty) \to (0, \infty)$ be two strictly increasing functions, $c \geqslant 0$, let $\{x_i\}_{i=1}^{n}$ be a finite sequence of real numbers such that $x_i \geqslant c$ for all i, A and B be positive real numbers, $\sum_{i=1}^{n} x_i \leqslant A$, and $\sum_{i=1}^{n} \varphi(x_i) \geqslant \varphi(B)$. If, for any $k \in \{i\}_{i=1}^{n}$ satisfying $k \geqslant \tau(\frac{A}{B})$, there are k numbers among $\{x_i\}_{i=1}^{n}$ whose sum is not less than B, then we call (φ, τ) a Steffensen pair on $[c, \infty)$.

In [1] and [2], the following Steffensen pairs were found:

$$(x^{\alpha}, x^{1/(\alpha-1)}), \quad \alpha \geqslant 2, \quad x \in [0, \infty);$$

$$(x \exp(x^{\alpha} - 1), (1 + \ln x)^{1/\alpha}), \quad \alpha \geqslant 1, \quad x \in [1, \infty).$$

It was verified in [5] that: Let a and b be real numbers satisfying $b > a > 1$ or $b > a^{-1} > 1$, and $\sqrt{ab} \geqslant e$, then

$$\left(x \int_a^b t^{\ln x - 1} \mathrm{d}t, x^{1/\ln \sqrt{ab}} \right) \tag{3}$$

is a Steffensen pair on $[1, +\infty)$. If a and b are real numbers satisfying $b > a > 1$ and $\sqrt{ab} \geqslant e$, then

$$\left(x \int_a^b (\ln t)^n t^{\ln x - 1} \mathrm{d}t, x^{\frac{n+2}{n+1} \cdot \frac{(\ln b)^{n+1} - (\ln a)^{n+1}}{(\ln b)^{n+2} - (\ln a)^{n+2}}} \right) \tag{4}$$

are Steffensen pairs on $[1, +\infty)$ for any positive integer n.

In this article, we will establish more general Steffensen pairs, that is

Theorem 1. *Let $a, b \in \mathbb{R}$, let $p \not\equiv 0$ be a nonnegative and integrable function and f a positive and integrable function on the interval $[a, b]$.*

(i) *If inequality*

$$\int_a^b p(u) \mathrm{d}u \leqslant \int_a^b p(u) \ln f(u) \mathrm{d}u \tag{5}$$

holds, then

$$\left(x \int_a^b p(u)[f(u)]^{\ln x} du, x^{\frac{\int_a^b p(u) du}{\int_a^b p(u) \ln f(u) du}} \right) \tag{6}$$

is a Steffensen pair on $[1, +\infty)$.

(ii) *If $f(u) \geqslant 1$ and inequality (5) holds, then*

$$\left(x \int_a^b p(u)[f(u)]^{\ln x} [\ln f(u)]^n du, x^{\frac{\int_a^b p(u)[\ln f(u)]^n du}{\int_a^b p(u)[\ln f(u)]^{n+1} du}} \right) \tag{7}$$

are Steffensen pairs on $[1, +\infty)$ for any positive integer n.

Remark 1. This theorem generalizes the Proposition 2 in [2] and the related results in [5].

2. Lemmas

Lemma 1 ([2]). *Let $\psi : [c, \infty) \to [0, \infty)$ be increasing and convex, $c \geqslant 0$. Assume that ψ satisfies $\psi(xy) \geqslant \psi(x)g(y)$ for all $x \geqslant c$ and $y \geqslant 1$, where $g : [1, \infty) \to [0, \infty)$ is strictly increasing. Set $\varphi(x) = x\psi(x)$, $\tau(x) = g^{-1}(x)$, where g^{-1} is the inverse function of g. Then (φ, τ) is a Steffensen pair on $[c, \infty)$.*

Define

$$h(t) = \int_a^b p(u)f^t(u)du, \quad t \in \mathbb{R}, \tag{8}$$

where $p(u)$ is a nonnegative and continuous function, $f(u)$ a positive and continuous function on the interval $[a, b]$, and $a, b \in \mathbb{R}$.

It is clear [4, 7] that, if $f(u) \geqslant 1$ on $[a, b]$, then

$$h^{(n)}(t) = \int_a^b p(u)f^t(u)[\ln f(u)]^n du \geqslant 0, \tag{9}$$

that is, $h(t)$ is an absolutely monotonic function, see [3, 4].

By the Cauchy-Schwarz-Buniakowski inequality, it is easy to obtain

Lemma 2. *For $n \geqslant 0$, if $f(u) \geqslant 1$ on $[a, b]$, then we have*

$$[h^{(n+1)}(x)]^2 \leqslant h^{(n)}(x)h^{(n+2)}(x), \quad x \in \mathbb{R}. \tag{10}$$

Let $a, b, r, s \in \mathbb{R}$, let $p \not\equiv 0$ be a nonnegative and integrable function and f a positive and integrable function on the interval between a and b. Then the generalized weighted mean values $M_{p,f}(r, s; a, b)$ of the function f with weight p and two parameters r and s are defined in [4] by

$$M_{p,f}(r, s; a, b) = \left(\frac{\int_a^b p(u)f^s(u)\mathrm{d}u}{\int_a^b p(u)f^r(u)\mathrm{d}u}\right)^{1/(s-r)} = \left(\frac{h(s)}{h(r)}\right)^{1/(s-r)}, \qquad (r - s)(a - b) \neq 0; \tag{11}$$

$$M_{p,f}(r, r; a, b) = \exp\left(\frac{\int_a^b p(u)f^r(u)\ln f(u)\mathrm{d}u}{\int_a^b p(u)f^r(u)\mathrm{d}u}\right) = \exp\left(\frac{h'(r)}{h(r)}\right), \qquad a - b \neq 0; \tag{12}$$

$$M(r, s; a, a) = f(a).$$

From the Cauchy-Schwarz-Buniakowski inequality again and standard argument, we have

Lemma 3 ([7]). *The generalized weighted mean values $M_{p,f}(r, s; a, b)$ are increasing with both r and s for any given continuous nonnegative weight p and continuous positive function f.*

Lemma 4. *For $n \geqslant 0$ and $x \geqslant 0$, if*

$$\int_a^b p(u)\mathrm{d}u \leqslant \int_a^b p(u)\ln f(u)\mathrm{d}u, \tag{13}$$

then we have

$$h^{(n+1)}(x) \geqslant h^{(n)}(x). \tag{14}$$

Proof. By Lemma 3, the mean values

$$\left(\frac{h(x+y)}{h(x)}\right)^{1/y}$$

are increasing with respect to x and y, then the function

$$F(x) = \frac{h(x+y)}{h(x)}$$

is increasing with x for fixed $y \geqslant 0$. Therefore

$$F'(x) = \frac{h'(x+y)h(x) - h(x+y)h'(x)}{[h(x)]^2} \geqslant 0.$$

Hence, the inequality

$$h'(x+y)h(x) - h(x+y)h'(x) \geqslant 0 \tag{15}$$

holds for all x and all $y \geqslant 0$.

Note that the inequality (15) can also be obtained from the Lemma in [6].

Taking $x = 0$ in inequality (15), we obtain

$$h'(y)h(0) - h(y)h'(0) \geqslant 0 \tag{16}$$

for all $y \geqslant 0$, and

$$h(0) = \int_a^b p(u)\mathrm{d}u, \tag{17}$$

$$h'(0) = \int_a^b p(u)\ln f(u)\mathrm{d}u. \tag{18}$$

Since inequality (13) means that $h'(0) \geqslant h(0)$, thus $h'(y) \geqslant h(y)$ for all $y \geqslant 0$.

By mathematical induction, assume that $h^{(n+1)}(x) \geqslant h^{(n)}(x)$ for $n \geqslant 2$ and $x \geqslant 0$. Then, from Lemma 2, we obtain

$$h^{(n)}(x)h^{(n+1)}(x) \leqslant [h^{(n+1)}(x)]^2 \leqslant h^{(n)}(x)h^{(n+2)}(x), \tag{19}$$

therefore

$$h^{(n+1)}(x) \leqslant h^{(n+2)}(x).$$

The proof is completed. □

3. New Steffensen Pairs

Now we give a proof of Theorem 1.

Set $\psi(x) = h^{(n)}(\ln x)$ for $x \geqslant 1$ and $n \geqslant 0$. Direct computation yields that $\psi'(x) = \frac{h^{(n+1)}(\ln x)}{x} > 0$ and $\psi''(x) = \frac{h^{(n+2)}(\ln x) - h^{(n+1)}(\ln x)}{x^2} \geqslant 0$. Hence $\psi(x)$ is increasing and convex.

Since $f(u) \geqslant 1$, for $n \geqslant 1$, by Lemma 3, we have

$$\frac{h^{(n)}(x+y)}{h^{(n)}(x)} = \frac{\int_a^b p(u)[f(u)]^{x+y}[\ln f(u)]^n du}{\int_a^b p(u)[f(u)]^x[\ln f(u)]^n du} \geqslant \exp\left(y \cdot \frac{\int_a^b p(u)[\ln f(u)]^{n+1} du}{\int_a^b p(u)[\ln f(u)]^n du}\right). \tag{20}$$

Therefore, for $x, y \geqslant 1$,

$$\frac{\psi(xy)}{\psi(x)} = \frac{h^{(n)}(\ln(xy))}{h^{(n)}(\ln x)} = \frac{h^{(n)}(\ln x + \ln y)}{h^{(n)}(\ln x)} \geqslant y^{\frac{\int_a^b p(u)[\ln f(u)]^{n+1} du}{\int_a^b p(u)[\ln f(u)]^n du}}. \tag{21}$$

Let $g(x) = x^{\frac{\int_a^b p(u)[\ln f(u)]^{n+1} du}{\int_a^b p(u)[\ln f(u)]^n du}}$ for $x \geqslant 1$, then $g^{-1}(x) = x^{\frac{\int_a^b p(u)[\ln f(u)]^n du}{\int_a^b p(u)[\ln f(u)]^{n+1} du}}$, $x \in [1, +\infty)$.

By Lemma 1, (φ, τ), where $\varphi(x) = x\psi(x) = xh^{(n)}(\ln x) = x\int_a^b p(u)[f(u)]^{\ln x}[\ln f(u)]^n du$ and $\tau(x) = x^{\frac{\int_a^b p(u)[\ln f(u)]^n du}{\int_a^b p(u)[\ln f(u)]^{n+1} du}}$ for $x \geqslant 1$ and $n \geqslant 1$, are Steffensen pairs on $[1, +\infty)$ for any given $n \geqslant 1$.

For $n = 0$, by Lemma 3, we have

$$\frac{h(x+y)}{h(x)} = \frac{\int_a^b p(u)[f(u)]^{x+y} du}{\int_a^b p(u)[f(u)]^x du} \geqslant \exp\left(y \cdot \frac{\int_a^b p(u)\ln f(u) du}{\int_a^b p(u) du}\right). \tag{22}$$

Therefore, for $x, y \geqslant 1$,

$$\frac{\psi(xy)}{\psi(x)} = \frac{h(\ln(xy))}{h(\ln x)} = \frac{h(\ln x + \ln y)}{h(\ln x)} \geqslant y^{\frac{\int_a^b p(u)\ln f(u) du}{\int_a^b p(u) du}}. \tag{23}$$

Let $g(x) = x^{\frac{\int_a^b p(u)\ln f(u) du}{\int_a^b p(u) du}}$ for $x \geqslant 1$, then $g^{-1}(x) = x^{\frac{\int_a^b p(u) du}{\int_a^b p(u)\ln f(u) du}}$, $x \in [1, +\infty)$.

By Lemma 1, (φ, τ), where $\varphi(x) = x\psi(x) = xh(\ln x) = x\int_a^b p(u)[f(u)]^{\ln x} du$ and $\tau(x) = x^{\frac{\int_a^b p(u) du}{\int_a^b p(u)\ln f(u) du}}$ for $x \geqslant 1$ and $n \geqslant 1$, are Steffensen pairs on $[1, +\infty)$ for any given $n \geqslant 1$.

The proof is complete.

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