On Convergence of Quadrature Methods for the Lipschitz-Continuous Functions

ON CONVERGENCE OF QUADRATURE METHODS FOR THE LIPSCHITZ-CONTINUOUS FUNCTIONS

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Abstract. In this article we present a new approach to quadrature methods where any quadrature formula is generated by a discontinuous function whose jumps are quadrature weights. The quadrature error is estimated for Lipschitz-continuous functions with the constant which can not be replaced by a smaller one.

1. Introduction

Let I be a linear functional $I : C^0 [a, b] \to \mathbb{R}$ defined by formula

$$I(f) = \int_a^b w(t)f(t)dt, \tag{1.1}$$

where $w \in C^0 [a, b] \cap L_1 [a, b]$.

Then each formula from the sequence

$$I_n(f) = -\sum_{k=0}^{n} f(x_k)s_{nk}, \tag{1.2}$$

defines a *numerical quadrature method (rule):*

$$I(f) \equiv \int_a^b f(t)w(t)\,dt \approx -\sum_{k=0}^{n} f(x_k)s_{nk} \equiv I_n(f), \tag{1.3}$$

where $a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ and the *weights* s_{nk} and *nodes* $x_k^{(n)}$ are real numbers.

The main problem in numerical integration is finding a system of nodes and weights such that the *quadrature error*

$$E_n(f) = I(f) - I_n(f) \to 0 \quad \text{as} \quad n \to \infty.$$

We say in such cases that the quadrature rule (1.2) *converges.* It is also important to estimate the quadrature error. Usually, such estimations are given using the high order derivatives of function f, for example, the error in Simpson's rule uses the fourth order derivative. The estimation of the quadrature error is more difficult if the integrand is only continuous.

The first article concerning the convergence of quadrature rules for continuous functions was written in 1884 by Stieltjes [1], where he proved the convergence of

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Gaussian quadratures for any continuous function but he doubted on the convergence of Newton-Cotes quadratures. The next step was made by G. Pòlya in 1933, [2], where he proved the sufficient conditions of convergence of quadrature rules for continuous and even for Riemann integrable functions. The sequence $I_n(f)$ for continuous functions converges if and only if it converges for every polynomial and if

$$\sum_{k=0}^{\infty} |s_{nk}| < C < \infty, (n = 1, 2, ...). \tag{1.4}$$

The necessity of the Pòlya condition follows from the *Uniform Boundedness Principle* and the proof of this can be found in almost any text of functional analysis, see for example [3] . In the same paper Pòlya proved that the Newton-Cotes method diverges for big n and that the condition (1.4) does not hold, since for large n:

$$|s_{2n,n}| > \frac{2^{2n}}{n^3}.$$

If n is odd then s_{nk} also tends to 0 as $n \to \infty$ (see for example [4], [5] or [6]).

As mentioned above, there are not many convenient estimations of quadrature errors for the continuous functions. One of such estimation can be given for Gaussian quadrature for Lipschitz continuous functions. This estimation is based on the Jackson theorem and has the form [7]:

$$|E_n(f)| \leq \frac{2Ld_0}{2n-1} \int_a^b |w(x)|\, dx,$$

where L is the Lipschitz constant of function f and the constant d_0 is not always known.

Another estimation of the quadrature error for a composite quadrature for a continuous function can be found in [6] and is directly proportional to the modulo of continuity of the function f.

In the recent articles [8] and [9], some convenient methods for evaluation of the rate of convergence error estimations were obtained for the functions from $C^1[a, b]$.

In what follows, we establish some new error estimates which are sometimes exact and convenient for evaluation of the quadrature error for the Lipschitz continuous functions using a new approach to numerical integration. The result can be generalized for continuous functions and some types of cubature formulae.

2. Estimation of the Error for an Arbitrary Quadrature Rule for a Lipschitz-Continuous Function

Let us consider the semi-normed space $\mathbf{Lip^1}\,[a, b]$ of all mappings $f : [a, b] \to \mathbb{R}$ with the semi-norm

$$\|f\|_{\mathbf{Lip}} = \sup_{x \neq y, x, y \in [a,b]} \frac{|f(x) - f(y)|}{|x - y|} < \infty. \tag{2.1}$$

To transform this space into a normed space we factorize it by the subspace of constant functions.

We conserve for this new space with the norm defined by (2.1) the same notation $\mathbf{Lip^1}\,[a, b]$. For simplicity we consider a positive representative f of $\mathbf{Lip^1}\,[a, b]$.

Lemma 1. *If f is a Lipschitz-continuous function and u is Riemann integrable on $[a,b]$, then*

$$\left|\int_a^b u(x)df(x)\right| \leq \|f\|_{\mathbf{Lip}} \cdot \int_a^b |u(t)|dt. \tag{2.2}$$

Proof. Firstly, note that from the theory of Stieltjes integral, it follows that the integral $\int_a^b u(t)df(t)$ exists.

If $\Delta_n = \left\{a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b\right\}$ is a sequence of partitions of $[a,b]$ with $\lambda(\Delta_n) = \max\limits_{i=1,...,n}(x_i^{(n)} - x_{i-1}^{(n)}) \to 0$ (for $n \to \infty$) and $\xi_i^{(n)} \in [x_{i-1}^{(n)}, x_i^{(n)}]$ then

$$\begin{aligned}
&\left|\int_a^b u(x)df(x)\right| \\
=\ &\left|\lim_{\lambda(\Delta_n)\to 0}\sum_{i=1}^n u\left(\xi_i^{(n)}\right)\left[f\left(x_i^{(n)}\right) - f\left(x_{i-1}^{(n)}\right)\right]\right| \\
\leq\ &\lim_{\lambda(\Delta_n)\to 0}\sum_{i=1}^n \left|\left(\xi_i^{(n)}\right)\right| \frac{\left|f\left(x_i^{(n)}\right) - f\left(x_{i-1}^{(n)}\right)\right|}{\left|x_i^{(n)} - x_{i-1}^{(n)}\right|}\left(x_i^{(n)} - x_{i-1}^{(n)}\right) \\
\leq\ &\|f\|_{\mathbf{Lip}} \cdot \lim_{\nu(\Delta_n)\to 0}\sum_{i=1}^n \left|u(\xi_i^{(n)})\right|\left(x_i^{(n)} - x_{i-1}^{(n)}\right) = \|f\|_{\mathbf{Lip}} \cdot \int_a^b |u(t)|dt,
\end{aligned}$$

and the inequality (2.2) is proved. ∎

Let $u(t,\mathbf{x})$ be a Riemann integrable function on $[a,b]$ depending on vector parameter $\mathbf{x} \in \mathbb{R}^{n+1}$ and let $f \in \mathbf{Lip^1}\,[a,b]$.

Let $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be an arbitrary subdivision from Δ_n. We denote by $h_k = x_k - x_{k-1}$, $(k = 1,2,...,n-1)$ the steps corresponding to this subdivision.

Consider the following function

$$\rho(t) = c + \int_a^t w(\tau)\,d\tau, \tag{2.3}$$

where $w \in C^0[a,b] \cap L_1[a,b]$.

For $n = 1,2,...$, consider the following function

$$u_n(t,\mathbf{x}) = \chi(a,b)\left[\rho(t) + \vartheta_n(t,\mathbf{x})\right], \tag{2.4}$$

where $\mathbf{x} = (x_0, x_1, ..., x_n)$, $\chi(a,b) = \theta^+(t-a)\theta^-(t-b)$ is the characteristic function of segment $[a,b]$

$$\theta^+(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases}$$

is Heaviside function,

$$\theta^-(t) = \begin{cases} 1 & \text{if} \quad t \leq 0 \\ 0 & \text{if} \quad t > 0 \end{cases}$$

and

$$\vartheta_n(t,\mathbf{x}) = \sum_{k=0}^{n-1} s_{nk}\theta^+(t - x_k). \tag{2.5}$$

The numbers s_{nk} $(k = 0, 1, ..., n)$ represent the jumps of $u_n(t, \mathbf{x})$ at points $x_0, x_1, ..., x_n$ that is for $k = 0, 1, ..., n$

$$s_{nk} = u_n(x_k + 0, \mathbf{x}) - u_n(x_k - 0, \mathbf{x}). \tag{2.6}$$

We suppose that

$$\sum_{k=0}^{n} s_{nk} + (\rho(b) - \rho(a)) = 0. \tag{2.7}$$

The derivative of $u_n(t, \mathbf{x})$ has the form:

$$\frac{du_n(t, \mathbf{x})}{dt} = \chi(a, b)\, w(t) + \sum_{k=0}^{n} s_{nk} \delta(t - x_k).$$

Consider the following Riemann-Stieltjes integral

$$\int_{-\infty}^{\infty} u_n(t, \mathbf{x}) df(t) \equiv \int_a^b u_n(t, \mathbf{x}) df(t).$$

Integrating by parts we obtain

$$\int_{-\infty}^{\infty} u_n(t, \mathbf{x}) df(t) = -\int_{-\infty}^{\infty} f(t) du_n(t, \mathbf{x}) = -\int_a^b f(t) w(t)\, dt - \sum_{k=0}^{n} f(x_k) s_{nk}$$

or

$$\int_{-\infty}^{\infty} u_n(t, \mathbf{x}) df(t) = -\int_a^b f(t) w(t)\, dt - \sum_{k=0}^{n} f(x_k) s_{nk}. \tag{2.8}$$

Theorem 1. *If $f \in \mathbf{Lip^1}[a, b]$ and u_n is defined by (2.4), then the error of the quadrature rule*

$$\int_a^b f(t) w(t)\, dt \approx -\sum_{k=0}^{n} f(x_k) s_{nk} \tag{2.9}$$

can be evaluated using the inequality

$$|E_n(f)| = \left| \int_a^b f(t) w(t)\, dt + \sum_{k=0}^{n} f(x_k) s_{nk} \right| \leq \|f\|_{\mathbf{Lip}} \int_a^b |u_n(t, x)| dt, \tag{2.10}$$

i.e., the quadrature rule converges if

$$\int_{-\infty}^{\infty} |u_n(t, \mathbf{x})|\, dt \to 0 \quad as \quad n \to \infty. \tag{2.11}$$

Proof. Using (2.8) and Lemma 1, we obtain

$$\left| \int_a^b f(t) w(t)\, dt + \sum_{k=0}^{n} f(x_k) s_{nk} \right| = \left| \int_a^b u_n(t, \mathbf{x}) df(x) \right| \leq \|f\|_{\mathbf{Lip}} \int_a^b |u_n(t, x)| dt,$$

which tends to zero as n tends to infinity. ∎

We call the function $u_n(t, x)$ a generating function corresponding to the quadrature rule (2.9). For the interpolatory quadrature formulae this function is the Peano Kernel.

Remark 1. *The constant $\|f\|_{\mathbf{Lip}}$ in (2.10) is sharp in the sense that it cannot be replaced by a smaller one. In fact, consider a quadrature rule defined by the function*

$$u_n(t,\mathbf{x}) = t + \frac{1}{2n} + \sum_{k=0}^{n-1} \left(-\frac{1}{n}\right) \theta^+(t - x_k),$$

where $x_k = \dfrac{k}{n}, 0 \le t \le 1$, and $w = 1$. We denote by $Eu_n(t,x)$ the periodical extension of $u_n(t,\mathbf{x})$ from $[0,1]$ on whole t-axes and apply the quadrature rule to the function

$$f(t) = \left| Eu_n\left(t - \frac{1}{2n}, \mathbf{x}\right)\right|.$$

Here $s_{nk} = -\frac{1}{n}$.

It is easy to check that the sum

$$\sum_{k=0}^{n} f(x_k) s_{nk} = 0.$$

Also, $\|f\|_{\mathbf{Lip}} = 1$ and we can rewrite (2.10) as follows

$$\left|\int_0^1 f(t)dt\right| \le \int_0^1 |u_n(t,\mathbf{x})|dt,$$

or by definition of $f(t)$ we have

$$\left|\int_0^1 \left|Eu_n(t - \frac{1}{2n}, \mathbf{x})\right| dt\right| \le \int_0^1 |u_n(t,\mathbf{x})|dt,$$

but the left-hand side and the right-hand side of the last inequality coincide.

Remark 2. *It is possible to prove that the condition (2.11) is, in certain sense, necessary for the convergence of the quadrature rule. To prove it, we can consider the function $u_n(t,\mathbf{x})$ as an element of the space $\left(\mathbf{Lip}^{\mathbf{1}}[a,b]\right)^*$ dual to $\boldsymbol{Lip}^{\mathbf{1}}[a,b]$ (this space was introduced by Kantorovich and Rubinstein* [3]*) and apply the uniform boundedness principle for the weak* convergence (for example in the form* [3] *or* [10]*) to get that the sequence $\{\int_a^b |u_n(t,\mathbf{x})|\}$ is bounded. Under some additional assumptions, this sequence tends to zero.*

Remark 3. *The meaning of Theorem 1 is that:*
If for the Lipschitz-continuous functions

$$\int_0^1 |u_n(t,\mathbf{x})|dt \to 0 \text{ as } n \to \infty,$$

then the error functional tends strongly to zero. It is not true for continuous functions. Namely, the error functional never tends strongly to zero (for all sets of nodes and weights). In fact, if $f(x) \in C[a,b]$ then its dual space is the space of functions with the bounded variation. Moreover,

$$\left|\int_a^b u_n(t,\mathbf{x}) df(x)\right| \le \|u_n\|_{C^*} \|f\|_C,$$

where

$$\|u_n\|_{C^*} = Var[u_n] = \int_a^b w(t)\, dt + \sum_{k=0}^{n} s_{nk}$$

which never tends to zero.

3. Application of the Grüss' Inequality for the Riemann-Stieltjes Integral.

Theorem 1 gives the possibility to evaluate the error of the quadrature method for Lipschitz-continuous functions. However, the estimation of the value $\int_a^b |u_n(t, \mathbf{x})|dt$ sometimes is not simple. Therefore, we suggest another possibility in evaluating the quadrature error of a quadrature rule based on the inequalities of Grüss type. These estimations are not sharp but they are much simpler to check.

The Grüss inequality was proved in 1935. It establishes a connection between the integral of the product of two functions and the product of the integrals:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma),$$

provided f and g are two integrable functions on $[a, b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi \text{ and } \gamma \leq g(x) \leq \Gamma \text{ for all } x \in [a, b].$$

The constant $\frac{1}{4}$ is the *best possible one* and is achieved for

$$f(x) = g(x) = sgn\left(x - \frac{a+b}{2}\right).$$

For other similar results, generalizations for positive linear functionals, discrete versions, determinantal versions etc. see the Chapter X of the book [11], where further references are given.

In the following we point out a Grüss' type inequality for the Riemann-Stieltjes integral and apply it for the convergence of quadrature methods (see also [12]).

Let $f : [a, b] \to \mathbb{R}$ be L-Lipschitzian on $[a, b]$, i.e. (2.1) holds. In other words,

$$|f(x) - f(y)| \leq L|x - y|, \quad \text{for all } x, y \in [a, b], \tag{3.1}$$

where $L = \|f\|_{\mathbf{Lip}}$.

The following result of Grüss' type holds:

Lemma 2. *Let a and b be finite real numbers and let $f, u : [a, b] \to \mathbb{R}$ be such that $f \in \boldsymbol{Lip^1}[a, b]$ and u is Riemann integrable on $[a, b]$. Assume that there exist the real numbers m, M with*

$$m \leq u(x) \leq M, \quad \textit{for all } x \in [a, b]. \tag{3.2}$$

Then the following inequality holds

$$\left|\int_a^b u(t)df(t) - \frac{f(b) - f(a)}{b-a}\int_a^b u(t)dt\right| \leq \frac{L}{2}(M - m)(b - a) \tag{3.3}$$

and the constant $\frac{1}{2}$ is sharp.

Lemma 2 is proved in [12].

Corollary 1. *Let* $f \in \boldsymbol{Lip^1}\,[a,b]$ *and* $u_n(t,\mathbf{x})$ *is the function defined by (2.4). Then the following inequality holds*

$$\left|\int_a^b f(t)w\,(t)\,dt + \sum_{k=0}^{n} f(x_k)s_{nk} + (f(b)-f(a))\bar{u}_n(\mathbf{x})\right| \leq \frac{L}{2}(M_n - m_n)(b-a), \tag{3.4}$$

where m_n *and* M_n *are lower and upper bounds of the function:*

$$m_n \leq u_n(t,\mathbf{x}) \leq M_n \tag{3.5}$$

and

$$\bar{u}_n(\mathbf{x}) = \frac{1}{b-a}\int_a^b u_n(t,\mathbf{x})dt.$$

Now, it is clear that the assumption (2.7) is natural due to inequality (3.4). In fact, if $f(t) \equiv 1$, then the Lipschitz constant $L = 0$ and (3.4) has the form

$$\left|-\int_a^b w\,(t)\,dt - \sum_{k=0}^{n} s_{nk}\right| \leq 0 \Rightarrow \int_a^b w\,(t)\,dt + \sum_{k=0}^{n} s_{nk} = 0,$$

which is equivalent to (2.7).

Corollary 2. *Let the functions* u *and* f *satisfy the conditions of Lemma 2 and the function* u *be such that*

$$\bar{u} \equiv \frac{1}{b-a}\int_a^b u(t,\mathbf{x})dt = 0,$$

then the following inequality holds:

$$\left|\int_a^b u(t)df(t)\right| \leq L\sqrt{-mM}(b-a). \tag{3.6}$$

Proof. We can write inequality (2.2) as

$$\left|\int_a^b u(t)df(t)\right| \leq L\left\|u(t)-\bar{u}\right\|_1 \leq L\sqrt{b-a}\left\|u\right\|_2 \tag{3.7}$$

and since a simple computation shows that

$$\begin{aligned}\frac{1}{b-a}\left\|u\right\|^2 &= -mM - \frac{1}{b-a}\int_a^b (M-u\,(t))\,(u\,(t)-m)\,dt \\ &\leq -mM,\end{aligned}$$

that is

$$\left\|u\right\|_2 \leq \sqrt{b-a}\sqrt{-mM}, \tag{3.8}$$

then from (3.7) and (3.8) we get (3.6). ∎

We shall prove below that the constant c in definition (2.3) of generating function can be chosen such that

$$\bar{u}_n(\mathbf{x}) \equiv \frac{1}{b-a}\int_a^b u_n(t,\mathbf{x})dt = 0. \tag{3.9}$$

We suppose in what follows that (3.9) is valid.

Theorem 2. *Let $u_n(t,\mathbf{x})$ be a function defined by (2.4) satisfying (3.9), and let f be a L-Lipschitzian on $[a,b]$ function and the function*

$$w \in C^0[a,b] \cap L_1[a,b].$$

Then the quadrature rule (2.9) converges either if

$$\int_a^b |u_n(t,\mathbf{x})|dt \to 0 \quad as \quad n \to \infty,$$

$$or\ if\ \sqrt{-m_n M_n} \to 0 \quad as \quad n \to \infty,$$

$$or\ if\ M_n - m_n \to 0 \quad as \quad n \to \infty.$$

More precisely, the following inequalities are true:

$$\begin{aligned} &\left| \int_a^b f(t) w(t)\, dt + \sum_{k=0}^{n} f(x_k) s_{nk} \right| \\ \leq\ & L \int_a^b |u_n(t,\mathbf{x})|dt \leq L\sqrt{-m_n M_n}(b-a) \\ \leq\ & \frac{L}{2}(M_n - m_n)(b-a). \end{aligned} \tag{3.10}$$

Remark 4. *The second and the third inequalities in (3.10) are sharp, for example*

$$\int_a^b |u_n(t,\mathbf{x})|dt = \sqrt{-m_n M_n}(b-a)$$

for

$$u_n = \chi(0,1)\left[-\frac{1}{2} + \theta^+(t-\frac{1}{2})\right]$$

for $a=0,\ b=1$ and the last inequality becomes an equality if $-m_n = M_n$.

Some applications of formula (3.3) which are different from those presented in this article can be found in [12].

We are going to prove now that the constant c can be chosen so that (3.9) would be true. We consider

$$(b-a)\,\bar{u}_n(\mathbf{x}) = \int_a^b u_n(t,\mathbf{x})dt = \int_a^b \rho(t)\, dt + \int_a^b \vartheta_n(t,\mathbf{x})dt. \tag{3.11}$$

To evaluate the last integral we use the definition (2.5). After elementary computation we obtain

$$\begin{aligned} \int_a^b \vartheta_n(t,\mathbf{x})dt &= \int_a^b \sum_{k=0}^{n-1} s_{nk}\theta^+(t-x_k)dt \\ &= \sum_{j=1}^{n} h_j \sum_{k=0}^{j-1} s_{nk} = \sum_{k=0}^{n} s_{nk}\,(b-x_k) \end{aligned} \tag{3.12}$$

and

$$\int_a^b \rho(t,\mathbf{x})dt = c(b-a) + \int_a^b (b-\tau)w(\tau)d\tau. \tag{3.13}$$

Thus

$$\bar{u}_n(\mathbf{x}) = c + \int_a^b \frac{b-\tau}{b-a} w(\tau)\, d\tau + \sum_{k=0}^{n} \frac{b-x_k}{b-a} s_{nk} \tag{3.14}$$

and if

$$c=-\int_a^b \frac{b-\tau}{b-a}w(\tau)\,d\tau-\sum_{k=0}^{n}\frac{b-x_k}{b-a}s_{nk} \tag{3.15}$$

then (3.9) holds.

The following are some weight functions that are of most interest in the quadrature formulae theory.

Example 1. *Interpolatory Method with Legendre weight function $w(t)=1$. For example, in the closed Newton-Cotes scheme the jumps are defined by formulae*

$$s_{nk}=-\frac{(b-a)(-1)^{n-k}}{n\,(k!)\,(n-k)!}\int_0^n \frac{\prod_{j=0}^{n}(t-j)}{t-k}\,dt. \tag{3.16}$$

Here,

$$\begin{aligned} c &= -\int_a^b \frac{b-\tau}{b-a}d\tau-\sum_{k=0}^{n}\frac{b-x_k}{b-a}s_{nk} \\ &= -\frac{b-a}{2}-\sum_{k=0}^{n}\frac{b-x_k}{b-a}s_{nk} \\ &= -\frac{b-a}{2}-\frac{b}{b-a}\sum_{k=0}^{n}s_{nk}+\frac{1}{b-a}\sum_{k=0}^{n}x_k s_{nk}. \end{aligned}$$

Since all interpolatory methods for $n\geq 1$ are exact for $f(t)\equiv 1$ and $f(t)=t$, then

$$\sum_{k=0}^{n}s_{nk}=-(b-a) \text{ and } \sum_{k=0}^{n}x_k s_{nk}=-\left(\frac{b^2-a^2}{2}\right)$$

and we have

$$c=-\frac{b-a}{2}+b-\frac{b+a}{2}=0.$$

Thus, in this case

$$u_n(t,\mathbf{x})=\chi(a,b)\left[\rho(t)+\vartheta(t,\mathbf{x})\right]=u_n(t,\mathbf{x})=\chi(a,b)\left[t-a+\vartheta(t,\mathbf{x})\right]$$

Example 2. *Consider the Chebyshev weight function*

$$\begin{aligned} w(t) &= \frac{1}{\sqrt{1-x^2}}, a=-1, b=1, \\ s_{nk} &= \frac{\pi}{n}, k=1,2,...,n, x_k=\cos\left[\frac{(2k-1)\pi}{2n}\right]. \end{aligned}$$

Since

$$\sum_{k=1}^{n}\cos\left[\frac{(2k-1)\pi}{2n}\right]=0$$

and

$$\int_{-1}^{-1}\frac{1-\tau}{2}\frac{1}{\sqrt{1-\tau^2}}d\tau=\frac{1}{2}\pi,$$

then

$$c=-\frac{\pi}{2}+\frac{\pi}{2n}\sum_{k=1}^{n}(1-x_k)=-\frac{\pi}{2n}\sum_{k=1}^{n}x_k=0.$$

In conclusion, we know the values of $\int_0^1 |u_n(t,\mathbf{x})|\,dt$ and $\dfrac{M_n - m_n}{2}$ for the generating functions of closed Newton-Cotes Method, Gauss-Legendre Method on the interval $[0,1]$ and for the simplest generating "saw"-function

$$\begin{aligned} u_n(t,\mathbf{x}) &= t + \frac{1}{2n} - \frac{1}{n}\theta^+(t) - \frac{1}{n}\theta^+\left(t - \frac{1}{n}\right) - \dots \\ &\quad -\frac{1}{n}\theta^+\left(t - \frac{n-1}{n}\right). \end{aligned}$$

Denote $B_n = \dfrac{M_n - m_n}{2}$; $S_n = \int_0^1 |u_n(t,\mathbf{x})|\,dt$. With these notations, we have the table:

	$N-C$	$G-L$	Saw	$N-C$	$G-L$	Saw
n	S_n	S_n	S_n	B_n	B_n	B_n
1	0.2500	0.2500	0.2500	0.5000	0.5000	0.5000
2	0.1667	0.1280	0.1250	0.3333	0.2887	0.2500
3	0.0868	0.0893	0.0833	0.2083	0.2222	0.1667
4	0.0738	0.6900	0.0625	0.1833	0.1700	0.1250
5	0.0537	0.0563	0.0500	0.1340	0.1422	0.1000
6	0.0627	0.0476	0.0417	0.1619	0.1193	0.0834
7	0.0417	0.0412	0.0357	0.1076	0.1045	0.0714
8	0.0845	0.0363	0.0312	0.2050	0.0896	0.0625
9	0.0433			0.1022		
10	0.1493			0.3569		
11	0.0669			0.1595		
12	0.3111			0.7794		
13	0.1270			0.2911		
14	0.7249			1.9520		
15	0.2755					

This table shows that for n not large, no method has advantage in the numerical integration of a Lipschitz-continuous function. For large n the Newton-Cotes method is not good and the Gauss-Legendre and "Saw"- methods have almost the same efficiency.

References

[1] T.I. Stieltjes, *Annales scientifiques de l'Ecole Normale Supérieure*, **3-1**, 1884, 409-426.
[2] G. Pòlya, Über die Konvergenz von Quadraturvervahren, *Math. Zeitschrift*, **37**, 264-286.
[3] L.V. Kantarovich and G.P. Akilov, *Functional Analysis,* Pergamon Press, 1982.
[4] V.I. Krylov, *Approximate calculation of integrals,* McMillan, New York, 1962.
[5] C. Cryer, *Numerical functional Analysis,* Oxford Univ. Press, Oxford, 1982.
[6] K.E. Atkinson, *An Introduction to Numerical Analysis,* Wiley, New York, 1989.
[7] E. Isaacson and H. Keller, *Analysis of Numerical Methods,* Wiley, New York,1966
[8] S.S. Dragomir and S. Wang, An inequalities Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Comp. Math. Applic.* **33** (11) (1997), 15-20.
[9] I. Fedotov, and S.S. Dragomir, An inequality of Ostrowski's type and its applications for Simpson's rule in numerical integration and special means, *Mathematical Inequalities and Applications,* **2** (4) (1999), 491-499.
[10] K. Yosida, *Functional Analysis,* 4th Ed., Berlin, Springer, 1974.

[11] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[12] S.S Dragomir and I. Fedotov, An inequality of Grüss type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4) (1998), 287-292.

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