A Trapezoid Type Inequality for Double Integrals

A TRAPEZOID TYPE INEQUALITY FOR DOUBLE INTEGRALS

N.S. BARNETT AND S.S. DRAGOMIR

Abstract. In this paper, we point out a trapezoid like inequality for double integrals and apply it in connection with the Grüss inequality.

1. Introduction

In the recent papers [1] and [2], the authors proved the following inequality of the Ostrowski type for double integrals.

Theorem 1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous on $[a,b]\times[c,d]$. Then we have the inequality:*

$$\begin{aligned}(1.1)\qquad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right. \\ & \left.-\frac{1}{d-c}\int_c^d f(x,s)\,ds-\frac{1}{b-a}\int_a^b f(t,y)\,dt+f(x,y)\right| \\ \leq & \begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right](b-a)(d-c)\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty \\ \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_\infty([a,b]\times[c,d]); \\ \frac{1}{(q+1)^{\frac{2}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\left[\left(\frac{y-c}{d-c}\right)^{q+1}+\left(\frac{d-y}{d-c}\right)^{q+1}\right]^{\frac{1}{q}} \\ \times[(b-a)(d-c)]^{\frac{1}{q}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p \\ \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1 \\ \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_1([a,b]\times[c,d])\end{cases}\end{aligned}$$

Date: 23rd August, 1999.
1991 *Mathematics Subject Classification.* Primary 26D15, 26D99; Secondary 41A55, 41A99.
Key words and phrases. Trapezoid Inequality, Double Integrals.

for all $(x,y)\in[a,b]\times[c,d]$, *where*

$$\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty := \sup_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|,$$

$$\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p := \left(\int_a^b\int_c^d\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|^p dsdt\right)^{\frac{1}{p}}$$

if $p\in[1,\infty)$.

The best inequality we can get from (1.1) is the one for which $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, obtaining:

Corollary 1. *With the assumptions in Theorem 1, we have the following mid-point type inequality:*

$$\begin{aligned}(1.2)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds\right.\\ &\left.-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\ &\le\begin{cases}\frac{1}{16}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty(b-a)(d-c) & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_\infty([a,b]\times[c,d]);\\[2mm] \frac{1}{4(q+1)^{\frac{2}{q}}}\left[(b-a)(d-c)\right]^{\frac{1}{q}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p & \\ \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_p([a,b]\times[c,d]), & \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \frac{1}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1 & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_1([a,b]\times[c,d]).\end{cases}\end{aligned}$$

For some applications of the above results in Numerical Integration for cubature formulae see [1] and [2].

Another result of Ostrowski type was proved in [4].

Theorem 2. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be a mapping as in Theorem 1. Then we have the inequality:*

$$\begin{aligned}(1.3)\quad &\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|\\ &\le M_1(x)+M_2(y)+M_3(x,y),\end{aligned}$$

where

$$M_1(x)=\begin{cases}\frac{\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]}{b-a}\left\|\frac{\partial f}{\partial t}\right\|_\infty, & \text{if } \frac{\partial f}{\partial t}\in L_\infty([a,b]\times[c,d]);\\[2mm] \frac{\left[\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}}}{(b-a)[(d-c)]^{\frac{1}{p_1}}}\left\|\frac{\partial f}{\partial t}\right\|_{p_1}, & \text{if } \frac{\partial f}{\partial t}\in L_{p_1}([a,b]\times[c,d]),\\ & p_1>1,\ \frac{1}{p_1}+\frac{1}{q_1}=1;\\[2mm] \frac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f}{\partial t}\right\|_1, & \text{if } \frac{\partial f}{\partial t}\in L_1([a,b]\times[c,d]).\end{cases}$$

$$M_2(y) = \begin{cases} \frac{\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]}{d-c}\left\|\frac{\partial f}{\partial s}\right\|_\infty, & \text{if } \frac{\partial f}{\partial s}\in L_\infty([a,b]\times[c,d]); \\ \frac{\left[\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right]^{\frac{1}{q_2}}}{[(b-a)]^{\frac{1}{p_2}}(d-c)}\left\|\frac{\partial f}{\partial s}\right\|_{p_2}, & \text{if } \frac{\partial f}{\partial s}\in L_{p_2}([a,b]\times[c,d]); \\ & p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1; \\ \frac{\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f}{\partial s}\right\|_1, & \text{if } \frac{\partial f}{\partial s}\in L_1([a,b]\times[c,d]); \end{cases}$$

and

$$M_3(x,y) = \begin{cases} \frac{\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty, \\ \text{if } \frac{\partial^2 f}{\partial s\partial t}\in L_\infty([a,b]\times[c,d]); \\ \frac{\left[\frac{(b-x)^{q_3+1}+(x-a)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}\left[\frac{(d-y)^{q_3+1}+(y-c)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3}, \\ \text{if } \frac{\partial^2 f}{\partial s\partial t}\in L_{p_3}([a,b]\times[c,d]),\ p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1; \\ \frac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1, \\ \text{if } \frac{\partial^2 f}{\partial s\partial t}\in L_1([a,b]\times[c,d]); \end{cases}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $\|\cdot\|_p$ $(1\le p\le\infty)$ *are the usual* $p-$*norms on* $[a,b]\times[c,d]$.

Corollary 2. *With the assumptions in Theorem 1, we have the inequality*

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \le \tilde{M}_1+\tilde{M}_2+\tilde{M}_3, \tag{1.4}$$

where

$$\tilde{M}_1 := \begin{cases} \frac{1}{4}(b-a)\left\|\frac{\partial f}{\partial t}\right\|_\infty, & \text{if } \frac{\partial f}{\partial t}\in L_\infty([a,b]\times[c,d]) \\ \frac{1}{2}\left[\frac{(b-a)^{\frac{1}{q_1}}}{(q_1+1)^{\frac{1}{q_1}}(d-c)^{\frac{1}{p_1}}}\right]\left\|\frac{\partial f}{\partial t}\right\|_{p_1}, & \text{if } \frac{\partial f}{\partial t}\in L_{p_1}([a,b]\times[c,d]) \\ & p_1>1,\ \frac{1}{p_1}+\frac{1}{q_1}=1; \\ \frac{1}{2(d-c)}\left\|\frac{\partial f}{\partial t}\right\|_1, & \text{if } \frac{\partial f}{\partial t}\in L_1([a,b]\times[c,d]) \end{cases}$$

$$\tilde{M}_2 := \begin{cases} \frac{1}{4}(d-c)\left\|\frac{\partial f}{\partial s}\right\|_\infty, & \text{if } \frac{\partial f}{\partial t}\in L_\infty([a,b]\times[c,d]) \\ \frac{1}{2}\left[\frac{(d-c)^{\frac{1}{q_2}}}{(q_2+1)^{\frac{1}{q_2}}(b-a)^{\frac{1}{p_2}}}\right]\left\|\frac{\partial f}{\partial s}\right\|_{p_2} & \text{if } \frac{\partial f}{\partial t}\in L_{p_2}([a,b]\times[c,d]) \\ & p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1; \\ \frac{1}{2(b-a)}\left\|\frac{\partial f}{\partial s}\right\|_1, & \text{if } \frac{\partial f}{\partial t}\in L_1([a,b]\times[c,d]) \end{cases}$$

and

$$\tilde{M}_3 := \begin{cases} \frac{1}{16}(b-a)(d-c)\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{\infty}, & \text{if } \frac{\partial^2 f}{\partial t\partial s} \in L_\infty([a,b]\times[c,d]); \\ \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q_3}}(d-c)^{\frac{1}{q_3}}}{(q_3+1)^{\frac{2}{q_3}}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3}, & \text{if } \frac{\partial^2 f}{\partial t\partial s} \in L_{p_3}([a,b]\times[c,d]); \\ & p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1, \\ \frac{1}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{1}, & \text{if } \frac{\partial^2 f}{\partial t\partial s} \in L_1([a,b]\times[c,d]). \end{cases}$$

2. Some Integral Equalities

Let us start with the following integral identity.

Theorem 3. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a continuous mapping on $[a,b]\times[c,d]$ such that $\frac{\partial f(a,\cdot)}{\partial y}$, $\frac{\partial f(b,\cdot)}{\partial y}$ are continuous on $[c,d]$, $\frac{\partial f(\cdot,c)}{\partial x}$, $\frac{\partial f(\cdot,d)}{\partial x}$ are continuous on $[a,b]$ and $\frac{\partial^2 f(\cdot,\cdot)}{\partial x\partial y}$ is continuous on $[a,b]\times[c,d]$. Then we have the identity:*

$$\begin{aligned} &\int_a^b\int_c^d f(x,y)\,dydx+(b-a)\int_c^d\left(y-\frac{c+d}{2}\right)\left[\frac{\frac{\partial f(a,y)}{\partial y}+\frac{\partial f(b,y)}{\partial y}}{2}\right]dy \\ &+(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)\left[\frac{\frac{\partial f(x,c)}{\partial x}+\frac{\partial f(x,d)}{\partial x}}{2}\right]dx \\ =\ &\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c) \\ &+\int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x\partial y}dydx. \end{aligned} \tag{2.1}$$

Proof. A simple integration by parts gives

$$\int_\alpha^\beta h(x)\,dx=\frac{h(\alpha)+h(\beta)}{2}(\beta-\alpha)-\int_\alpha^\beta\left(x-\frac{\alpha+\beta}{2}\right)h'(x)\,dx, \tag{2.2}$$

provided that $h:[\alpha,\beta]\to\mathbb{R}$ is absolutely continuous on $[\alpha,\beta]$.

Using (2.2), we can write:

$$\int_a^b f(x,y)\,dx=(b-a)\frac{f(a,y)+f(b,y)}{2}-\int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f(x,y)}{\partial x}dx \tag{2.3}$$

for all $y\in[c,d]$.

Integrating (2.3) on the interval $[c,d]$, we obtain

$$\begin{aligned} \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy &= \frac{1}{2}(b-a)\left[\int_c^d f(a,y)\,dy+\int_c^d f(b,y)\,dy\right] \\ &\quad-\int_c^d\left(\int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f(x,y)}{\partial x}dx\right)dy. \end{aligned}$$

Using Fubini's theorem, we can state:

$$\int_a^b \int_c^d f(x,y)\,dydx = \frac{1}{2}(b-a)\left[\int_c^d f(a,y)\,dy + \int_c^d f(b,y)\,dy\right] - \int_a^b \left(x - \frac{a+b}{2}\right)\left(\int_c^d \frac{\partial f(x,y)}{\partial x}dy\right)dx. \tag{2.4}$$

By the identity (2.2), we can also state:

$$\int_c^d f(a,y)\,dy = \frac{f(a,c)+f(a,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(a,y)}{\partial y}dy, \tag{2.5}$$

$$\int_c^d f(b,y)\,dy = \frac{f(b,c)+f(b,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(b,y)}{\partial y}dy, \tag{2.6}$$

and

$$\int_c^d \frac{\partial f(x,y)}{\partial x}dy = \frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x} + \frac{\partial f(x,d)}{\partial x}\right](d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x \partial y}dy. \tag{2.7}$$

Now, using (2.4) and (2.5)-(2.7), we have successively

$$\begin{aligned}
&\int_a^b \int_c^d f(x,y)\,dydx \\
= \;& \frac{1}{2}(b-a)\left[\frac{f(a,c)+f(a,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(a,y)}{\partial y}dy\right. \\
&\left. + \frac{f(b,c)+f(b,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(b,y)}{\partial y}dy\right] \\
& - \int_a^b \left(x - \frac{a+b}{2}\right)\left[\frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x} + \frac{\partial f(x,d)}{\partial x}\right](d-c)\right. \\
&\left. - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x \partial y}dy\right]dx
\end{aligned}$$

$$= \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)$$
$$-\frac{1}{2}(b-a)\int_c^d\left(y-\frac{c+d}{2}\right)\frac{\partial f(a,y)}{\partial y}dy$$
$$-\frac{1}{2}(b-a)\int_c^d\left(y-\frac{c+d}{2}\right)\frac{\partial f(b,y)}{\partial y}dy$$
$$-\frac{1}{2}(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f(x,c)}{\partial x}dx$$
$$-\frac{1}{2}(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f(x,d)}{\partial x}dx$$
$$+\int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x\partial y}dydx$$

and the identity (2.1) is proved. □

The following corollary holds:

Corollary 3. *Let $f,g:[a,b]\to\mathbb{R}$ be differentiable on (a,b) and such that f', $g'\in L_1(a,b)$. Then we have the equality:*

$$(b-a)\int_a^b f(x)g(x)\,dx-\int_a^b f(x)\,dx\int_a^b g(x)\,dx \tag{2.8}$$
$$= \frac{[f(a)-f(b)][g(a)-g(b)]}{4}(b-a)^2$$
$$+\int_a^b\left(x-\frac{a+b}{2}\right)\left(\frac{g(a)+g(b)}{2}-g(x)\right)f'(x)\,dx$$
$$+\int_a^b\left(x-\frac{a+b}{2}\right)\left(\frac{f(a)+f(b)}{2}-f(x)\right)g'(x)\,dx$$
$$-\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx\int_a^b\left(x-\frac{a+b}{2}\right)g'(x)\,dx.$$

Proof. The above identity can be proved by direct computation. We give here a proof based on the previous identity (2.1).

Consider the mapping $h:[a,b]^2\to\mathbb{R}$ given by $h(x,y):=(f(x)-f(y))(g(x)-g(y))$ and write the equality (2.1) for h and the interval $[a,b]^2$ to get

$$\int_a^b\int_a^b h(x,y)\,dxdy+\frac{1}{2}(b-a)\int_a^b\left(y-\frac{a+b}{2}\right)\left[\frac{\partial h(a,y)}{\partial y}+\frac{\partial h(b,y)}{\partial y}\right]dy \tag{2.9}$$
$$+\frac{1}{2}(b-a)\int_a^b\left(x-\frac{a+b}{2}\right)\left[\frac{\partial h(x,a)}{\partial x}+\frac{\partial h(x,b)}{\partial x}\right]dx$$
$$= \frac{h(a,a)+h(a,b)+h(b,a)+h(b,b)}{4}(b-a)^2$$
$$+\int_a^b\int_a^b\left(x-\frac{a+b}{2}\right)\left(y-\frac{a+b}{2}\right)\frac{\partial^2 h(x,y)}{\partial x\partial y}dxdy.$$

Note that

$$\begin{aligned}\frac{\partial h(x,y)}{\partial x} &= f'(x)(g(x)-g(y))+g'(x)(f(x)-f(y))\\ \frac{\partial h(x,y)}{\partial y} &= f'(y)(g(y)-g(x))+g'(y)(f(y)-f(x))\end{aligned}$$

and then

$$\begin{aligned}&\frac{1}{2}\left[\frac{\partial h(a,y)}{\partial y}+\frac{\partial h(b,y)}{\partial y}\right]\\ &= f'(y)\left(g(y)-\frac{g(a)+g(b)}{2}\right)+g'(y)\left(f(y)-\frac{f(a)+f(b)}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\left[\frac{\partial h(x,a)}{\partial x}+\frac{\partial h(x,b)}{\partial x}\right]\\ &= f'(x)\left(g(x)-\frac{g(a)+g(b)}{2}\right)+g'(x)\left(f(x)-\frac{f(a)+f(b)}{2}\right).\end{aligned}$$

In addition, we note that

$$\frac{\partial^2 h(x,y)}{\partial x\partial y}=-f'(x)g'(y)-f'(y)g'(x).$$

We have

$$\begin{aligned}I &:= \frac{1}{2}\int_a^b\left(y-\frac{a+b}{2}\right)\left[\frac{\partial h(a,y)}{\partial y}+\frac{\partial h(b,y)}{\partial y}\right]dy\\ &= \int_a^b\left(y-\frac{a+b}{2}\right)\left[f'(y)\left(g(y)-\frac{g(a)+g(b)}{2}\right)\right.\\ &\quad\left.+g'(y)\left(f(y)-\frac{f(a)+f(b)}{2}\right)\right]dy\\ &= \int_a^b\left(x-\frac{a+b}{2}\right)\left(g(x)-\frac{g(a)+g(b)}{2}\right)f'(x)\,dx\\ &\quad+\int_a^b\left(x-\frac{a+b}{2}\right)\left(f(x)-\frac{f(a)+f(b)}{2}\right)g'(x)\,dx\end{aligned}$$

and, similarly,

$$\frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)\left[\frac{\partial h(x,a)}{\partial x}+\frac{\partial h(x,b)}{\partial x}\right]dx=I.$$

On the other hand,

$$\frac{h(a,a)+h(a,b)+h(b,a)+h(b,b)}{4}=\frac{(f(a)-f(b))(g(a)-g(b))}{2}$$

and

$$\begin{aligned} & \int_a^b \int_a^b \left(x - \frac{a+b}{2}\right)\left(y - \frac{a+b}{2}\right) \frac{\partial^2 h(x,y)}{\partial x \partial y} dxdy \\ = & -\int_a^b \int_a^b \left(x - \frac{a+b}{2}\right)\left(y - \frac{a+b}{2}\right) \left[f'(x) g'(y) + f'(y) g'(x)\right] dxdy \\ = & -2\int_a^b \int_a^b \left(x - \frac{a+b}{2}\right)\left(y - \frac{a+b}{2}\right) f'(x) g'(y) dxdy \\ = & -2\int_a^b \left(x - \frac{a+b}{2}\right) f'(x) dx \int_a^b \left(x - \frac{a+b}{2}\right) g'(x) dx. \end{aligned}$$

Now, by (2.9) (dividing by 2), we get the identity:

$$\begin{aligned} & \frac{1}{2}\int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) dxdy \\ & + \int_a^b \left(x - \frac{a+b}{2}\right)\left(g(x) - \frac{g(a)+g(b)}{2}\right) f'(x) dx \\ & + \int_a^b \left(x - \frac{a+b}{2}\right)\left(f(x) - \frac{f(a)+f(b)}{2}\right) g'(x) dx \\ = & \frac{(f(a) - f(b))(g(a) - g(b))}{4}(b-a)^2 \\ & - \int_a^b \left(x - \frac{a+b}{2}\right) f'(x) dx \int_a^b \left(x - \frac{a+b}{2}\right) g'(x) dx. \end{aligned}$$

As it is well known that

$$\begin{aligned} & \frac{1}{2}\int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) dxdy \\ = & (b-a)\int_a^b f(x) g(x) dx - \int_a^b f(x) dx \int_a^b g(x) dx, \end{aligned}$$

the inequality (2.8) is completely proved. □

3. Some Integral Inequalities for $\|\cdot\|_\infty$ −Norm

The following inequality holds.

Theorem 4. *Let f be as in Theorem 3 and assume that*

$$\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)} \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| < \infty.$$

Then we have the estimation

$$
\begin{aligned}
(3.1)\qquad & \left|\int_a^b\int_c^d f(x,y)\,dydx+(b-a)\int_c^d\left(y-\frac{c+d}{2}\right)f_2(y)\,dy\right. \\
& +(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)f_1(x)\,dx \\
& \left.-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right| \\
\leq\ & \frac{1}{16}(b-a)^2(d-c)^2\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty,
\end{aligned}
$$

where

$$f_1(x):=\frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x}+\frac{\partial f(x,d)}{\partial x}\right],\ x\in[a,b]$$

and

$$f_2(y):=\frac{1}{2}\left[\frac{\partial f(a,y)}{\partial y}+\frac{\partial f(b,y)}{\partial y}\right],\ y\in[c,d].$$

Proof. Using the identity (2.1) and the properties of the integral, we can state:

$$
\begin{aligned}
& \left|\int_a^b\int_c^d f(x,y)\,dydx+(b-a)\int_c^d\left(y-\frac{c+d}{2}\right)f_2(y)\,dy\right. \\
& +(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)f_1(x)\,dx \\
& \left.-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right| \\
\leq\ & \int_a^b\int_c^d\left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right|\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|dydx \\
\leq\ & \left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty\int_a^b\left|x-\frac{a+b}{2}\right|dx\int_c^d\left|y-\frac{c+d}{2}\right|dy \\
=\ & \left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty\frac{(b-a)^2}{4}\cdot\frac{(d-c)^2}{4}
\end{aligned}
$$

and the inequality (3.1) is proved. □

Another inequality which employs the $\|\cdot\|_\infty$ −norm of f_1 and f_2 which can be useful in practice is embodied in the following theorem:

Theorem 5. *Let f be as in Theorem 4 and assume that*

$$\|f_1\|_\infty:=\sup_{x\in[a,b]}|f_1(x)|<\infty,\ \ \|f_2\|_\infty:=\sup_{x\in[a,b]}|f_2(x)|<\infty.$$

Then

$$\left|\int_a^b \int_c^d f(x,y)\,dydx - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right| \tag{3.2}$$
$$\leq \frac{1}{4}(b-a)(d-c)$$
$$\times\left[(b-a)\|f_1\|_\infty + (d-c)\|f_2\|_\infty + \frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_\infty\right].$$

Proof. As in Theorem 4, we have, by the identity (2.1) that

$$\left|\int_a^b \int_c^d f(x,y)\,dydx - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right|$$
$$\leq (b-a)\int_c^d \left|y-\frac{c+d}{2}\right| |f_2(y)|\,dy + (d-c)\int_a^b \left|x-\frac{a+b}{2}\right| |f_1(x)|\,dx$$
$$+\int_a^b\int_c^d \left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right|\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right| dydx$$
$$\leq (b-a)\|f_2\|_\infty \frac{(d-c)^2}{4} + (d-c)\|f_1\|_\infty \frac{(b-a)^2}{4}$$
$$+\frac{(b-a)^2}{4}\cdot\frac{(d-c)^2}{4}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty$$
$$= \frac{1}{4}(b-a)(d-c)$$
$$\times\left[(d-c)\|f_2\|_\infty + (b-a)\|f_1\|_\infty + \frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty\right].$$

Hence, the proof is completed. □

Remark 1. *If we know that*

$$\left\|\frac{\partial f}{\partial x}\right\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial f(x,y)}{\partial x}\right| < \infty$$

and

$$\left\|\frac{\partial f}{\partial y}\right\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial f(x,y)}{\partial y}\right| < \infty,$$

then, obviously

$$\|f_1\|_\infty \leq \left\|\frac{\partial f}{\partial x}\right\|_\infty,\ \|f_2\|_\infty \leq \left\|\frac{\partial f}{\partial y}\right\|_\infty$$

and by (3.2) we deduce

$$\left|\int_a^b \int_c^d f(x,y)\,dydx - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right| \tag{3.3}$$
$$\leq \frac{1}{4}(b-a)(d-c)$$
$$\times\left[(b-a)\left\|\frac{\partial f}{\partial x}\right\|_\infty + (d-c)\left\|\frac{\partial f}{\partial y}\right\|_\infty + \frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty\right].$$

Now, let us recall Grüss' inequality:
If $f,g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$m_1 \le f(x) \le M_1,\quad m_2 \le g(x) \le M_2 \text{ for all } x\in[a,b], \tag{3.4}$$

then we have the inequality:

$$\begin{aligned} &\left|(b-a)\int_a^b f(x)\,g(x)\,dx - \int_a^b f(x)\,dx \int_a^b g(x)\,dx\right| \\ \le\ & \frac{1}{4}(M_2-m_2)(M_1-m_1)(b-a)^2 \end{aligned} \tag{3.5}$$

and the constant $\frac{1}{4}$ is the best possible one.
Using Corollary 3, we can state a similar result.

Theorem 6. *Let f,g be as in Corollary 3 and assume that $f',\ g',\ f,g\in L_\infty[a,b]$. Then we have the inequality:*

$$\begin{aligned} &\left|(b-a)\int_a^b f(x)\,g(x)\,dx - \int_a^b f(x)\,dx\cdot\int_a^b g(x)\,dx\right| \\ \le\ & \frac{1}{4}(b-a)^2\left[\left|(f(a)-f(b))(g(a)-g(b))\right| + \left\|\frac{g(a)+g(b)}{2}-g\right\|\|f'\|_\infty\right. \\ &\left. +\left\|\frac{f(a)+f(b)}{2}-f\right\|_\infty\|g'\|_\infty + \frac{(b-a)^2}{4}\|f'\|_\infty\|g'\|_\infty\right], \end{aligned} \tag{3.6}$$

where, by $\left\|\frac{g(a)+g(b)}{2}-g\right\|_\infty$, we understand

$$\left\|\frac{g(a)+g(b)}{2}-g\right\|_\infty = \sup_{x\in[a,b]}\left|\frac{g(a)+g(b)}{2}-g(x)\right|.$$

Proof. Using the identity (2.8) and the properties of modulus, we have

$$\begin{aligned} &\left|(b-a)\int_a^b f(x)\,g(x)\,dx - \int_a^b f(x)\,dx\int_a^b g(x)\,dx\right| \\ \le\ & \frac{(b-a)^2}{4}\left|(f(a)-f(b))(g(a)-g(b))\right| \\ &+\int_a^b\left|x-\frac{a+b}{2}\right|\left|\frac{g(a)+g(b)}{2}-g(x)\right||f'(x)|\,dx \\ &+\int_a^b\left|x-\frac{a+b}{2}\right|\left|\frac{f(a)+f(b)}{2}-f(x)\right||g'(x)|\,dx \\ &+\int_a^b\left|x-\frac{a+b}{2}\right||f'(x)|\,dx\int_a^b\left|x-\frac{a+b}{2}\right||g'(x)|\,dx \end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{(b-a)^2}{4}\left|\left(f(a)-f(b)\right)\left(g(a)-g(b)\right)\right| \\
& +\left\|\frac{g(a)+g(b)}{2}-g\right\|\left\|f'\right\|_\infty \int_a^b\left|x-\frac{a+b}{2}\right|dx \\
& +\left\|\frac{f(a)+f(b)}{2}-f\right\|_\infty \left\|g'\right\|_\infty \int_a^b\left|x-\frac{a+b}{2}\right|dx \\
& +\left\|f'\right\|_\infty\left\|g'\right\|_\infty\left(\int_a^b\left|x-\frac{a+b}{2}\right|dx\right)^2
\end{aligned}
$$

and the inequality (3.6) is proved. □

4. Some Integral Inequalities for $\|\cdot\|_p$ −Norm, $p \in [1, \infty)$.

The following result also holds.

Theorem 7. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be as in Theorem 3 and assume that*

$$
\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_p := \left(\int_a^b\int_c^d\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|^p dydx\right)^{\frac{1}{p}} < \infty,\ \ p\in[1,\infty). \tag{4.1}
$$

Then we have the estimate

$$
\begin{aligned}
& \left|\int_a^b\int_c^d f(x,y)\,dydx + (b-a)\int_c^d\left(y-\frac{c+d}{2}\right)f_2(y)\,dy\right. \\
& +(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)f_1(x)\,dx \\
& \left. -\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right| \\
\leq & \begin{cases}
\frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_1 & \text{if } \frac{\partial^2 f}{\partial x\partial y}\in L_1[a,b]; \\[2ex]
\left[\frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}}\right]\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_p, & \text{if } \frac{\partial^2 f}{\partial x\partial y}\in L_p[a,b] \text{ and} \\
& p>1,\ \frac{1}{p}+\frac{1}{q}=1,
\end{cases}
\end{aligned} \tag{4.2}
$$

where

$$
f_1(x)=\frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x}+\frac{\partial f(x,d)}{\partial x}\right],\ \ x\in[a,b]
$$

and

$$
f_2(y)=\frac{1}{2}\left[\frac{\partial f(a,y)}{\partial y}+\frac{\partial f(b,y)}{\partial y}\right],\ \ y\in[c,d].
$$

Proof. It is easy to see that

$$\begin{aligned}
&\int_a^b \int_c^d \left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right|\left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| dydx \\
\leq\; &\sup_{(x,y)\in[a,b]\times[c,d]} \left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right| \int_a^b \int_c^d \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| dydx \\
=\; &\frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_1
\end{aligned}$$

and, by Hölder's inequality for double integrals, ($p>1$, $\frac{1}{p}+\frac{1}{q}=1$), we have

$$\begin{aligned}
&\int_a^b \int_c^d \left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right|\left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| dydx \\
\leq\; &\left(\int_a^b \int_c^d \left|x-\frac{a+b}{2}\right|^q\left|y-\frac{c+d}{2}\right|^q dydx\right)^{\frac{1}{q}}\left(\int_a^b \int_c^d \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right|^p dydx\right)^{\frac{1}{p}} \\
=\; &\left[\frac{(b-a)^{1+\frac{1}{q}}}{2^q(q+1)}\right]^{\frac{1}{q}}\left[\frac{(d-c)^{1+\frac{1}{q}}}{2^q(q+1)}\right]^{\frac{1}{q}}\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_p \\
=\; &\frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}}\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_p .
\end{aligned}$$

Using the representation (2.1) we get the desired result. □

Remark 2. *Similar results to those in Theorems 5 and 6 may be stated, but we omit the details.*

References

[1] N.S. BARNETT and S.S. DRAGOMIR, An Ostrowski type inequality for double integrals and applications for cubature formulae, *Soochow J. of Math.*, (in press).

[2] S.S. DRAGOMIR, N.S. BARNETT and P. CERONE, An Ostrowski type inequality for double integrals in terms of L_p-norms and applications in numerical integration, *Anal. Num. Theor. Approx.* (Romania), (in press).

[3] N.S. BARNETT and S.S. DRAGOMIR, A note on bounds for the estimation error variance of a continuous stream with stationary variogram, *J. KSIAM,* **2**(2) (1998), 49-56.

[4] S.S. DRAGOMIR, P. CERONE, N.S. BARNETT and J. ROUMELIOTIS, An inequality of the Ostrowski type for double integrals and applications for cubature formulae.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC 8001, Victoria, Australia

E-mail address: neil@sci.vu.edu.au
URL: http://sci.vu.edu.au/staff/neilb.html

E-mail address: sever@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html