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CHEBYCHEV FUNCTIONAL BOUNDS USING OSTROWSKI SEMINORMS

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ABSTRACT. Bounds are obtained for the Chebychev functional using what is termed as the *Ostrowski seminorm* which is related to an inequality developed by Ostrowski. The Ostrowski seminorm is also compared to the Δ -seminorm introduced in earlier work by the authors.

1. THE OSTROWSKI SEMINORM

For a measurable function $f : [a, b] \rightarrow \mathbb{R}$, we define the functional

$$(1.1) \quad \|f\|_p^\theta := \left(\int_a^b \int_a^b \left| \frac{f(x) - f(y)}{x - y} \right|^p dx dy \right)^{\frac{1}{p}}, \quad p \geq 1$$

provided the above integral exists and is finite.

It is obvious that

- (i) $\|f\|_p^\theta \geq 0$ and $\|f\|_p^\theta = 0$ implies $f = \text{constant a.e. on } [a, b]$;
- (ii) $\|\alpha f\|_p^\theta = |\alpha| \|f\|_p^\theta$ for $\alpha \in \mathbb{R}$,
- (iii) $\|f + g\|_p^\theta \leq \|f\|_p^\theta + \|g\|_p^\theta$,

showing that $\|\cdot\|_p^\theta$ is a seminorm on the space of all measurable functions for which the integral

$$(1.2) \quad \int_a^b \int_a^b \left| \frac{f(x) - f(y)}{x - y} \right|^p dx dy$$

is finite.

We will call $\|\cdot\|_p^\theta$ the *Ostrowski seminorm*, because of the following fundamental inequality established by Ostrowski in [15]

$$(1.3) \quad \int_a^b \int_a^b \left| \frac{f(x) - f(y)}{x - y} \right|^p dx dy \leq (b - a) \log 4 \int_a^b |f'(t)|^p dt,$$

where $f : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous function on $[a, b]$ and $p \geq 1$. The constant $\log 4$ is best for $p = 1$. Fink [12] applies (1.3) and a generalisation to averages of divided differences.

Consequently, if we use the classical notations for Lebesgue norms, we may state the following theorem.

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Theorem 1. (*Ostrowski*) Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$ and $p \geq 1$. Then

$$(1.4) \quad \|f\|_p^\theta \leq (b-a)^{\frac{1}{p}} [\log 4]^{\frac{1}{p}} \|f'\|_p.$$

The constant $\log 4$ is sharp for $p = 1$.

2. COMPARING THE OSTROWSKI SEMINORM WITH Δ -SEMINORMS

For $f \in L_p[a, b]$ ($p \in [1, \infty)$) we can define the functional (see also [2])

$$(2.1) \quad \|f\|_p^\Delta := \left(\int_a^b \int_a^b |f(t) - f(s)|^p dt ds \right)^{\frac{1}{p}}$$

and for $f \in L_\infty[a, b]$, we can define

$$(2.2) \quad \|f\|_\infty^\Delta := \operatorname{ess\,sup}_{(t,s) \in [a,b]^2} |f(t) - f(s)|.$$

If we consider $f_\Delta : [a, b]^2 \rightarrow \mathbb{R}$,

$$f_\Delta(t, s) = f(t) - f(s),$$

then, obviously,

$$(2.3) \quad \|f\|_p^\Delta = \|f_\Delta\|_p, \quad p \in [1, \infty],$$

where $\|\cdot\|_p$ are the usual Lebesgue p -norms on $[a, b]^2$.

Using the properties of the Lebesgue p -norms, we may deduce the following semi-norm properties of $\|\cdot\|_p^\Delta$:

- (i) $\|f\|_p^\Delta \geq 0$ for $f \in L_p[a, b]$ and $\|f\|_p^\Delta = 0$ implies that $f = c$ (c is a constant) a.e. on $[a, b]$;
- (ii) $\|f + g\|_p^\Delta \leq \|f\|_p^\Delta + \|g\|_p^\Delta$ if $f, g \in L_p[a, b]$,
- (iii) $\|\alpha f\|_p^\Delta = |\alpha| \|f\|_p^\Delta$ if $f \in L_p[a, b]$, $\alpha \in \mathbb{R}$.

We note that if $p = 2$, then

$$\begin{aligned} \|f\|_2^\Delta &= \left(\int_a^b \int_a^b (f(t) - f(s))^2 dt ds \right)^{\frac{1}{2}} \\ &= \sqrt{2} \left[(b-a) \|f\|_2^2 - \left(\int_a^b f(t) dt \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$, then we can point out the following bounds for $\|f\|_p^\Delta$ in terms of $\|f'\|_p$.

Theorem 2. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$.

(i) If $p \in [1, \infty)$, then we have the inequality:

$$(2.4) \quad \|f\|_p^\Delta \leq \begin{cases} \frac{2^{\frac{1}{p}} (b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|f'\|_\infty & \text{if } f' \in L_\infty[a, b] \\ \frac{(2\beta^2)^{\frac{1}{\beta}} (b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}} \|f'\|_\alpha & \text{if } f' \in L_\alpha[a, b], \\ & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ (b-a)^{\frac{2}{p}} \|f'\|_1. & \end{cases}$$

(ii) If $p = \infty$, then we have the inequality:

$$(2.5) \quad \|f\|_p^\Delta \leq \begin{cases} (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty[a, b] \\ (b-a)^{\frac{1}{\beta}} \|f'\|_\alpha & \text{if } f' \in L_\alpha[a, b], \\ & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|f'\|_1. & \end{cases}$$

For a short proof of the above inequalities, see for example [2].

The following result in comparing the Δ -seminorm with the Ostrowski-seminorm holds.

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a measurable function such that $\|f\|_p^\theta$ ($p \geq 1$) exists. Then we have the inequality:

$$(2.6) \quad \|f\|_p^\Delta \leq \frac{2^{\frac{1}{p\beta}} (b-a)^{1+\frac{2}{p\beta}}}{[(p\beta+1)(p\beta+2)]^{\frac{1}{p\beta}}} \|f\|_{p\alpha}^\theta, \quad \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Proof. We have that

$$\begin{aligned} \|f\|_p^\Delta &= \left(\int_a^b \int_a^b |f(x) - f(y)|^p dx dy \right)^{\frac{1}{p}} \\ &= \left(\int_a^b \int_a^b \left| \frac{f(x) - f(y)}{x-y} \right|^p \cdot |x-y|^p dx dy \right)^{\frac{1}{p}} \\ &\leq \left(\int_a^b \int_a^b \left| \frac{f(x) - f(y)}{x-y} \right|^{p\alpha} dx dy \right)^{\frac{1}{p\alpha}} \left(\int_a^b \int_a^b |x-y|^{p\beta} dx dy \right)^{\frac{1}{p\beta}} \\ &= \|f\|_{p\alpha}^\theta \left[\frac{2(b-a)^{p\beta+2}}{(p\beta+1)(p\beta+2)} \right]^{\frac{1}{p\beta}} = \frac{2^{\frac{1}{p\beta}} (b-a)^{1+\frac{2}{p\beta}}}{[(p\beta+1)(p\beta+2)]^{\frac{1}{p\beta}}} \|f\|_{p\alpha}^\theta \end{aligned}$$

which proves (2.6). ■

Corollary 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have the inequality:

$$(2.7) \quad \|f\|_p^\Delta \leq \frac{2^{\frac{1}{p\beta}} (\log 4)^{\frac{1}{p\alpha}} (b-a)^{1+\frac{2}{p\beta}+\frac{1}{p\alpha}}}{[(p\beta+1)(p\beta+2)]^{\frac{1}{p\beta}}} \|f'\|_{p\alpha},$$

for $p \geq 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\alpha > 1$.

Proof. Using Ostrowski's inequality (1.3), we have from (1.1)

$$\|f\|_{p\alpha}^\theta \leq (\log 4)^{\frac{1}{p\alpha}} (b-a)^{\frac{1}{p\alpha}} \|f'\|_{p\alpha}$$

and then, by (2.6) we get (2.7). ■

3. SOME BOUNDS FOR THE CHEBYCHEV FUNCTIONAL

For two measurable functions $f, g : [a, b] \rightarrow \mathbb{R}$, we define the Chebychev-functional as follows

$$(3.1) \quad T(f, g; a, b) := \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt.$$

It is well known that if f, g are synchronous, i.e.,

$$(f(t) - f(s))(g(t) - g(s)) \geq 0 \text{ for a.e. } t, s \in [a, b],$$

then

$$(3.2) \quad T(f, g; a, b) \geq 0$$

which is called in the literature the Chebychev inequality for synchronous functions.

If $m \leq f \leq M$, $n \leq g \leq N$ a.e. on $[a, b]$, m, M, n, N are real functions, then the following Grüss inequality holds (see for example [13] or [16] and [14])

$$(3.3) \quad |T(f, g; a, b)| \leq \frac{1}{4} (M - m)(N - n).$$

The constant $\frac{1}{4}$ in (3.3) is best possible.

If $f, g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous and $f', g' \in L_\infty[a, b]$, that is,

$$\|f'\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |f'(t)| < \infty$$

and similarly $\|g'\|_\infty < \infty$, then we can state another result by Chebychev

$$(3.4) \quad |T(f, g; a, b)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2,$$

where the constant $\frac{1}{12}$ is the best possible.

If we assume that the derivatives $f', g' \in L_2[a, b]$, then we may point out the following bound in terms of the euclidean norm due to Lupas:

$$(3.5) \quad |T(f, g; a, b)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a)$$

for which the constant $\frac{1}{\pi^2}$ is sharp.

For these and related work involving bounds of the Chebychev functional, see the references [1] – [10].

Now, we point out some bounds for the modulus of Chebychev's functional in terms of the Ostrowski seminorm introduced above.

Theorem 4. Assume that $f, g : [a, b] \rightarrow \mathbb{R}$ are measurable and $p, r, q > 0$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $p > 1$. Then

$$(3.6) \quad |T(f, g; a, b)| \leq \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}}}{[(2p+1)(2p+2)]^{\frac{1}{p}}} \|f\|_q^\theta \|g\|_r^\theta,$$

provided that all the integrals involved exist.

Proof. Using Korkine's identity for integrals, i.e.,

$$T(f, g; a, b) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(t) - f(s))(g(t) - g(s)) dt ds$$

we have, by Hölder's inequality for p, q, r with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $p > 1$, that

$$\begin{aligned} & |T(f, g; a, b)| \\ & \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b |f(t) - f(s)| |g(t) - g(s)| dt ds \\ & = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s)^2 \left| \frac{f(t) - f(s)}{t-s} \right| \left| \frac{g(t) - g(s)}{t-s} \right| dt ds \\ & \leq \frac{1}{2(b-a)^2} \left(\int_a^b \int_a^b (t-s)^{2p} dt ds \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b \left| \frac{f(t) - f(s)}{t-s} \right|^q dt ds \right)^{\frac{1}{q}} \\ & \quad \times \left(\int_a^b \int_a^b \left| \frac{f(t) - f(s)}{t-s} \right|^r dt ds \right)^{\frac{1}{r}} \\ & = \frac{1}{2(b-a)^2} \left[\frac{2(b-a)^{2p+2}}{(2p+1)(2p+2)} \right]^{\frac{1}{p}} \|f\|_q^\theta \|g\|_r^\theta \\ & = \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}}}{[(2p+1)(2p+2)]^{\frac{1}{p}}} \|f\|_q^\theta \|g\|_r^\theta \end{aligned}$$

and the theorem is proved. ■

Using Ostrowski's theorem, we may state the following corollary concerning f' with an upper bound of the modulus of Chebychev's functional in terms of the usual Lebesgue norms of the derivative.

Corollary 2. *Assume that the functions $f, g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous and p, r, q are as in Theorem 2. Then*

$$(3.7) \quad |T(f, g; a, b)| \leq \frac{2^{\frac{1}{p}-1} (\log 4)^{1-\frac{1}{p}} (b-a)^{1+\frac{1}{p}}}{[(2p+1)(2p+2)]^{\frac{1}{p}}} \|f'\|_q \|g'\|_r,$$

Proof. By Ostrowski's inequality (1.3) we have

$$\|f\|_q^\theta \leq (b-a)^{\frac{1}{q}} (\log 4)^{\frac{1}{q}} \|f'\|_q$$

and

$$\|g\|_r^\theta \leq (b-a)^{\frac{1}{r}} (\log 4)^{\frac{1}{r}} \|g'\|_r.$$

Thus, using (3.6) we deduce (3.7). ■

Remark 1. *If $p \rightarrow 1$ in (3.7), then Chebychev's result (3.4) is recaptured.*

Rather than using coarser upper bounds as provided by Corollary 2, for certain functions the Ostrowski functional may be evaluated directly.

Corollary 3. Assume that the mapping $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$. Further, let $p, q, r > 0$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $p > 1$, then

$$\begin{aligned}
 (3.8) \quad & \left| \frac{1}{b-a} \int_a^b f(t) dt - \left[\left(\frac{z-a}{b-a} \right) f(a) + \left(\frac{b-z}{b-a} \right) f(b) \right] \right. \\
 & \left. - \left(z - \frac{a+b}{2} \right) S(f; a, b) \right| \\
 & \leq \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p} + \frac{2}{q} - 1}}{[(2p+1)(2p+2)]^{\frac{1}{p}}} \|f'\|_r^\theta \\
 & \leq \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p} + \frac{2}{q} + \frac{1}{r} - 1}}{[(2p+1)(2p+2)]^{\frac{1}{p}}} (\log 4)^{\frac{1}{r}} (\|f''\|_r)^r,
 \end{aligned}$$

provided all the integrals involved exist, where

$$S(f; a, b) = \frac{f(b) - f(a)}{b-a},$$

is the secant slope.

Proof. If we take

$$(3.9) \quad h(t) = \frac{t-z}{b-a}, \quad z \in [a, b]$$

and associate f with h and g with f' in (3.1), then

$$\begin{aligned}
 (3.10) \quad -T(h, f'; a, b) &= \frac{1}{b-a} \int_a^b f(t) dt - \left[\left(\frac{z-a}{b-a} \right) f(a) \right. \\
 &\quad \left. + \left(\frac{b-z}{b-a} \right) f(b) \right] - \left(z - \frac{a+b}{2} \right) S(f; a, b)
 \end{aligned}$$

where a simple integration by parts has been preformed.

Now, from (1.1) and (3.9)

$$(3.11) \quad \|h\|_q^\theta = \left(\int_a^b \int_a^b \left| \frac{h(x) - h(y)}{x-y} \right|^q dx dy \right)^{\frac{1}{q}} = (b-a)^{\frac{2}{q}-1}.$$

Hence, using (3.10), (3.11) and (3.6) gives the first inequality in (3.8).

Further, using the Ostrowski inequality (1.3) gives, on utlising (1.1),

$$\|f'\|_r^\theta \leq (b-a) \log 4 \int_a^b |f''(t)|^r dt$$

and thus the second inequality is obtained on noting that

$$\int_a^b |f''(t)|^r dt = (\|f''\|_r)^r,$$

where $\|\cdot\|_r$ is the usual Lebesgue norm. ■

Remark 2. We note that the bounds are independent of the point z . If we take $z = \frac{a+b}{2}$, then the standard trapezoidal rule is obtained from the perturbed rule (3.8) on multiplying both sides by $b-a$.

Corollary 4. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping for which $f^{(n-1)}$ ($n \geq 1$) is absolutely continuous on $[a, b]$ and $p, q, r > 0$ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $p > 1$ then,

$$\begin{aligned}
 (3.12) \quad & \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(z-a)^{k+1} f^{(k)}(a) + (-1)^k (b-z)^{k+1} f^{(k)}(b) \right] \right. \\
 & \left. - \frac{(z-a)^{n+1} + (-1)^n (b-z)^{n+1}}{(n+1)!} S_n(f; a, b) \right| \\
 & \leq \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}+1}}{[(2p+1)(2p+2)]^{\frac{1}{p}}} \cdot \frac{1}{n!} \left(\int_a^b \int_a^b \left| \frac{(z-x)^n - (z-y)^n}{x-y} \right|^q dx dy \right)^{\frac{1}{q}} \|f^{(n)}\|_r^\theta \\
 & \leq \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}+1}}{[(2p+1)(2p+2)]^{\frac{1}{p}} [(n-1)!]^p} \left[\frac{(z-a)^{(n-1)p+1} + (b-z)^{(n-1)p+1}}{(n-1)p+1} \right] \|f^{(n)}\|_r^\theta,
 \end{aligned}$$

where $S_n(f; a, b) = \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a}$.

Proof. The proof follows closely that of the previous corollary.

Cerone et al. [3] obtained the identity

$$\begin{aligned}
 (3.13) \quad \int_a^b f(t) dt &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(z-a)^{k+1} f^{(k)}(a) \right. \\
 & \quad \left. + (-1)^k (b-z)^{k+1} f^{(k)}(b) \right] + \int_a^b H_n(t) f^{(n)}(t) dt
 \end{aligned}$$

where

$$(3.14) \quad H_n(t) = \frac{(z-t)^n}{n!}, \quad z \in [a, b].$$

Now, if we associate f with H_n and g with $f^{(n)}$ in (3.1), then

$$\begin{aligned}
 (3.15) \quad & T(H_n, f^{(n)}; a, b) \\
 &= \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(z-a)^{k+1} f^{(k)}(a) \right. \\
 & \quad \left. + (-1)^k (b-z)^{k+1} f^{(k)}(b) \right] - \int_a^b H_n(t) dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) dt.
 \end{aligned}$$

Further, from (3.14)

$$\int_a^b H_n(t) dt = \frac{(z-a)^{n+1} + (-1)^n (b-z)^{n+1}}{(n+1)!}$$

and

$$\frac{1}{b-a} \int_a^b f^{(n)}(t) dt = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a},$$

giving from (3.1) and (3.15) the left hand side of (3.15). Now, from (3.6) and (3.15) and using (3.14) gives

$$(3.16) \quad \|H_n\|_q^\theta = \frac{1}{n!} \left(\int_a^b \int_a^b \left| \frac{(z-x)^n - (z-y)^n}{x-y} \right|^q dx dy \right)^{\frac{1}{q}}$$

and so the first inequality in (3.12) results.

Now, for the second inequality (3.12) we use Ostrowski's inequality on $H_n(t)$ as defined by (3.14) to give

$$\begin{aligned} \int_a^b \int_a^b \left| \frac{H_n(x) - H_n(y)}{x - y} \right|^p dx dy &\leq (b - a) \log 4 \int_a^b |H'_n(t)|^p dt \\ &= (b - a) \log 4 \int_a^b \left| \frac{(z - t)^{n-1}}{(n-1)!} \right|^p dt, \end{aligned}$$

which, upon explicit evaluation gives the second inequality. ■

Remark 3. *The second inequality in (3.12) is coarser than the first. For particular values of n the double integral in the first inequality may be evaluated explicitly. From (3.16),*

$$\begin{aligned} (3.17) \quad \|H_2\|_q^\theta &= \frac{1}{2!} \left(\int_a^b \int_a^b \left| \frac{(z-x)^2 - (z-y)^2}{x-y} \right|^q dx dy \right)^{\frac{1}{q}} \\ &= \frac{1}{2} \left(\int_a^b \int_a^b |z-x+z-y|^q dx dy \right)^{\frac{1}{q}}. \end{aligned}$$

Let

$$J := \int_a^b \int_a^b |z-x+z-y|^q dx dy$$

then

$$\begin{aligned} (3.18) \quad J &= \int_a^z \int_a^z |z-x+z-y|^q dx dy + \int_z^b \int_a^z |z-x+z-y|^q dx dy \\ &\quad + \int_a^z \int_z^b |z-x+z-y|^q dx dy + \int_z^b \int_z^b |z-x+z-y|^q dx dy \\ &:= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Now, it may be verified by some straightforward algebra that

$$(3.19) \quad D_q(W) = \int_0^W \int_0^W |u-v|^q du dv = \frac{2W^{q+2}}{(q+1)(q+2)}$$

and

$$(3.20) \quad A_q(W) = \int_0^W \int_0^W |u+v|^q du dv = \frac{2(2^{q+1}-1)W^{q+2}}{(q+1)(q+2)}.$$

Thus, by making some obvious transformations, it may be determined that

$$(3.21) \quad J_1 = A_q(z-a) \quad \text{and} \quad J_4 = A_q(b-z).$$

Further, it may be ascertained by symmetry that $J_2 = J_3$ and hence we will consider the evaluation of J_2 in some detail. Now,

$$J_2 = \begin{cases} D_q(z-a) + \int_{z-a}^{b-z} \int_0^{z-a} (v-u)^q du dv, & z < \frac{a+b}{2}, \\ D_q(b-z) + \int_0^{b-z} \int_{b-z}^{z-a} (u-v)^q du dv, & z > \frac{a+b}{2}, \end{cases}$$

and so, after some algebra

$$(3.22) \quad J_2 = \frac{(z-a)^{q+2} + (b-z)^{q+2} - 2^{q+2} \left| z - \frac{a+b}{2} \right|^{q+2}}{(q+1)(q+2)} (= J_3).$$

Combining the results (3.18) – (3.22) into (3.17) gives

$$(3.23) \quad \|H_2\|_q^\theta = \frac{2^{\frac{2}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}} \left[(z-a)^{q+2} + (b-z)^{q+2} - 2 \left| z - \frac{a+b}{2} \right|^{q+2} \right]^{\frac{1}{q}}$$

and so from (3.12) with $n = 2$

$$\begin{aligned} & \left| \int_a^b f(t) dt - \left[(z-a)f(a) + (b-z)f(b) + \frac{1}{2!}(z-a)^2 f'(a) - \frac{1}{2!}(b-z)^2 f'(b) \right] \right. \\ & \quad \left. - \frac{(z-a)^3 + (b-z)^3}{3!} \frac{[f'(b) - f'(a)]}{b-a} \right| \\ &= \left| \int_a^b f(t) dt - [(z-a)f(a) + (b-z)f(b)] - \frac{(z-a)^2}{6(b-a)} (z-a-3(b-a)) f'(a) \right. \\ & \quad \left. + \frac{(b-z)^2}{6(b-a)} (b-z-3(b-a)) f'(b) \right| \\ &\leq \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}+1}}{[(2p+1)(2p+2)]^{\frac{1}{p}}} \|H_2\|_q^\theta \|f''\|_r^\theta, \end{aligned}$$

where $\|H_2\|_q^\theta$ is as given by (3.23).

If $z = \frac{a+b}{2}$, then some simplification occurs to give

$$\begin{aligned} & \left| \int_a^b f(t) dt - \frac{b-a}{2} [f(a) + f(b)] + \frac{5}{48} (b-a) [f'(b) - f'(a)] \right| \\ &\leq \frac{2^{2(\frac{1}{p}+\frac{1}{q}-1)} (b-a)^{2(\frac{1}{p}+\frac{1}{q}-1)}}{[(2p+1)(2p+2)]^{\frac{1}{p}} [(q+1)(q+2)]^{\frac{1}{q}}} \|f''\|_r^\theta. \end{aligned}$$

Remark 4. Using the definition of the functional (1.1), Ostrowski's inequality (1.3) could be used to obtain upper bounds on the inequalities (3.12) using

$$\left\| f^{(n)} \right\|_r^\theta \leq (b-a) \log 4 \left(\left\| f^{(n+1)} \right\|_r \right)^r$$

in terms of the usual Lebesgue norm for $f^{(n+1)}$.

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