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A SHORT NOTE ON AN INTEGRAL INEQUALITY

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Abstract. In this short note, we give a positive answer to an open problem posed by F. Qi in the paper *Several integral inequalities*, Journal of Inequalities in Pure and Applied Mathematics, **1** (2000), no. 2, Article 19. `http://jipam.vu.edu.au/v1n2.html/001_00.html`. RGMIA Research Report Collection **2** (1999), no. 7, Article 9. `http://rgmia.vu.edu.au/v2n7.html`.

1. Introduction

In [3], the second author of this note obtained the following new inequality which is not found in [1], [2], [4] and [5]:

Theorem A. *Suppose that $f(x)$ has a continuous derivative of the n-th order on $[a,b]$, $f^{(i)}(a) \geq 0$ and $f^{(n)}(x) \geq n!$, where $0 \leq i \leq n-1$. Then*

$$\int_a^b [f(x)]^{n+2} dx \geq \left[\int_a^b f(x)dx\right]^{n+1}. \tag{1}$$

Next, he proposed the following open problem

Open problem. *Under what conditions does the inequality*

$$\int_a^b [f(x)]^t dx \geq \left[\int_a^b f(x)dx\right]^{t-1}$$

hold for $t > 1$?

In this note, we are going to give an affirmative answer to the above problem, that is

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Theorem 1. *Suppose that f is a continuous function on $[a,b]$. If $\int_a^b f(x)dx \geq (b-a)^{t-1}$ for given $t>1$, then*

$$\int_a^b [f(x)]^t dx \geq \left[\int_a^b f(x)dx\right]^{t-1} \tag{2}$$

holds.

Corollary 1. *Suppose that f is a continuous function on $[a,b]$, where $b-a \leq 1$. If $\int_a^b f(x)dx \geq 1$, then inequality* (2) *holds for all $t>1$.*

Corollary 2. *Suppose that f is a continuous function on $[a,b]$, where $b-a \leq 1$. If $f(x) \geq 1/(b-a)$ for all $x \in [a,b]$, then inequality* (2) *holds for all $t>1$.*

2. Lemmae

The basic tool we use here is the integral version of Jensen's inequality and a lemma of convexity.

Lemma 1 (Jensen's inequality [6])**.** *Let μ be a positive finite measure on a σ-algebra $\mathcal{M}$ in (a,b). If f is a real function in $L^1(a,b)$ and φ is convex in (a,b), then*

$$\varphi\left(\frac{\int_a^b f(x)d\mu(x)}{\int_a^b d\mu}\right) \leq \frac{\int_a^b \varphi(f(x))d\mu(x)}{\int_a^b d\mu}. \tag{3}$$

Lemma 2 ([6])**.** *Suppose that φ is a real differentiable function in (a,b). Then φ is convex in (a,b) if and only if for any u and v such that $a<u<v<b$ we have $\varphi'(u) \leq \varphi'(v)$.*

3. Proof of Theorem 1

Consider the function $\varphi(x) = x^t$ in (a,b) for $t>1$. It is easy to check that $\varphi'(u) \leq \varphi'(v)$ for $a<u<v<b$. Hence, by Lemma 2, the function φ is convex in (a,b).

Therefore, by the Jensen inequality (3) in Lemma 1, we have

$$\frac{\int_a^b \varphi(f(x))dx}{\int_a^b dx} \geq \varphi\left(\frac{\int_a^b f(x)dx}{\int_a^b dx}\right), \tag{4}$$

and then

$$\int_a^b f^t(x)dx \geq \frac{\left[\int_a^b f(x)dx\right]^t}{(b-a)^{t-1}}. \tag{5}$$

Since

$$\int_a^b f(x)dx \geq (b-a)^{t-1} \tag{6}$$

for given $t > 1$, it follows that

$$\frac{\int_a^b f(x)dx}{(b-a)^{t-1}} \geq 1 \tag{7}$$

for fixed $t > 1$. Hence, the desired result follows from (5) and (7).

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