On a Generalization of Martins' Inequality

ON A GENERALIZATION OF MARTINS' INEQUALITY

TSZ HO CHAN, PENG GAO, AND FENG QI

ABSTRACT. Let $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers. Under certain conditions on this sequence we prove the following inequality

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}},$$

where $n, m \in \mathbb{N}$ and r is a positive number, $a_i!$ denotes $\prod_{i=1}^{n} a_i$. The upper bound is best possible. This inequality generalizes the Martins' inequality. A special case of the above inequality solves an open problem by F. Qi in *Generalization of H. Alzer's Inequality*, J. Math. Anal. Appl. **240** (1999), 294–297.

1. Introduction

It is well-known that the following inequality

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \qquad (1.1)$$

holds for $r > 0$ and $n \in \mathbb{N}$. We call the left hand side of inequality (1.1) H. Alzer's inequality [1], and the right hand side of inequality (1.1) J. S. Martins' inequality [5].

The Alzer's inequality has invoked the interest of several mathematicians, we refer the reader to [3, 9, 11] and the references therein.

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Recently, F. Qi and L. Debnath in [10] proved that: Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers satisfying

$$\frac{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}{(k+1)a_{k+1}^r - ka_k^r} \geq \left(\frac{a_{k+2}}{a_{k+1}}\right)^r \tag{1.2}$$

for any given positive real number r and $k \in \mathbb{N}$. Then

$$\frac{a_n}{a_{n+m}} \leq \left(\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r}. \tag{1.3}$$

The lower bound of (1.3) is best possible.

In [8, 12, 13], F. Qi and others proved the following inequalities:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} < \sqrt{\frac{n+k}{n+m+k}}, \tag{1.4}$$

where $n, m \in \mathbb{N}$ and k is a nonnegative integer.

In [7, 10], F. Qi proved that: Let n and m be natural numbers, k a nonnegative integer. Then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Bigg/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{1.5}$$

where r is any given positive real number. The lower bound is best possible.

An open problem in [6, 7] asked for the validity of the following inequality:

$$\left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Bigg/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}}, \tag{1.6}$$

where $r > 0$, $n, m \in \mathbb{N}$, $k \in \mathbb{Z}^+$.

The purpose of this paper is to verify and generalize the above inequality (1.6), that is

Theorem 1. *Let $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers and*

(1) *for any positive integer $\ell > 1$,*

$$\frac{a_\ell}{a_{\ell+1}} \geq \frac{a_{\ell-1}}{a_\ell}; \tag{1.7}$$

(2) *for any positive integer $\ell > 1$,*

$$\left(\frac{a_{\ell+1}}{a_\ell}\right)^{\ell} \geq \left(\frac{a_\ell}{a_{\ell-1}}\right)^{\ell-1}. \tag{1.8}$$

Then, for any natural numbers n and m, we have

$$\left(\frac{1}{n}\sum_{i=1}^{n}a_i^r\Big/\frac{1}{n+m}\sum_{i=1}^{n+m}a_i^r\right)^{1/r}<\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}},\tag{1.9}$$

where r is a positive number, $n,m\in\mathbb{N}$, and $a_i!$ denotes $\prod_{i=1}^{n}a_i$. The upper bound is best possible.

Notice that if a positive sequence $\{a_i\}_{i=1}^{\infty}$ satisfies inequality (1.7), then we call it a logarithmically concave sequence.

The proof of Theorem 1 is motivated by [5].

As a corollary of Theorem 1, we have:

Corollary 1. *Let a and b be positive real numbers, k a nonnegative integer, and $m,n\in\mathbb{N}$. Then, for any real number $r>0$, we have*

$$\left(\frac{1}{n}\sum_{i=k+1}^{n+k}(ai+b)^r\Big/\frac{1}{n+m}\sum_{i=k+1}^{n+m+k}(ai+b)^r\right)^{1/r}<\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(ai+b)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(ai+b)}}.\tag{1.10}$$

By letting $a=1$ and $b=0$ in (1.10), we recover inequality (1.6).

2. Lemmas

To prove our main results, the following lemmas are neccessary.

Lemma 1. *For any positive integers ℓ and n such that $2\le\ell\le n$, let $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers satisfying inequality (1.8), then we have*

$$\frac{a_\ell}{\big(a_{\ell-1}!\big)^{1/(\ell-1)}}\le\frac{a_n}{\big(a_{n-1}!\big)^{1/(n-1)}}.\tag{2.1}$$

Proof. It suffices to show

$$\frac{a_n}{\big(a_{n-1}!\big)^{1/(n-1)}}\le\frac{a_{n+1}}{\big(a_n!\big)^{1/n}}.\tag{2.2}$$

The above expression is equivalent to

$$\frac{a_{n+1}}{a_n}\ge\frac{\big(a_n!\big)^{1/n}}{\big(a_{n-1}!\big)^{1/(n-1)}},\tag{2.3}$$

which is further equivalent to

$$\left(\frac{a_{n+1}}{a_n}\right)^n\ge\frac{a_n}{\big(a_{n-1}!\big)^{1/(n-1)}}.\tag{2.4}$$

Now we prove (2.4) by induction. For $n = 2$ it follows from inequality (1.8) directly.

Suppose inequality (2.4) holds for $n = m$. Then

$$\left(\frac{a_{m+1}}{a_m}\right)^m \ge \frac{a_m}{\left(a_{m-1}!\right)^{1/(m-1)}} \tag{2.5}$$

is equivalent to

$$\left(\frac{a_{m+1}}{a_m}\right)^{m(m-1)} \Big/ a_m^m \ge \frac{1}{a_m!}. \tag{2.6}$$

By inequality (1.8), we have

$$\left(\frac{a_{m+2}}{a_{m+1}}\right)^{m+1} \ge \left(\frac{a_{m+1}}{a_m}\right)^m, \tag{2.7}$$

which implies

$$\left(\frac{a_{m+2}}{a_{m+1}}\right)^{m(m+1)} \ge \left(\frac{a_{m+1}}{a_m}\right)^{m(m-1)} \left(\frac{a_{m+1}}{a_m}\right)^m. \tag{2.8}$$

Therefore, from inequality (2.6), we obtain

$$\frac{\left(a_{m+2}/a_{m+1}\right)^{m(m+1)}}{a_{m+1}^m} \ge \frac{\left(a_{m+1}/a_m\right)^{m(m-1)}}{a_m^m} \ge \frac{1}{a_m!}. \tag{2.9}$$

Dividing by a_{m+1} on both sides of inequality (2.9) yields

$$\frac{\left(a_{m+2}/a_{m+1}\right)^{m(m+1)}}{a_{m+1}^{m+1}} \ge \frac{1}{a_{m+1}!}, \tag{2.10}$$

that is

$$\left(\frac{a_{m+2}}{a_{m+1}}\right)^{m+1} \ge \frac{a_{m+1}}{\left(a_m!\right)^{1/m}}, \tag{2.11}$$

which completes the induction. ∎

Lemma 2. *For any positive integers ℓ and n such that $1 \le \ell \le n$, let $\{a_i\}_{i=1}^\infty$ be an increasing sequence of positive real numbers satisfying inequalities* (1.7) *and* (1.8), *then we have*

$$\frac{a_\ell}{\left(a_\ell!\right)^{1/\ell}} \le \frac{a_n}{\left(a_n!\right)^{1/n}}. \tag{2.12}$$

Proof. Since $1 \le \ell \le n$, by inequality (1.7) in Theorem 1, we have

$$\frac{a_\ell}{a_{\ell+1}} \le \frac{a_n}{a_{n+1}}, \tag{2.13}$$

and, from Lemma 1, we have

$$\frac{a_\ell}{a_{\ell+1}} \cdot \frac{a_{\ell+1}}{\left(a_\ell!\right)^{1/\ell}} \le \frac{a_n}{a_{n+1}} \cdot \frac{a_{n+1}}{\left(a_n!\right)^{1/n}}. \tag{2.14}$$

The proof is complete. ∎

Lemma 3 (König's inequality [2, p. 149]). *Let* $\{a_i\}_{i=1}^n$ *and* $\{b_i\}_{i=1}^n$ *be decreasing nonnegative* n*-tuples such that*

$$\prod_{i=1}^{k} b_i \le \prod_{i=1}^{k} a_i, \quad 1 \le k \le n, \tag{2.15}$$

then, for $r > 0$*, we have*

$$\sum_{i=1}^{k} b_i^r \le \sum_{i=1}^{k} a_i^r, \quad 1 \le k \le n. \tag{2.16}$$

This is a well-known result due to König used to give a proof of Weyl's inequality (cf. Corollary 1.b.8 of [4, p. 24]).

3. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. Inequality (1.9) holds for $n = 1$ by the arithmetic-geometric mean inequality.

For $n \ge 2$, inequality (1.9) is equivalent to

$$\left(\frac{1}{n} \sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+1} \sum_{i=1}^{n+1} a_i^r \right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}}, \tag{3.1}$$

which is equivalent to

$$\frac{1}{n} \sum_{i=1}^{n} \left(\frac{a_i}{\sqrt[n]{a_n!}} \right)^r < \frac{1}{n+1} \sum_{i=1}^{n+1} \left(\frac{a_i}{\sqrt[n+1]{a_{n+1}!}} \right)^r. \tag{3.2}$$

Set

$$b_{jn+1} = b_{jn+2} = \cdots = b_{jn+n} = \frac{a_{n+1-j}}{\sqrt[n+1]{a_{n+1}!}}, \quad 0 \le j \le n; \tag{3.3}$$

$$c_{j(n+1)+1} = c_{j(n+1)+2} = \cdots = c_{j(n+1)+(n+1)} = \frac{a_{n-j}}{\sqrt[n]{a_n!}}, \quad 0 \le j \le n-1. \tag{3.4}$$

Direct calculation yields

$$\begin{aligned} \sum_{i=1}^{n(n+1)} b_i^r &= \sum_{j=0}^{n} \sum_{k=1}^{n} b_{jn+k}^r \\ &= n \sum_{j=0}^{n} \left(\frac{a_{n+1-j}}{\sqrt[n+1]{a_{n+1}!}} \right)^r \\ &= n \sum_{i=1}^{n+1} \left(\frac{a_i}{\sqrt[n+1]{a_{n+1}!}} \right)^r \end{aligned} \tag{3.5}$$

and

$$\sum_{i=1}^{n(n+1)} c_i^r = (n+1)\sum_{i=1}^{n}\left(\frac{a_i}{\sqrt[n]{a_n!}}\right)^r. \tag{3.6}$$

Since $\{a_i\}_{i=1}^{\infty}$ is increasing, the sequence $\{b_i\}_{i=1}^{n(n+1)}$ and $\{c_i\}_{i=1}^{n(n+1)}$ are decreasing. Therefore, by Lemma 3, to obtain inequality (3.2), it is sufficient to prove inequality

$$b_m! \geq c_m! \tag{3.7}$$

for $1 \leq m \leq n(n+1)$.

It is easy to see that $b_{n(n+1)}! = c_{n(n+1)}! = 1$. Thus, inequality (3.7) is equivalent to

$$\prod_{i=m}^{n(n+1)} b_i \leq \prod_{i=m}^{n(n+1)} c_i \tag{3.8}$$

for $1 \leq m \leq n(n+1)$.

For $0 \leq \ell \leq n$ and $0 \leq j \leq n-1$, we have $1 \leq (n-\ell)n + (n-j) = (n-\ell)(n+1) + (\ell - j) \leq n(n+1)$. Then

$$\prod_{i=(n-\ell)n+(n-j)}^{n(n+1)} b_i = \frac{(a_{\ell+1})^{j+1}(a_\ell!)^n}{(a_{n+1}!)^{\frac{\ell n+j+1}{n+1}}}; \tag{3.9}$$

$$\prod_{i=(n-\ell)(n+1)+(\ell-j)}^{n(n+1)} c_i = \frac{(a_\ell)^{n-\ell+j+2}(a_{\ell-1}!)^{n+1}}{(a_n!)^{\frac{\ell n+j+1}{n}}}, \quad \ell > j; \tag{3.10}$$

$$\begin{aligned}\prod_{i=(n-\ell)(n+1)+(\ell-j)}^{n(n+1)} c_i &= \prod_{i=(n-\ell-1)(n+1)+(n+1+\ell-j)}^{n(n+1)} c_i \\ &= \frac{(a_{\ell+1})^{j-\ell+1}(a_\ell!)^{n+1}}{(a_n!)^{\frac{\ell n+j+1}{n}}}, \quad \ell \leq j;\end{aligned} \tag{3.11}$$

where $a_0 = 1$.

The last term in (3.11) is bigger than the right term in (3.10), so, without loss of generality, we can assume $j < \ell$. Therefore, from formulae (3.9) and (3.10), inequality (3.8) is reduced to

$$\frac{(a_{\ell+1})^{j+1}(a_\ell!)^n(a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{(a_{n+1}!)^\ell} \leq \frac{(a_\ell)^{n-\ell+j+2}(a_{\ell-1}!)^{n+1}}{(a_n!)^\ell(a_n!)^{\frac{j+1}{n}}}, \tag{3.12}$$

that is

$$\frac{(a_{\ell+1})^{j+1}(a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{(a_\ell!)(a_\ell)^{j-\ell+1}} \leq \frac{(a_{n+1})^\ell(a_n!)^{\frac{-\ell}{n}}}{(a_n!)^{\frac{j-\ell+1}{n}}}, \tag{3.13}$$

this is further equivalent to

$$\frac{(a_{\ell+1})^{j+1}(a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{a_\ell!(a_\ell)^{j-\ell+1}(a_n!)^{\frac{\ell-j-1}{n}}} \le \frac{(a_{n+1})^\ell}{(a_n!)^{\frac{\ell}{n}}}. \tag{3.14}$$

Using inequality (2.12) and inequality (1.7) yields

$$\frac{(a_{n+1}!)^{\frac{1}{n+1}}}{(a_n!)^{\frac{1}{n}}} \le \frac{a_{n+1}}{a_n} \le \frac{a_{\ell+1}}{a_\ell} \tag{3.15}$$

for $\ell \le n$. Thus, in order to prove (3.14), it suffices to prove the following inequality

$$\frac{(a_{\ell+1})^{j+1}}{(a_\ell!)(a_\ell)^{j-\ell+1}}\left(\frac{a_{\ell+1}}{a_\ell}\right)^{\ell-j-1} \le \frac{(a_{n+1})^\ell}{(a_n!)^{\frac{\ell}{n}}}, \tag{3.16}$$

which is equivalent to

$$\frac{a_{\ell+1}}{(a_\ell!)^{\frac{1}{\ell}}} \le \frac{a_{n+1}}{(a_n!)^{\frac{1}{n}}}. \tag{3.17}$$

This follows from inequality (2.1) in Lemma 1. Inequality (1.9) follows.

Note that, since the a_i's are not all equal, inequality (1.9) is strict.

By the L'Hospital rule, easy calculation produces

$$\lim_{r\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} = \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{3.18}$$

thus, the upper bound is best possible. The proof is complete. ∎

Proof of Corollary 1. It suffices to show the sequence $\{a_i\}_{i=1}^\infty = \{a(k+i)+b\}_{i=1}^\infty$ satisfies the inequalities (1.7) and (1.8) for any nonnegative integer k.

It is easy to show

$$\frac{a(\ell+k+1)+b}{a(\ell+k)+b} \le \frac{a(\ell+k)+b}{a(\ell+k-1)+b} \tag{3.19}$$

for any positive integer $\ell > 1$ and nonnegative integer k. Inequality (1.7) holds for the sequence $\{a_i\}_{i=1}^\infty = \{a(k+i)+b\}_{i=1}^\infty$.

Now consider the function

$$f(x) = x\ln\left(1+\frac{1}{x+c}\right), \quad x>0 \tag{3.20}$$

with $c \ge 0$ a constant. Then

$$f'(x) = \ln\left(1+\frac{1}{x+c}\right) - \frac{x}{(x+c)(x+c+1)}, \tag{3.21}$$

$$f''(x) = -\frac{(2c+1)x+2c(c+1)}{(x+c)^2(x+c+1)^2} < 0. \tag{3.22}$$

Thus $f'(x)$ is decreasing. From $\lim_{x\to\infty} f'(x) = 0$, we deduce $f'(x) > 0$ and $f(x)$ is increasing, and the function

$$\left(1 + \frac{1}{x + k + b/a}\right)^x \tag{3.23}$$

is increasing for $x > 0$. Hence

$$\left(\frac{a(\ell + k + 1) + b}{a(\ell + k) + b}\right)^\ell \geq \left(\frac{a(\ell + k) + b}{a(\ell + k - 1) + b}\right)^{\ell - 1} \tag{3.24}$$

holds for any positive integer $\ell > 1$ and nonnegative integer k. Inequality (1.8) holds for the sequence $\{a_i\}_{i=1}^\infty = \{a(k+i) + b\}_{i=1}^\infty$.

Corollary 1 follows. The proof is complete. ∎

Remark 1. The main result in [10], inequality (1.2) and (1.3) of this paper, can be further generalized to the following form, and we will leave the proof to the reader since it is similar to the one in [10].

Theorem 2. *Let* $n, m \in \mathbb{N}$, $\Lambda_n = \sum_{i=1}^n \lambda_i$, $\lambda_i > 0$ *and* $\{a_i\}_{i=1}^\infty$ *be an increasing sequence of positive real numbers satisfying:*

$$\frac{\Lambda_{k+2} a_{k+2} - \Lambda_{k+1} a_{k+1}}{\Lambda_{k+1} a_{k+1} - \Lambda_k a_k} \geq \frac{\lambda_{k+2}}{\lambda_{k+1}} \cdot \frac{a_{k+2}}{a_{k+1}} \tag{3.25}$$

for any given positive real number r *and* $k \in \mathbb{N}$, *then the following inequality holds*

$$\frac{a_n}{a_{n+m}} \leq \frac{\frac{1}{\Lambda_n} \sum_{i=1}^n \lambda_i a_i}{\frac{1}{\Lambda_{n+m}} \sum_{i=1}^{n+m} \lambda_i a_i}. \tag{3.26}$$

The lower bound of (3.26) *is best possible.*

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