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DIFFERENCES BETWEEN MEANS WITH BOUNDS FROM A RIEMANN-STIELTJES INTEGRAL

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ABSTRACT. An identity for the difference between two integral means is obtained in terms of a Riemann-Stieltjes integral. This enables bounds to be procured when the integrand is of bounded variation, Lipschitzian and monotonic. If f is absolutely continuous, bounds are also obtained for $f' \in L_p[a, b]$, $1 \leq p < \infty$, the usual Lebesgue norms. This supplements earlier results involving $f' \in L_\infty[a, b]$.

1. INTRODUCTION

The following theorem was proved in Barnett et al. [2].

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping with the property that $f' \in L_\infty[a, b]$, i.e.,*

$$\|f'\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |f'(t)| < \infty.$$

Then for $a \leq c < d \leq b$, we have the inequality

$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(u) du \right| \\ \leq \left\{ \frac{1}{4} + \left[\frac{\frac{a+b}{2} - \frac{c+d}{2}}{(b-a) - (d-c)} \right]^2 \right\} [(b-a) - (d-c)] \|f'\|_\infty \\ \leq \frac{1}{2} [(b-a) - (d-c)] \|f'\|_\infty.$$

The constant $\frac{1}{4}$ is best possible in the first inequality and $\frac{1}{2}$ is best in the second one.

They utilised the identity

$$(1.2) \quad \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(u) du = \int_a^b K_{c,d}(s) f'(s) ds,$$

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where

$$(1.3) \quad K_{c,d}(s) := \begin{cases} \frac{a-s}{b-a} & \text{if } s \in [a, c]; \\ \frac{s-c}{d-c} + \frac{a-s}{b-a} & \text{if } s \in (c, d); \\ \frac{b-s}{b-a} & \text{if } s \in [d, b]. \end{cases}$$

It was demonstrated that the Ostrowski inequality [12], represented by the following theorem, could be recaptured by using some limiting procedure:

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and assume that $|f'(x)| \leq M$ for all $x \in (a, b)$. Then we have the inequality*

$$(1.4) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

For some generalisations and related results, see the book [11, p. 468 - 484], the papers [1] – [12] and the website <http://rgmia.vu.edu.au/> where many papers devoted to this inequality can be accessed on line.

It is the aim of the current article to obtain bounds on

$$(1.5) \quad D(f; a, c, d, b) := \mathcal{M}(f; a, b) - \mathcal{M}(f; c, d), \quad a \leq c < d \leq b,$$

where

$$\mathcal{M}(f; a, b) := \frac{1}{b-a} \int_a^b f(t) dt,$$

in terms of the Lebesgue norms $\|f'\|_p$, $1 \leq p < \infty$ with $f' \in L_p[a, b]$ implying

$$\left(\int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.$$

Further bounds on $|D(f; a, c, d, b)|$ will be obtained under less restrictive assumptions than absolute continuity on f . Bounds are obtained for f Hölder continuous in Section 2 while in Section 3 bounds are obtained for f of bounded variation, Lipschitzian and monotonic.

2. RESULTS FOR $f' \in L_p[a, b]$, $1 \leq p < \infty$

The following theorem holds (see also Cerone and Dragomir [3]).

Theorem 3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping. Then for $a \leq c < d \leq b$ the inequalities*

$$(2.1) \quad |D(f; a, c, d, b)| \leq \begin{cases} \frac{b-a}{(q+1)^{\frac{1}{q}}} \left[1 + \left(\frac{\rho}{1-\rho} \right)^q \right]^{\frac{1}{q}} [\nu^{q+1} + \lambda^{q+1}]^{\frac{1}{q}} \|f'\|_p, \\ \quad f' \in L_p[a, b], \quad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1; \\ [1 - \rho + |\nu + \rho + \lambda|] \|f'\|_1, \quad f' \in L_1[a, b], \end{cases}$$

where $(b-a)\nu = c-a$, $(b-a)\rho = d-c$, $(b-a)\lambda = b-d$.

Proof. From (1.2) and (1.5) we have on using Hölder's integral inequality that

$$(2.2) \quad |D(f; a, c, d, b)| \leq \left(\int_a^b |K_{c,d}(s)|^q dt \right)^{\frac{1}{q}} \|f'\|_p, \quad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Now,

$$(2.3) \quad \int_a^c |s-a|^q ds = \frac{(c-a)^{q+1}}{q+1} \quad \text{and} \quad \int_d^b |b-s|^q ds = \frac{(b-d)^{q+1}}{q+1}.$$

Further,

$$\begin{aligned} M & : = \int_c^d \left| \frac{s-c}{d-c} + \frac{a-s}{b-a} \right|^q ds \\ & = \frac{1}{(b-a)(d-c)} \int_c^d |[(b-a) - (d-c)]s - cb + ad|^q ds \\ & = \frac{b-a-(d-c)}{(b-a)(d-c)} \int_c^d |s-s_0|^q ds \end{aligned}$$

since $b-a > d-c$ and

$$(2.4) \quad s_0 = \frac{cb-ad}{(b-a)-(d-c)} \in [c, d].$$

Hence, as $c-s_0 < 0$ and $d-s_0 > 0$,

$$\begin{aligned} M & = \frac{b-a-(d-c)}{(b-a)(d-c)} \left[\int_c^{s_0} (s_0-s)^q ds + \int_{s_0}^d (s-s_0)^q ds \right] \\ & = \frac{b-a-(d-c)}{(b-a)(d-c)} \cdot \frac{(s_0-c)^{q+1} + (d-s_0)^{q+1}}{q+1}. \end{aligned}$$

Further simplification may be accomplished since

$$s_0 - c = \frac{(c-a)(d-c)}{(b-a)(d-c)} \quad \text{and} \quad d - s_0 = \frac{(d-c)(b-d)}{(b-a)(d-c)},$$

giving

$$M = \frac{(d-c)^q}{(q+1)(b-a)[(b-a)-(d-c)]^q} \left[(c-a)^{q+1} + (b-d)^{q+1} \right].$$

Thus, combining the expression for M with (2.3) and using (2.2) gives (2.1) after some algebra.

Now, for the second inequality

$$|D(f; a, c, d, b)| \leq \sup_{s \in [a, b]} |K_{c,d}(s)| \int_a^b |f'(s)| ds.$$

From (1.3) it is easily seen that $K_{c,d}(s)$ is a piecewise linear and continuous function. It is negative on (a, s_0) and positive on (s_0, b) . It reaches its extremities at c and d . Thus,

$$\begin{aligned} \sup_{s \in [a, b]} |K_{c,d}(s)| & = \max \left\{ \frac{c-a}{b-a}, \frac{b-d}{b-a} \right\} \\ & = \frac{1}{b-a} \left\{ \frac{c-a+b-d}{2} + \left| \frac{(c-a)-(b-d)}{2} \right| \right\}. \end{aligned}$$

A simple rearrangement gives the result as stated. ■

Remark 1. *If we take $q = 1$ in (2.1) then the first inequality in (1.1) is recaptured.*

3. SOME INEQUALITIES FOR MAPPINGS OF HÖLDER TYPE

If we drop the assumption of absolute continuity and allow f to be Hölder continuous, then the following result is valid.

Theorem 4. *Assume that the mapping $f : [a, b] \rightarrow \mathbb{R}$ is of $r - H$ -Hölder type, i.e.,*

$$(3.1) \quad |f(t) - f(s)| \leq H |t - s|^r \quad \text{for all } t, s \in [a, b],$$

where $r \in (0, 1]$ and $H > 0$ are given.

Then for $a < c < d < b$, we have the inequality

$$(3.2) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(t) dt \right| \leq \frac{(c-a)^{r+1} + (b-d)^{r+1}}{[(b-a) - (d-c)](r+1)} \cdot H.$$

The inequality (2.2) is best in the sense that we cannot put in the right hand side a constant K less than 1.

Proof. Write

$$\frac{1}{b-a} \int_a^b f(t) dt = \int_0^1 f(ub + (1-u)a) du$$

and similarly for the second term.

Then

$$\begin{aligned} I & : = \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(t) dt \\ & = \int_0^1 [f(ub + (1-u)a) - f(ud + (1-u)c)] du. \end{aligned}$$

Using the fact that f is of $r - H$ -Hölder type, we have

$$(3.3) \quad \begin{aligned} |I| & \leq \int_0^1 |f(ub + (1-u)a) - f(ud + (1-u)c)| du \\ & \leq H \int_0^1 |ub + (1-u)a - ud - (1-u)c|^r du \\ & = H \int_0^1 |u[b-a - (d-c)] - (c-a)|^r du. \end{aligned}$$

Now, as $b - a > d - c$, then $u_0 := \frac{c-a}{(b-a)-(d-c)} \in (0, 1)$ and

$$\begin{aligned}
& \int_0^1 |u[b-a-(d-c)] - (c-a)|^r du \\
&= \int_0^{u_0} [(c-a) - u[b-a-(d-c)]]^r du + \int_{u_0}^1 [u[b-a-(d-c)] - (c-a)]^r du \\
&= -\frac{1}{b-a-(d-c)} \frac{((c-a) - u[(b-a)-(d-c)])^{r+1}}{r+1} \Big|_0^{u_0} \\
&\quad + \frac{1}{b-a-(d-c)} \frac{(u[(b-a)-(d-c)] - (c-a))^{r+1}}{r+1} \Big|_{u_0}^1 \\
&= -\frac{1}{(b-a)-(d-c)} \frac{((c-a) - u_0[(b-a)-(d-c)])^{r+1}}{r+1} \\
&\quad + \frac{(c-a)^{r+1}}{(r+1)(b-a-(d-c))} + \frac{1}{(b-a)-(d-c)} \frac{([b-a-(d-c)] - c+a)^{r+1}}{r+1} \\
&\quad - \frac{(u_0[b-a-(d-c)] - (c-a))^{r+1}}{[b-a-(d-c)](r+1)} \\
&= \frac{(c-a)^{r+1} + (b-d)^{r+1}}{[(b-a)-(d-c)](r+1)}.
\end{aligned}$$

Using (3.3) we deduce (3.2).

Assume that now, the inequality (2.2) holds with a constant $K > 0$, i.e.,

$$(3.4) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(t) dt \right| \leq K \cdot \frac{(c-a)^{r+1} + (b-d)^{r+1}}{[(b-a)-(d-c)](r+1)} H.$$

Choose $f_0 : [0, 1] \rightarrow \mathbb{R}$, $f_0(t) = t^r$, $r \in (0, 1]$. Then

$$|f_0(t) - f_0(s)| \leq |t-s|^r \quad \text{for all } t, s \in [0, 1],$$

which shows that f_0 is of r - H -Hölder type with $H = 1$.

Now, choose in (3.4) $a = 0$, $b = 1$, $c \in (0, 1)$, $d = c + \varepsilon$, ε small and such that $c + \varepsilon \in (0, 1)$. Then we get

$$\left| \frac{1}{r+1} - \frac{(c+\varepsilon)^{r+1} - c^{r+1}}{\varepsilon(r+1)} \right| \leq K \cdot \frac{c^{r+1} + (1-c-\varepsilon)^{r+1}}{(1-\varepsilon)(r+1)},$$

which is clearly equivalent with

$$(3.5) \quad \left| 1 - \frac{(c+\varepsilon)^{r+1} - c^{r+1}}{\varepsilon} \right| \leq K \cdot \frac{c^{r+1} + (1-c-\varepsilon)^{r+1}}{1-\varepsilon}.$$

Now, if in (3.5) we let $\varepsilon \rightarrow 0+$, then we get

$$(3.6) \quad |1 - (r+1)c^r| \leq K [c^{r+1} + (1-c)^{r+1}] \quad \text{for all } c \in (0, 1).$$

If in (3.6) we let $c \rightarrow 0+$, then we get $1 \leq K$, and the theorem is completely proved. ■

4. RESULTS FOR THE RIEMANN-STIELTJES INTEGRAL

The results obtained to date for bounds for differences of integral means assume that f is differentiable. That is, f is absolutely continuous. This assumption may be relaxed somewhat and bounds on $D(f; a, c, d, b)$ may still be procured. The following lemma holds.

Lemma 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be of bounded variation on $[a, b]$, then*

$$(4.1) \quad D(f; a, c, d, b) = \int_a^b K_{c,d}(s) df(s),$$

where $K_{c,d}$ is as given by (1.3) and $D(f; a, c, d, b)$ is as defined by (1.5).

Proof. The proof follows closely that used in obtaining (1.2). The integration by parts formula is used for Riemann-Stieltjes integrals to give

$$\begin{aligned} & (b-a)(d-c) \int_a^b K_{c,d}(s) df(s) \\ = & (d-c) \int_a^c (a-s) df(s) + \int_c^d [(b-a)(s-c) - (d-c)(s-a)] df(s) \\ & + (d-c) \int_d^b (b-s) df(s) \\ = & (d-c) \left\{ (a-s)f(s) \Big|_a^c + \int_a^c f(s) ds \right\} + [(b-a)(s-c) - (d-c)(s-a)] f(s) \Big|_c^d \\ & - [(b-a) - (d-c)] \int_c^d f(s) ds + (d-c) \left\{ (b-s)f(s) \Big|_d^b + \int_d^b f(s) ds \right\} \\ = & (d-c) \left\{ (a-c)f(c) + \int_a^c f(s) ds \right\} + [(b-a)(d-c) - (d-c)(d-a)] f(d) \\ & + (d-c)(c-a)f(c) - [(b-a) - (d-c)] \int_c^d f(s) ds \\ & + (d-c) \left[\int_c^d f(s) ds - (b-d)f(d) \right] \\ = & (d-c) \int_c^d f(s) ds - (b-a) \int_c^d f(s) ds. \end{aligned}$$

Division by $(b-a)(d-c)$ produces (4.1) on noting the definition (1.5). ■

The following well known lemmas involving Riemann-Stieltjes integrals are well known. They are stated here for clarity. (See Cerone and Dragomir [4] where they were applied to three point rules in numerical integration.)

Lemma 2. *Let $g, v : [a, b] \rightarrow \mathbb{R}$ be such that g is continuous on $[a, b]$ and v is of bounded variation on $[a, b]$. Then the Riemann-Stieltjes integral $\int_a^b g(t) dv(t)$ exists and is such that*

$$(4.2) \quad \left| \int_a^b g(t) dv(t) \right| \leq \sup_{t \in [a,b]} |g(t)| \bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ is the total variation of v on $[a, b]$.

Lemma 3. Let $g, v : [a, b] \rightarrow \mathbb{R}$ be such that g is Riemann integrable on $[a, b]$ and v is L -Lipschitzian on $[a, b]$. Then

$$(4.3) \quad \left| \int_a^b g(t) dv(t) \right| \leq L \int_a^b |g(t)| dt$$

with v being L -Lipschitzian if it satisfies

$$|v(x) - v(y)| \leq L|x - y|$$

for all $x, y \in [a, b]$.

Lemma 4. Let $g, v \in [a, b] \rightarrow \mathbb{R}$ be such that g is Riemann integrable on $[a, b]$ and v is monotonic nondecreasing on $[a, b]$. Then

$$(4.4) \quad \left| \int_a^b g(t) dv(t) \right| \leq \int_a^b |g(t)| dv(t).$$

Theorem 5. Let $f : [a, b] \rightarrow \mathbb{R}$ be of bounded variation on $[a, b]$. The following bounds hold

$$(4.5) \quad |D(f; a, c, d, b)| \leq \begin{cases} \left[\frac{b-a-(d-c)}{2} + \left| \frac{c+d}{2} - \frac{a+b}{2} \right| \right] \frac{V_a^b(f)}{b-a}, \\ \frac{(c-a)^2 + (b-d)^2}{2[(b-a) - (d-c)]} L, & \text{for } f \text{ } L\text{-Lipschitzian;} \\ \left(\frac{b-d}{b-a} \right) f(b) - \left(\frac{c-a}{b-a} \right) f(a) \\ + \left[\frac{c+d-(a+b)}{b-a} \right] f(s_0), & \text{for } f \text{ monotonic} \\ & \text{nondecreasing,} \end{cases}$$

where $S_0 = \frac{cb-ad}{(b-a)-(d-c)}$.

Proof. Using Lemma 2, we have from (4.2)

$$(4.6) \quad \left| \int_a^b K_{c,d}(s) df(s) \right| \leq \sup_{s \in [a,b]} |K_{c,d}(s)| \bigvee_a^b(f).$$

Now, $K_{c,d}(a) = K_{c,d}(b) = K_{c,d}(s_0) = 0$. Further, $K_{c,d}(s)$ consists of straight line segments on $[a, c]$, $[c, d]$ and $[d, b]$. The extreme values occur at c and d . Thus,

$$\begin{aligned} \sup_{s \in [a,b]} |K_{c,d}(s)| &= \max \{ |K_{c,d}(c)|, |K_{c,d}(d)| \} = \max \left\{ \frac{c-a}{b-a}, \frac{b-d}{b-a} \right\} \\ &= \frac{1}{b-a} \left[\frac{c-a+b-d}{2} + \left| \frac{c-a-(b-d)}{2} \right| \right], \end{aligned}$$

which on rearrangement and using (4.6) gives the first inequality in (4.5).

Now, for the second inequality, we use Lemma 3 and so from (4.3)

$$(4.7) \quad \left| \int_a^b K_{c,d}(s) df(s) \right| \leq L \int_a^b |K_{c,d}(s)| ds,$$

where

$$(4.8) \quad \int_a^b |K_{c,d}(s)| ds = \frac{1}{b-a} \int_a^c (s-a) ds + \frac{(b-a)-(d-c)}{(b-a)(d-c)} \int_c^d |s-s_0| ds + \frac{1}{b-a} \int_d^b (b-s) ds$$

with S_0 as given by (2.4).

Now,

$$\begin{aligned} \frac{1}{b-a} \int_a^c (s-a) ds &= \frac{(c-a)^2}{2(b-a)}, \\ \frac{1}{b-a} \int_d^b (b-s) ds &= \frac{(b-d)^2}{2(b-a)} \end{aligned}$$

and

$$\begin{aligned} \int_c^d |s-s_0| ds &= \int_c^{s_0} (s_0-s) ds + \int_{s_0}^d (s-s_0) ds \\ &= \frac{(s_0-c)^2 + (d-s_0)^2}{2}. \end{aligned}$$

That is, using (2.4), we have

$$\begin{aligned} &\frac{(b-a)-(d-c)}{(b-a)(d-c)} \int_c^d |s-s_0| ds \\ &= \frac{(d-c)}{2(b-a)[(b-a)-(d-c)]} [(c-a)^2 + (b-d)^2], \end{aligned}$$

and so combining the above results and using (4.7) and (4.8) gives the second inequality.

For the final inequality in (4.5) we use Lemma 4 giving from (4.4), for f monotonic nondecreasing,

$$(4.9) \quad \left| \int_a^b K_{c,d}(s) df(s) \right| \leq \int_a^b |K_{c,d}(s)| df(s).$$

Using the properties of $K_{c,d}(S)$ discussed earlier that $K_{c,d}(S) < 0$ for $S \in (c, s_0)$ and $K_{c,d}(S) > 0$ for $S \in (s_0, b)$ and zero at a, s_0 and b , then from (1.3)

$$(4.10) \quad \begin{aligned} &\int_a^b |K_{c,d}(s)| df(s) \\ &= \int_a^c \left(\frac{s-a}{b-a} \right) df(s) + \int_c^{s_0} \left(\frac{s-a}{b-a} + \frac{s-c}{d-c} \right) df(s) \\ &\quad + \int_{s_0}^d \left(\frac{s-c}{d-c} + \frac{a-s}{b-a} \right) df(s) + \int_d^b \left(\frac{b-s}{b-a} \right) df(s). \end{aligned}$$

Now, integration by parts of the Riemann-Stieltjes integrals on the right of (4.10) produces, after some simplification,

$$\begin{aligned} \int_a^b |K_{c,d}(s)| df(s) &= -\frac{1}{b-a} \int_a^c f(s) ds + \left(\frac{1}{d-c} - \frac{1}{b-a} \right) \int_c^{s_0} f(s) ds \\ &\quad - \left(\frac{1}{d-c} - \frac{1}{b-a} \right) \int_{s_0}^d f(s) ds + \frac{1}{b-a} \int_d^b f(s) ds \\ &\leq -\left(\frac{c-a}{b-a} \right) f(a) + \left(\frac{1}{d-c} - \frac{1}{b-a} \right) (s_0 - c) f(s_0) \\ &\quad - \left(\frac{1}{d-c} - \frac{1}{b-a} \right) (d - s_0) f(s_0) + \left(\frac{b-d}{b-a} \right) f(b), \end{aligned}$$

where we have used the fact that f is monotonic nondecreasing to obtain the last inequality. Thus, from (3.5) we obtain the third inequality in (4.5) on grouping terms and simplifying. ■

Remark 2. From (4.10) we may use the fact that $\sup_{s \in [a,b]} |K_{c,d}(S)|$ occurs at c for $s \in [a, s_0]$ and at d for $s \in [s_0, b]$ to give for f monotonic nondecreasing:

$$\begin{aligned} &\left| \int_a^b K_{c,d}(s) df(s) \right| \\ &\leq \left(\frac{c-a}{b-a} \right) (f(c) - f(a)) + \left(\frac{c-a}{b-a} \right) (f(s_0) - f(c)) \\ &\quad + \left(1 - \frac{d-a}{b-a} \right) (f(d) - f(s_0)) + \left(\frac{b-d}{b-a} \right) (f(b) - f(d)) \\ &= \left(\frac{c-a}{b-a} \right) (f(s_0) - f(a)) + \left(\frac{b-d}{b-a} \right) (f(b) - f(s_0)) \\ &= \left(\frac{b-d}{b-a} \right) f(b) - \left(\frac{c-a}{b-a} \right) f(a) + \frac{c+d-(a+b)}{b-a} f(s_0). \end{aligned}$$

Remark 3. If we put $r = 1$ and $H = L$ in (4.2), then we obtain the second inequality in (4.5). It should also be noted that if the parallelogram identity

$$2(x^2 + y^2) = (x - y)^2 + (x + y)^2, \quad x, y \in \mathbb{R}$$

is used then

$$\frac{(c-a)^2 + (b-d)^2}{2[(b-a) - (d-c)]} = \left[\frac{1}{4} + \left(\frac{\frac{a+b}{2} - \frac{c+d}{2}}{(b-a) - (d-c)} \right)^2 \right] [(b-a) - (d-c)],$$

with $\frac{1}{4}$ being the best constant attained when $\frac{a+b}{2} = \frac{c+d}{2}$.

Remark 4. If we assume that there is a point $x \in (a, b)$ for which the functions is continuous, then we may recapture bounds for the Ostrowski functional

$$\theta(f)(x) := f(x) - \mathcal{M}(f).$$

Indeed, if we assume that $c = x \in (a, b)$, $d = x + \varepsilon$ where $x + \varepsilon \in (a, b)$, then from (2.1) for example

$$\begin{aligned} & |D(f; a, x, x + \varepsilon, b)| \\ &= \left| \mathcal{M}(f; a, b) - \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(u) du \right| \\ &\leq \frac{b-a}{(q+1)^{\frac{1}{q}}} \left[1 + \left(\frac{\varepsilon}{b-a-\varepsilon} \right)^q \right] \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{b-x-\varepsilon}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \|f'\|_p. \end{aligned}$$

Now, taking the limit as $\varepsilon \rightarrow 0+$ gives

$$\begin{aligned} |\theta(f)(x)| &\leq \frac{b-a}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{b-x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \|f'\|_p, \\ f' &\in L_p[a, b], \quad 1 \leq p < \infty, \end{aligned}$$

recapturing a result of Dragomir and Wang [9].

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