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# INEQUALITIES FOR MATHIEU'S SERIES

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ABSTRACT. In the article, using the integral expression of Mathieu's series and by some integral inequalities involving periodic functions, several new inequalities and estimates for the Mathieu's series are presented.

## CONTENTS

1. Introduction	1
2. Lemmae	2
3. Main Results	3
4. Open Problems	6
References	7

## 1. INTRODUCTION

In 1890, Mathieu defined in [8]  $S(r)$  as

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2}, \quad r > 0, \quad (1)$$

and conjectured that  $S(r) < 1/r^2$ . We call formula (1) Mathieu's series.

In [7], Makai proved

$$\frac{1}{r^2 + 1/2} < S(r) < \frac{1}{r^2}. \quad (2)$$

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H. Alzer, J. L. Brenner and O. G. Ruehr in [1] obtained

$$\frac{1}{x^2 + 1/(2\zeta(3))} < S(x) < \frac{1}{x^2 + 1/6}, \quad (3)$$

where  $\zeta$  denotes the zeta function.

The integral form of Mathieu's series (1) was given in [3, 4] by

$$S(r) = \frac{1}{r} \int_0^\infty \frac{x}{e^x - 1} \sin(rx) dx. \quad (4)$$

There has been a rich literature on the study of Mathieu's series and its inequalities, we can find many more interesting refinements and extensions of Mathieu's inequality in [1]–[11].

In this paper, using the integral expression (4) of Mathieu's series and certain inequalities involving periodic function, some new inequalities for Mathieu's series are established. At last, three open problems are proposed.

## 2. LEMMAE

**Lemma 2.1.** *Let  $\psi(x)$  be an integrable function satisfying  $\psi(x) = -\psi(x + T)$ , where  $T$  is a given positive number, and  $\psi(x) \geq 0$  for  $x \in [0, T]$ , let  $f(x)$  and  $g(x)$  be two integrable functions on  $[0, 2T]$  such that*

$$f(x) - g(x) \geq f(x + T) - g(x + T) \quad (5)$$

on  $[0, T]$ . Then

$$\int_0^{2T} \psi(x)f(x) dx \geq \int_0^{2T} \psi(x)g(x) dx. \quad (6)$$

*Proof.* By easy computation, it is deduced that

$$\begin{aligned} & \int_0^{2T} \psi(x)[f(x) - g(x)] dx \\ &= \int_0^T \psi(x)[f(x) - g(x)] dx + \int_T^{2T} \psi(x)[f(x) - g(x)] dx \\ &= \int_0^T \psi(x)[f(x) - g(x)] dx + \int_0^T \psi(x + T)[f(x + T) - g(x + T)] dx \\ &= \int_0^T \psi(x)\{[f(x) - g(x)] - [f(x + T) - g(x + T)]\} dx \\ &\geq 0. \end{aligned}$$

The proof is complete. ■

**Corollary 2.1.** *Let  $\psi(x) \not\equiv 0$  be an integrable periodic function with period  $2T > 0$  satisfying  $\psi(x) = -\psi(x + T)$  and  $\psi(x) \geq 0$  for  $x \in [0, T]$ . If  $f(x)$  is an integrable function such that  $f(x) \geq f(x + T)$  on  $[0, T]$ , then*

$$\int_0^{2T} \psi(x)f(x) dx \geq 0. \quad (7)$$

**Corollary 2.2.** *Let  $f(x)$  be an integrable function such that  $f(x) \geq f(x + \pi)$  on  $[0, \pi]$ , then*

$$\int_0^{2\pi} f(x) \sin x dx \geq 0. \quad (8)$$

### 3. MAIN RESULTS

As a direct consequence of Lemma 2.1, we have

**Theorem 3.1.** *Let  $\Phi_1$  and  $\Phi_2$  be two integral functions such that  $\frac{x}{e^x - 1} - \Phi_1(x)$  and  $\Phi_2(x) - \frac{x}{e^x - 1}$  are increasing respectively. Then, for any positive number  $r$ , we have*

$$\frac{1}{r} \int_0^\infty \Phi_2(x) \sin(rx) dx \leq \sum_{n=1}^\infty \frac{2n}{(n^2 + r^2)^2} \leq \frac{1}{r} \int_0^\infty \Phi_1(x) \sin(rx) dx. \quad (9)$$

*Proof.* The function  $\psi(x) = \sin(rx)$  has a period  $\frac{2\pi}{r}$ , and  $\psi(x) = -\psi(x + \frac{\pi}{r})$ .

Since  $f(x) = \frac{x}{e^x - 1} - \Phi_1(x)$  is increasing, for any  $\alpha > 0$ , we have  $f(x + \alpha) \geq f(x)$ .

Therefore, from Corollary 2.1, we obtain

$$\int_{2k\pi/r}^{2(k+1)\pi/r} \left[ \frac{x}{e^x - 1} - \Phi_1(x) \right] \sin(rx) dx \leq 0, \quad (10)$$

$$\int_{2k\pi/r}^{2(k+1)\pi/r} \frac{x}{e^x - 1} \sin(rx) dx \leq \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x) \sin(rx) dx. \quad (11)$$

Then, from formula (4), we have

$$\begin{aligned} S(r) &= \frac{1}{r} \sum_{k=0}^\infty \int_{2k\pi/r}^{2(k+1)\pi/r} \frac{x}{e^x - 1} \sin(rx) dx \\ &\leq \frac{1}{r} \sum_{k=0}^\infty \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x) \sin(rx) dx \\ &= \frac{1}{r} \int_0^\infty \Phi_1(x) \sin(rx) dx. \end{aligned} \quad (12)$$

The right hand side of inequality (9) follows.

Similar arguments yield the left hand side of inequality (9). ■

**Proposition 3.1.** *The function*

$$g(x) = \frac{x}{e^x - 1} - \frac{x^2}{e^{3x} - e^x} \quad (13)$$

*is decreasing with  $x > 0$ .*

*Proof.* By straightforward computation, we have

$$\begin{aligned} g'(x) &= -\frac{e^{-x}(e^x + e^{2x} - e^{3x} - e^{4x} - 2x + 3x e^{2x} + 2x e^{3x} + x e^{4x} - 3x^2 e^{2x} + x^2)}{(e^{2x} - 1)^2} \\ &\equiv -\frac{h(x) e^{-x}}{(e^{2x} - 1)^2}, \end{aligned} \quad (14)$$

$$h'(x) = -2 + e^x + 5e^{2x} - e^{3x} - 3e^{4x} + 6x e^{3x} + 4x e^{4x} - 6x^2 e^{2x}, \quad (15)$$

$$h''(x) = 2 + e^x + 10e^{2x} + 3e^{3x} - 8e^{4x} - 12x e^{2x} + 18x e^{3x} + 16x^2 e^{4x} - 12x^2 e^{2x}, \quad (16)$$

$$h'''(x) = e^x(1 + 8e^x + 27e^{2x} - 16e^{3x} - 48x e^x + 54x e^{2x} + 64x e^{3x} - 24x^2 e^x), \quad (17)$$

$$\left(\frac{h'''(x)}{e^x}\right)' = 4e^x(-10 + 27e^x + 4e^{2x} - 24x + 27x e^x + 48x e^{2x} - 6x^2), \quad (18)$$

$$\left(\frac{(h'''(x)/e^x)'}{4e^x}\right)' = (54e^x - 24) + 56e^{2x} + (27x e^x - 12x) + 96x e^{2x} > 0. \quad (19)$$

Therefore, the function  $(h'''(x)/e^x)'/4e^x$  is increasing. Since  $[(h'''(x)/e^x)'/4e^x]|_{x=0} = 21$ , then  $(h'''(x)/e^x)' > 0$ , and  $h'''(x)/e^x$  is increasing,  $[h'''(x)/e^x]|_{x=0} = 20$ ,  $h'''(x) > 0$ ,  $h''(x)$  increases,  $h''(0) = 8$ ,  $h''(x) > 0$ ,  $h'(x)$  increases,  $h'(0) = 0$ ,  $h'(x) \geq 0$ ,  $h(x)$  increases,  $h(0) = 0$ , then  $h(x) \geq 0$ ,  $g'(x) \leq 0$  for  $x > 0$ , the function  $g(x)$  is decreasing. The proof follows. ■

**Corollary 3.1.** *For any positive number  $r > 0$ , we have*

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} \geq \frac{1}{8r(1 + r^2)^3} \left[ 16r(r^2 - 3) + \pi^3(r^2 + 1)^3 \operatorname{sech}^2\left(\frac{\pi r}{2}\right) \tanh\left(\frac{\pi r}{2}\right) \right]. \quad (20)$$

*Proof.* In [12, p. 356], the following formula is given

$$\int_0^\infty \frac{x^{2m} \sin(ax)}{e^{(2n+1)\alpha x} - e^{(2n-1)\alpha x}} dx = (-1)^m \frac{\partial^{2m}}{\partial a^{2m}} \left[ \frac{\pi}{4\alpha} \tanh \frac{a\pi}{2\alpha} - \sum_{\nu=1}^n \frac{a}{a^2 + (2\nu-1)^2 \alpha^2} \right], \quad (21)$$

where  $\alpha > 0$  and  $m, n = 0, 1, 2, \dots$ . If  $n = 0$  in formula (21), then the summation terms are omitted.

Therefore, we have

$$\begin{aligned} \int_0^\infty \frac{x^2 \sin(rx)}{e^{3x} - e^x} dx &= -\frac{\partial^2}{\partial r^2} \left[ \frac{\pi}{4} \tanh \frac{\pi r}{2} - \frac{r}{r^2 + 1} \right] \\ &= \frac{1}{8(1+r^2)^3} \left[ 16r(r^2 - 3) + \pi^3(r^2 + 1)^3 \operatorname{sech}^2 \left( \frac{\pi r}{2} \right) \tanh \left( \frac{\pi r}{2} \right) \right]. \end{aligned} \quad (22)$$

From Theorem 3.1 and Proposition 3.1, inequality (20) follows. ■

*Remark 3.1.* By numerical calculation, we can find that the lower bound of the inequality (20) is better than that of (3) if  $0.5625 < r < 1.784$ .

**Theorem 3.2.** *Suppose  $c$  is a positive number, then, for any positive real number  $\alpha$ , we have*

$$\frac{1}{c^2 + \frac{1}{2}} < \sum_{n=1}^{\infty} \frac{2n^{\alpha/2}}{(n^\alpha + c^2)^2} < \frac{1}{c^2}. \quad (23)$$

*Proof.* By standard argument, we obtain

$$\begin{aligned} & \frac{1}{(n^{\alpha/2} - \frac{1}{2})^2 + c^2 - \frac{1}{4}} - \frac{1}{(n^{\alpha/2} + \frac{1}{2})^2 + c^2 - \frac{1}{4}} \\ &= \frac{2n^{\alpha/2}}{(n^\alpha + c^2 - n^{\alpha/2})(n^\alpha + c^2 + n^{\alpha/2})} \\ &> \frac{2n^{\alpha/2}}{(n^\alpha + c^2)^2} \\ &> \frac{2n^{\alpha/2}}{(n^\alpha + c^2)^2 + c^2 + \frac{1}{4}} \\ &= \frac{1}{(n^{\alpha/2} - \frac{1}{2})^2 + c^2 + \frac{1}{4}} - \frac{1}{(n^{\alpha/2} + \frac{1}{2})^2 + c^2 + \frac{1}{4}}, \end{aligned}$$

the proof is complete. ■

## 4. OPEN PROBLEMS

In [5], Professor B.-N. Guo proposed the following

**Open Problem 4.1.** *Let*

$$S(r, t) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^{t+1}}, \quad (24)$$

where  $t > 0$  and  $r > 0$ . Can one obtain an integral representation of  $S(r, t)$  similar to (4)?

Now we propose two similar open problems below

**Open Problem 4.2.** *Let*

$$S(r, \alpha) = \sum_{n=1}^{\infty} \frac{2n^{\alpha/2}}{(n^{\alpha} + r^2)^2}, \quad (25)$$

where  $r > 0$  and  $\alpha > 0$ . Can one establish an integral expression of  $S(r, \alpha)$ ?

**Open Problem 4.3.** *Let*

$$S(r, t, \alpha) = \sum_{n=1}^{\infty} \frac{2n^{\alpha/2}}{(n^{\alpha} + r^2)^{t+1}} \quad (26)$$

for  $t > 0$ ,  $r > 0$  and  $\alpha > 0$ . Can one obtain an integral expression of  $S(r, t, \alpha)$ ?

In [12, p. 356], the following formula is given

$$\int_0^{\infty} \frac{x^{2m} \sin(ax)}{e^{2n\alpha x} - e^{(2n-2)\alpha x}} dx = (-1)^m \frac{\partial^{2m}}{\partial a^{2m}} \left[ \frac{\pi}{4\alpha} \coth \frac{a\pi}{2\alpha} - \frac{1}{2a} - \sum_{\nu=1}^{n-1} \frac{a}{a^2 + (2\nu)^2 \alpha^2} \right], \quad (27)$$

where  $a > 0$ ,  $\alpha > 0$ ,  $m = 0, 1, 2, \dots$  and  $n = 1, 2, \dots$ . If  $n = 1$  in formula (27), then the summation terms are omitted.

**Open Problem 4.4.** *Find suitable ranges of numbers  $a$ ,  $\alpha$ ,  $m$  and  $n$  such that*

$$\frac{x}{e^x - 1} - \frac{x^{2m}}{e^{(2n+1)\alpha x} - e^{(2n-1)\alpha x}}, \quad \alpha > 0 \quad \text{and} \quad m, n = 0, 1, 2, \dots \quad (28)$$

or

$$\frac{x}{e^x - 1} - \frac{x^{2m}}{e^{2n\alpha x} - e^{(2n-2)\alpha x}}, \quad \alpha > 0, \quad m = 0, 1, 2, \dots \quad \text{and} \quad n = 1, 2, \dots \quad (29)$$

are monotonic in  $x$  respectively.

## REFERENCES

- [1] H. Alzer, J. L. Brenner, and O. G. Ruehr, *On Mathieu's inequality*, J. Math. Anal. Appl. **218** (1998), 607–610.
- [2] P. S. Bullen, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics **97**, Addison Wesley Longman Limited, 1998.
- [3] A. Elbert, *Asymptotic expansion and continued fraction for Mathieu's series*, Period. Math. Hungar. **13** (1982), no. 1, 1–8.
- [4] O. E. Emersleben, *Über die Reihe  $\sum_{k=1}^{\infty} k(k^2 + c^2)^{-2}$* , Math. Ann. **125** (1952), 165–171.
- [5] B.-N. Guo, *Note on Mathieu's inequality*, RGMIA Res. Rep. Coll. **3** (2000), no. 3, Article 5.  
<http://rgmia.vu.edu.au/v3n3.html>.
- [6] J.-Ch. Kuang, *Changyong Budengshi* (Applied Inequalities), 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)
- [7] E. Makai, *On the inequality of Mathieu*, Publ. Math. Debrecen **5** (1957), 204–205.
- [8] E. Mathieu, *Traité de physique mathématique, VI–VII: Théorie de l'élasticité des corps solides*, Gauthier-Villars, Paris, 1890.
- [9] D. S. Mitrinović, *Analytic Inequalities*, Springer, New York, 1970.
- [10] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
- [11] D. C. Russell, *A note on Mathieu's inequality*, Aequationes Mathematicae, **36** (1988), 294–302.
- [12] F.-W. Zou, Zh.-Zh. Liu, and H.-Ch. Zhou, *Jifen Biao Huibian* (Collection of Integral Formulae), Space Navigation Press, Beijing, China, 1992. (Chinese)

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