3-Dimensional L-Summing Method

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Abstract

In this short note, we apply L-Summing Method on some 3-dimensional multiplication tables to yield some new identities involving Riemann zeta function.

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Let a_{ij} be an $n \times n$ array. The 2-dimensional $L-$Summing Method is the following rearrange:

$$\sum_{1\leq i,j\leq n} a_{ij} = \sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_{ik} + \sum_{j=1}^{k} a_{kj} - a_{kk}\right).$$

Specially, when $a_{ij} = (ij)^{-s}$, we yield [2]:

$$\sum_{k=1}^{n}\frac{\zeta_k(s)}{k^s} = \frac{\zeta_n^2(s)+\zeta_n(2s)}{2}, \qquad (s\in\mathbb{C}).$$

The base of this array was 2-dimensional multiplication table and since we can generalize multiplication table to hight dimensional versions [1], we can generalize $L-$Summing Method, and we are going to do this generalization in $\mathbb{R}^3$. In this case, $L-$Summing Elements are 3-dimensional as the following figure:

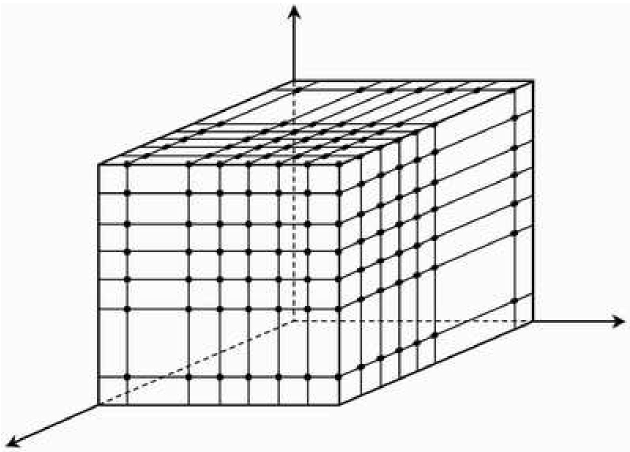

Figure 1: $L-$Summing Elements in $\mathbb{R}^3$

For start, let proceed on $\mathrm{MT}^3_{n\times n}$ [1]; i.e. let $a_{ijk} = ijk$, in which $1 \leq i,j,k \leq n$. Easily, we have

$$S = \sum_{1\leq i,j,k\leq n} ijk = \Big(\frac{n(n+1)}{2}\Big)^3.$$

By $L-$Summing Method; according to above figure, $L-$Summing Element in this table is:

$$L_k = 3k\Big(\frac{k(k+1)}{2}\Big)^2 - 3k^2\Big(\frac{k(k+1)}{2}\Big) + k^3 = \frac{3}{4}k^5 + \frac{1}{4}k^3.$$

So, we have the following known identity:

$$\sum_{k=1}^{n} \frac{3}{4}k^5 + \frac{1}{4}k^3 = \Big(\frac{n(n+1)}{2}\Big)^3.$$

Now, suppose $s \in \mathbb{C}$ and let $a_{ijk} = (ijk)^{-s}$. It is clear that

$$S = \sum_{1\leq i,j,k\leq n} (ijk)^{-s} = \Big(\sum_{k=1}^{n} \frac{1}{k^s}\Big)^3 = \zeta_n^3(s).$$

By $L-$Summing Method we have:

$$L_k = 3\frac{\zeta_k^2(s)}{k^s} - 3\frac{\zeta_k(s)}{k^{2s}} + \frac{1}{k^{3s}},$$

and we can reform $\sum L_k = S$ as follows:

$$\sum_{k=1}^{n} \frac{\zeta_k^2(s)}{k^s} - \frac{\zeta_k(s)}{k^{2s}} = \frac{\zeta_n^3(s) - \zeta_n(3s)}{3}.$$

Note that if $\Re(s) > 1$, then $\lim_{n\to\infty} \zeta_n(s) = \zeta(s)$. So, for $\Re(s) > 1$ we have

$$\sum_{k=1}^{\infty} \frac{\zeta_k^2(s)}{k^s} - \frac{\zeta_k(s)}{k^{2s}} = \frac{\zeta^3(s) - \zeta(3s)}{3}.$$

Also, if $s = 1$, then $\zeta_n(1) = H_n = \sum_{k=1}^{n} \frac{1}{k}$, and so, we have

$$\sum_{k=1}^{n} \frac{H_k^2}{k} - \frac{H_k}{k^2} = \frac{H_n^3 - \zeta_n(3)}{3}.$$

References

[1] M. Hassani, A Generalization of Multiplication Table, *RGMIA Research Report Collection*, **7**(4), Article 13, 2004.
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[2] M. Hassani and S. Rahimpour, L-Summing Method, *RGMIA Research Report Collection*, **7**(4), Article 10, 2004.
Available at: http://rgmia.vu.edu.au/v7n4.html