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INTEGRAL INEQUALITIES ON INFINITE INTERVALS

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ABSTRACT. Inequalities concerning the distance between a function and some integrals on infinite intervals are given.

1. INTRODUCTION

Let $-\infty \leq a < b \leq +\infty$ and $w \in L(a, b)$ a Lebesgue integrable function on (a, b) with $\int_a^b w(s) ds \neq 0$.

The following identity holding for locally absolutely continuous functions $f : (a, b) \rightarrow \mathbb{R}$, where (a, b) is finite or infinite, is known in the literature as the *weighted Montgomery identity*:

$$\begin{aligned} (1.1) \quad f(x) - \frac{1}{\int_a^b w(s) ds} \int_a^b w(t) f(t) dt \\ = \frac{1}{\int_a^b w(s) ds} \int_a^x \left(\int_a^t w(s) ds \right) f'(t) dt \\ - \frac{1}{\int_a^b w(s) ds} \int_x^b \left(\int_t^b w(s) ds \right) f'(s) ds \end{aligned}$$

for any $x \in (a, b)$.

For a simple proof of this fact we refer to the monograph [2, p. 376] where further similar results are provided.

For generalisations to the case of n -time differentiable functions we refer to [3].

In [1] a different representation for the left hand side of (1.1) has been provided

$$\begin{aligned} (1.2) \quad f(x) - \frac{1}{\int_a^b w(s) ds} \int_a^b w(t) f(t) dt \\ = \frac{1}{\int_a^b w(s) ds} \int_a^b w(t) (x - t) \left(\int_0^1 f'[(1 - \lambda)x + \lambda t] d\lambda \right) dt \end{aligned}$$

for any $x \in (a, b)$.

If $a = 0$, $b = \infty$, $w(t) = e^{-t}$, then from (1.1) we obtain the identity:

$$(1.3) \quad f(x) - \int_0^\infty e^{-t} f(t) dt = \int_0^x (1 - e^{-t}) f'(t) dt - \int_x^\infty e^{-t} f'(t) dt$$

for any $x \in [0, \infty)$, provided the involved integrals exist for f locally absolutely continuous on $[0, \infty)$.

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Taking the modulus in (1.3), we get

$$(1.4) \quad \left| f(x) - \int_0^\infty e^{-t} f(t) dt \right| \leq \int_0^x (1 - e^{-t}) |f'(t)| dt + \int_x^\infty e^{-t} |f'(t)| dt := I(x),$$

for $x \in [0, \infty)$.

One can obtain various upper bounds for I . For instance,

$$(1.5) \quad \begin{aligned} I(x) &\leq \operatorname{ess\,sup}_{t \in [0, x]} |f'(t)| \int_0^x (1 - e^{-t}) dt + \operatorname{ess\,sup}_{t \in [x, \infty)} |f'(t)| \int_x^\infty e^{-t} dt \\ &= (e^{-x} + x - 1) \|f'\|_{[0, \infty), \infty} + e^{-x} \|f'\|_{[x, \infty)} \\ &\leq (2e^{-x} + x - 1) \|f'\|_{[0, \infty), \infty}, \quad x \in [0, \infty), \end{aligned}$$

provided $f' \in L_\infty[0, \infty)$.

The inequalities between the first and last term in (1.5) have been pointed out in [2, p. 377].

Also,

$$(1.6) \quad \begin{aligned} I(x) &\leq \sup_{t \in [0, x]} (1 - e^{-t}) \|f'\|_{[0, x], 1} + \sup_{t \in [x, \infty)} e^{-t} \|f'\|_{[x, \infty), 1} \\ &= (1 - e^{-x}) \|f'\|_{[0, x], 1} + e^{-x} \|f'\|_{[x, \infty), 1} \\ &\leq \max \{1 - e^{-x}, e^{-x}\} \|f'\|_{[0, \infty), 1} \\ &= \frac{1 + |1 - 2e^{-x}|}{2} \|f'\|_{[0, \infty), 1}, \quad x \in [0, \infty), \end{aligned}$$

provided $f' \in L_1[0, \infty)$.

If one uses Hölder type inequalities, then one may deduce other bounds for $I(x)$ in terms of the p -norms of f' , $p > 1$.

Now, if we use (1.2) for $a = 0$, $b = \infty$ and $w(t) = e^{-t}$, then we may state

$$(1.7) \quad f(x) - \int_0^\infty e^{-t} f(t) dt = \int_0^\infty e^{-t} (x - t) \left(\int_0^1 f'[(1 - \lambda)x + \lambda t] d\lambda \right) dt$$

for any $x \in (0, \infty)$, provided that the involved integrals exist.

Taking the modulus on (1.7) we have

$$(1.8) \quad \left| f(x) - \int_0^\infty e^{-t} f(t) dt \right| \leq \int_0^\infty e^{-t} |x - t| \left(\int_0^1 |f'[(1 - \lambda)x + \lambda t]| d\lambda \right) dt =: J(x)$$

for $x \in [0, \infty)$.

On making use of similar arguments outlined above, we may produce various bounds for $J(x)$ in terms of the p -norms $\|f'\|_p$. If $|f'|$ is convex on $(0, \infty)$, then

$$|f'[(1 - \lambda)x + \lambda t]| \leq (1 - \lambda) |f'(x)| + \lambda |f'(t)|$$

for any $x, y \in [0, \infty)$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned} J(x) &\leq \int_0^\infty e^{-t} |x-t| \left[|f'(x)| \int_0^1 (1-\lambda) dx + |f'(t)| \int_0^1 \lambda d\lambda \right] \\ &= \frac{1}{2} \int_0^\infty e^{-t} |x-t| |f'(t)| dt + \frac{1}{2} |f'(x)| \int_0^\infty e^{-t} |x-t| dt \\ &\leq \frac{1}{2} \left[\|f'\|_{[0,\infty),\infty} + |f'(x)| \right] \int_0^\infty e^{-t} |x-t| dt \\ &= \frac{1}{2} \left[\|f'\|_{[0,\infty),\infty} + |f'(x)| \right] (2e^{-x} + x - 1), \end{aligned}$$

for any $x \in [0, \infty)$, which is an improvement on the result

$$(1.9) \quad \left| f(x) - \int_0^\infty e^{-t} f(t) dt \right| \leq (2e^{-x} + x - 1) \|f'\|_{[0,\infty),\infty}, \quad x \geq 0$$

that has been obtained in [2, p. 377].

We note that for $x \rightarrow \infty$ the bound (1.9) is tending to ∞ as well, showing that for large $x \in (0, \infty)$, $\int_0^\infty e^{-t} f(t) dt$ is far from $f(x)$ even if $f' \in L_\infty(0, \infty)$.

It is natural to enquire how we can modify the expression under the integral such that its absolute distance from $f(x)$ will remain finite for any $x \in [0, \infty)$.

The aim of this paper is to provide some inequalities for which the absolute value of the difference between a function and an integral transform of it remain finite for any x in an infinite interval.

2. THE RESULTS

The following result holds.

Theorem 1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a locally absolutely continuous function on \mathbb{R} . Then for any $x \in \mathbb{R}$ we have the inequalities*

$$\begin{aligned} (2.1) \quad &\left| f(x) - \int_0^\infty \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv \right| \\ &\leq \frac{1}{2} \left[\int_{-\infty}^x e^{t-x} |f'(t)| dt + \int_x^\infty |f'(t)| dt \right] \\ &\leq \begin{cases} \frac{1}{2} \left[\|f'\|_{(-\infty,x],\infty} + \|f'\|_{[x,\infty),\infty} \right] & \text{if } f' \in L_\infty(\mathbb{R}); \\ \frac{1}{2 \cdot q^{\frac{1}{q}}} \left[\|f'\|_{(-\infty,x],p} + \|f'\|_{[x,\infty),p} \right] & \text{if } f' \in L_p(\mathbb{R}), \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \|f'\|_{\mathbb{R},1} & \text{if } f' \in L(\mathbb{R}), \end{cases} \\ &\leq \begin{cases} \|f'\|_{\mathbb{R},\infty} & \text{if } f' \in L_\infty(\mathbb{R}); \\ \frac{1}{2^{\frac{1}{p} \cdot \frac{1}{q}}} \|f'\|_{\mathbb{R},p} & \text{if } f' \in L_p(\mathbb{R}), p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \|f'\|_{\mathbb{R},1} & \text{if } f' \in L(\mathbb{R}). \end{cases} \end{aligned}$$

Proof. Define the function $p : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$(2.2) \quad p(t, x) := \begin{cases} \exp(t - x) & \text{if } -\infty < t \leq x < \infty, \\ -\exp(x - t) & \text{if } -\infty < x < t < \infty, \end{cases}$$

then we have

$$(2.3) \quad \begin{aligned} \int_{-\infty}^{\infty} p(x, t) f'(t) dt &= \int_{-\infty}^x e^{t-x} f'(t) dt + \int_x^{\infty} e^{x-t} f'(t) dt \\ &= e^{t-x} f(t) \Big|_{-\infty}^x - \int_{-\infty}^x e^{t-x} f(t) dt \\ &\quad - \left[e^{x-t} f(t) \Big|_x^{\infty} + \int_x^{\infty} e^{x-t} f(t) dt \right] \\ &= f(x) - \int_{-\infty}^x e^{t-x} f(t) dt + f(x) - \int_x^{\infty} e^{x-t} f(t) dt \\ &= 2f(x) - \left[\int_{-\infty}^x e^{t-x} f(t) dt + \int_x^{\infty} e^{x-t} f(t) dt \right]. \end{aligned}$$

On the other hand, by changing the variable $t - x = u$, we have

$$\int_{-\infty}^x e^{t-x} f(t) dt = \int_{-\infty}^0 e^u f(x+u) du$$

and by $v = -u$, we deduce

$$\begin{aligned} \int_{-\infty}^0 e^u f(x+u) du &= \int_{\infty}^0 e^{-v} f(x-v) d(-v) \\ &= \int_0^{\infty} e^{-v} f(x-v) dv. \end{aligned}$$

Also, if we choose $v = t - x$ in the second integral, we have

$$\int_x^{\infty} e^{x-t} f(t) dt = \int_0^{\infty} e^{-v} f(x+v) dv$$

and thus, by (2.3), we get the following identity that is of interest in itself as well

$$(2.4) \quad f(x) - \int_0^{\infty} \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv = \frac{1}{2} \int_{-\infty}^{\infty} p(x, t) f'(t) dt$$

for any $x \in \mathbb{R}$.

Now, if we take the modulus in (2.4), we deduce

$$\begin{aligned} \left| f(x) - \int_0^{\infty} \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv \right| \\ \leq \frac{1}{2} \left[\int_{-\infty}^x e^{t-x} |f'(t)| dt + \int_x^{\infty} e^{x-t} |f'(t)| dt \right], \end{aligned}$$

and the first inequality in (2.1) is proven.

Now, if $f' \in L_{\infty}(\mathbb{R})$, then obviously

$$\begin{aligned} \int_{-\infty}^x e^{t-x} |f'(t)| dt &\leq \|f'\|_{(-\infty, x], \infty} \int_{-\infty}^x e^{t-x} dt \\ &= \|f'\|_{(-\infty, x], \infty} \end{aligned}$$

and

$$\int_x^\infty e^{x-t} |f'(t)| dt \leq \|f'\|_{(-\infty, x], \infty} \int_x^\infty e^{x-t} dt = \|f'\|_{(-\infty, x], \infty}.$$

If $f' \in L_p(\mathbb{R})$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then, by Hölder's inequality, we have

$$\begin{aligned} \int_{-\infty}^x e^{t-x} |f'(t)| dt &\leq \left(\int_{-\infty}^x e^{q(t-x)} dt \right)^{\frac{1}{q}} \left(\int_{-\infty}^x |f'(t)|^p dt \right)^{\frac{1}{p}} \\ &= \frac{1}{q^{\frac{1}{q}}} \|f'\|_{(-\infty, x], p} \end{aligned}$$

and, similarly,

$$\int_x^\infty e^{x-t} |f'(t)| dt \leq \frac{1}{q^{\frac{1}{q}}} \|f'\|_{[x, \infty), p},$$

getting the second part of the second inequality in (2.1).

Also, since

$$\begin{aligned} \int_{-\infty}^x e^{t-x} |f'(t)| dt &\leq \sup_{-\infty < t \leq x} e^{t-x} \int_{-\infty}^x |f'(t)| dt = \|f'\|_{(-\infty, x], 1}, \\ \int_x^\infty e^{x-t} |f'(t)| dt &\leq \|f'\|_{[x, \infty), 1} \end{aligned}$$

and

$$\|f'\|_{(-\infty, x], 1} + \|f'\|_{[x, \infty), 1} = \|f'\|_{\mathbb{R}, 1},$$

then the last part of the second inequality in (2.1) is also proven.

Now, since

$$\begin{aligned} \frac{1}{2} \left[\|f'\|_{(-\infty, x], \infty} + \|f'\|_{[x, \infty), \infty} \right] &\leq \max \left\{ \|f'\|_{(-\infty, x], \infty}, \|f'\|_{[x, \infty), \infty} \right\} \\ &= \|f'\|_{\mathbb{R}, \infty} \end{aligned}$$

the first part of the third inequality in (2.1) is proved.

Using the elementary inequality

$$\alpha + \beta \leq 2^{\frac{1}{q}} (\alpha^p + \beta^p)^{\frac{1}{p}}, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha, \beta \geq 0,$$

we deduce that

$$\begin{aligned} \|f'\|_{(-\infty, x], p} + \|f'\|_{[x, \infty), p} &\leq 2^{\frac{1}{q}} \left(\|f'\|_{(-\infty, x], p}^p + \|f'\|_{[x, \infty), p}^p \right)^{\frac{1}{p}} \\ &= 2^{\frac{1}{q}} \|f'\|_{\mathbb{R}, p} \end{aligned}$$

and the second part of the third inequality is also proven.

The proof is completed. ■

The following result may be stated as well.

Theorem 2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a locally absolutely continuous function on \mathbb{R} such that there exist the constants $\gamma, \Gamma \in \mathbb{R}$ such that

$$(2.5) \quad \gamma \leq f'(t) \leq \Gamma \quad \text{for a.e. } t \in \mathbb{R},$$

then

$$(2.6) \quad \left| f(x) - \int_0^\infty \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv \right| \leq \frac{1}{2} (\Gamma - \gamma),$$

for any $x \in \mathbb{R}$.

Proof. From the proof of Theorem 1, we know that

$$(2.7) \quad f(x) - \int_0^\infty \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv = \frac{1}{2} \left[\int_{-\infty}^x e^{t-x} f'(t) dt - \int_x^\infty e^{x-t} f'(t) dt \right].$$

Utilising (2.5) we have, for a fixed $x \in \mathbb{R}$, that

$$\gamma e^{t-x} \leq e^{t-x} f'(t) \leq \Gamma e^{t-x} \quad \text{for a.e. } t \in (-\infty, x]$$

and

$$-\Gamma e^{x-t} \leq -f'(t) e^{x-t} \leq -\gamma e^{x-t} \quad \text{for a.e. } t \in [x, \infty),$$

which gives, by integration,

$$\gamma e^{-x} \int_{-\infty}^x e^t dt \leq \int_{-\infty}^x e^{t-x} f'(t) dt \leq \Gamma e^{-x} \int_{-\infty}^x e^t dt$$

and

$$-\Gamma e^x \int_x^\infty e^{-t} dt \leq -\int_x^\infty e^{x-t} f'(t) dt \leq -\gamma e^x \int_x^\infty e^{-t} dt$$

i.e.,

$$\gamma \leq \int_{-\infty}^x e^{t-x} f'(t) dt \leq \Gamma$$

and

$$-\Gamma \leq -\int_x^\infty e^{x-t} f'(t) dt \leq -\gamma,$$

which, by addition, provide the desired inequality (2.6). ■

Remark 1. The inequality (2.6) is better than the inequality

$$\left| f(x) - \int_0^\infty \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv \right| \leq \|f'\|_{\mathbb{R}, \infty},$$

which has been obtained in (2.1). This follows by the fact that, if (2.5) holds true, then $-\|f'\|_{\mathbb{R}, \infty} \leq \gamma$ and $\Gamma \leq \|f'\|_{\mathbb{R}, \infty}$, where $\|f'\|_{\mathbb{R}, \infty} := \text{ess sup}_{t \in \mathbb{R}} |f'(t)|$.

The case of convex functions is incorporated in the following theorem.

Theorem 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on \mathbb{R} and $f'_+(x)$, $f'_-(x)$ the lateral derivatives in x , $x \in \mathbb{R}$, then

$$(2.8) \quad f(x) - \int_0^\infty \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv \leq \frac{1}{2} [f'_-(x) - f'_+(x)] \leq 0$$

for any $x \in \mathbb{R}$.

Proof. Since f is convex, hence $f'(t) \leq f'_-(x)$ for a.e. $t \in (-\infty, x]$ and $f'(t) \geq f'_+(x)$ for a.e. $t \in [x, \infty)$. This implies that,

$$(2.9) \quad \int_{-\infty}^x e^{t-x} f'(t) dt \leq \int_{-\infty}^x e^{t-x} f'_-(x) dt = f'_-(x)$$

and

$$(2.10) \quad -\int_x^\infty e^{x-t} f'(t) dt \leq -\int_x^\infty e^{x-t} f'_+(x) dt = -f'_+(x)$$

for any $x \in \mathbb{R}$.

Adding (2.9) to (2.10) and utilising the representation (2.7), we deduce the desired inequality (2.8). ■

Remark 2. If f is convex on \mathbb{R} , then we have the inequality:

$$(2.11) \quad \int_0^\infty \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv \geq f(x)$$

for each $x \in \mathbb{R}$. This inequality may be proved on using the definition of convexity as well. Namely, since

$$\frac{f(x-v) + f(x+v)}{2} \geq f(x),$$

then

$$\int_0^\infty \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv \geq f(x) \int_0^\infty e^{-v} dv = f(x),$$

which is exactly (2.11).

Note that in general (2.8) is a better result than (2.11) since, for instance, if one considers the convex function $f(t) := |t - x|$, $t \in \mathbb{R}$, then $\frac{1}{2} [f'_-(x) - f'_+(x)] = -1$.

REFERENCES

- [1] S.S. DRAGOMIR, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.*, **3**(5) (2002), Art. 68. [ONLINE <http://jipam.vu.edu.au/article.php?sid=220>]
- [2] S.S. DRAGOMIR and Th.M. RASSIAS (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.
- [3] M. MATIĆ, J. PEČARIĆ and N. UJEVIĆ, Generalisations of weighted version of Ostrowski's inequality and some related results, *J. Inequal. Appl.*, **8**(5) (2000), 639-666.
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