



**VICTORIA UNIVERSITY**  
MELBOURNE AUSTRALIA

*Certain Logarithmically N-Alternating Monotonic Functions Involving Gamma and q-Gamma Functions*

This is the Published version of the following publication

Qi, Feng (2005) Certain Logarithmically N-Alternating Monotonic Functions Involving Gamma and q-Gamma Functions. Research report collection, 8 (3).

The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository <https://vuir.vu.edu.au/17432/>

# CERTAIN LOGARITHMICALLY $N$ -ALTERNATING MONOTONIC FUNCTIONS INVOLVING GAMMA AND $q$ -GAMMA FUNCTIONS

FENG QI

ABSTRACT. In the paper, three basic properties of the logarithmically  $N$ -alternating monotonic functions are established and the monotonicity results of some functions involving the gamma and  $q$ -gamma functions, which are obtained in [W. E. Clark and M. E. H. Ismail, *Inequalities involving gamma and psi functions*, Anal. Appl. (Singap.) **1** (2003), no. 1, 129–140.], are generalized to the logarithmically  $N$ -alternating monotonicity.

## 1. INTRODUCTION

Recall that the definition of completely monotonic functions is well-known, and can be stated as follows.

**Definition 1.** A function  $f$  is called *completely monotonic* on an interval  $I$  if  $f$  has derivatives of all orders on  $I$  and

$$0 \leq (-1)^k f^{(k)}(x) < \infty \quad (1)$$

for all  $k \geq 0$  on  $I$ .

The class of completely monotonic functions on  $I$  is denoted by  $\mathcal{C}[I]$ .

In 2004, the paper [15] explicitly introduces the following notion or terminology.

---

2000 *Mathematics Subject Classification.* Primary 33B15; Secondary 26A48, 26A51.

*Key words and phrases.* Completely monotonic function, logarithmically completely monotonic function, reciprocally completely monotonic function,  $N$ -alternating monotonic function, logarithmically  $N$ -alternating monotonic function, reciprocally  $N$ -alternating monotonic function,  $N$ -alternating monotonic function to  $\alpha$ -power, completely monotonic function to  $\alpha$ -power, gamma function, psi function,  $q$ -gamma function.

The author was supported in part by the Science Foundation of Project for Fostering Innovation Talents at Universities of Henan Province, China.

This paper was typeset using  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$ .

**Definition 2.** A positive function  $f$  is called *logarithmically completely monotonic* on an interval  $I$  if  $f$  has derivatives of all orders on  $I$  and its logarithm  $\ln f$  satisfies

$$0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty \quad (2)$$

for all  $k \in \mathbb{N}$  on  $I$ .

The set of logarithmically completely monotonic functions on an interval  $I$  is denoted by  $\mathcal{L}[I]$ .

Among other things, it is proved in [14, 15, 22] that a logarithmically completely monotonic function is always completely monotonic, that is,  $\mathcal{L}[I] \subset \mathcal{C}[I]$ , but not conversely. Motivated by the papers [15, 19], among other things, it is further revealed in [3] that  $\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0, \infty)] \subset \mathcal{C}[(0, \infty)]$ , where  $\mathcal{S}$  denotes the set of Stieltjes transforms. In [3, Theorem 1.1] and [8, 18] it is pointed out that logarithmically completely monotonic functions on  $(0, \infty)$  can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [9, Theorem 4.4]. In [16], among other things, a basic property of the logarithmically completely monotonic functions is obtained: If  $h'(x) \in \mathcal{C}[I]$  and  $f(x) \in \mathcal{L}[h(I)]$ , then  $f(h(x)) \in \mathcal{L}[I]$ . For more information on the logarithmically completely monotonic functions defined by Definition 2, please refer to [3, 8, 14, 17, 18, 19], especially [16, 22], and the references therein.

The following definition can be found in [6, 11, 12, 22].

**Definition 3.** A function  $f$  is called  *$N$ -alternating monotonic* on an interval  $I$  if there exists some nonnegative integer  $N$  such that inequality (1) holds for all  $0 \leq k \leq N + 1$  on  $I$ .

The class of  $N$ -alternating monotonic functions on an interval  $I$  will be denoted by  $\mathcal{C}_{N+1}[I]$ . Note that functions in  $\mathcal{C}_N[I]$  are called “monotonic of order  $N$ ” in [11, 12]. Here, we adopt the terminology “ $N$ -alternating monotonic” coined in [6]. It is obvious that  $\mathcal{C}_\infty[I] \triangleq \lim_{N \rightarrow \infty} \mathcal{C}_N[I] = \mathcal{C}[I]$ .

Further, by slightly modifying of corresponding classes of functions in [21, 22, 23] and formally assigning of names, we pose the following definitions.

**Definition 4.** A positive function  $f$  is said to be *logarithmically  $N$ -alternating monotonic* on an interval  $I$  if there exists some nonnegative integer  $N$  such that inequality (2) holds for all  $1 \leq k \leq N + 1$  on  $I$ .

**Definition 5.** For some nonnegative integer  $N$ , a function  $f$  is called  *$N$ -alternating monotonic to  $\alpha$ -power* on an interval  $I$  if either  $f \geq 0$  and  $f^\alpha \in \mathcal{C}_{N+1}[I]$  for  $\alpha > 0$  or  $f > 0$  and  $f^\alpha \in \mathcal{C}_{N+1}[I]$  for  $\alpha < 0$ . In particular, a positive function  $f$  is said to be *reciprocally  $N$ -alternating monotonic* on  $I$  if  $1/f \in \mathcal{C}_{N+1}[I]$ .

**Definition 6.** For some nonnegative integer  $N$ , a function  $f$  is said to be *completely monotonic to  $\alpha$ -power* on an interval  $I$  if either  $f \geq 0$  and  $f^\alpha \in \mathcal{C}[I]$  for  $\alpha > 0$  or  $f > 0$  and  $f^\alpha \in \mathcal{C}[I]$  for  $\alpha < 0$ . In particular, a positive function  $f$  is called *reciprocally completely monotonic* on  $I$  if  $1/f \in \mathcal{C}[I]$ .

The sets of logarithmically  $N$ -alternating monotonic functions,  $N$ -alternating monotonic functions to  $\alpha$ -power and completely monotonic functions to  $\alpha$ -power on an interval  $I$  are respectively denoted by  $\mathcal{L}_{N+1}[I]$ ,  $\mathcal{C}_{N+1}^\alpha[I]$  and  $\mathcal{C}^\alpha[I]$ . It is easy to see that  $\mathcal{L}_\infty[I] \triangleq \lim_{N \rightarrow \infty} \mathcal{L}_N[I] = \mathcal{L}[I]$ ,  $\mathcal{C}_\infty^\alpha[I] \triangleq \lim_{N \rightarrow \infty} \mathcal{C}_{N+1}^\alpha[I] = \mathcal{C}^\alpha[I]$ .

In [20, 21, 22, 23] the following classes of functions are also defined:

$$\mathcal{D}_N^\alpha[I] = \{f(x) > 0 \mid [f^\alpha(x)]' \in \mathcal{C}_{N-1}[I], N \geq 1, \alpha < 0\}, \quad (3)$$

$$\mathcal{K}_N[I] = \{f(x) \mid f'(x) \in \mathcal{C}_{N-1}[I], N \geq 1\}, \quad (4)$$

$$\mathcal{D}^\alpha[I] = \mathcal{D}_\infty^\alpha[I] = \lim_{N \rightarrow \infty} \mathcal{D}_N^\alpha[I], \quad \alpha < 0, \quad (5)$$

$$\mathcal{K}[I] = \mathcal{K}_\infty[I] = \lim_{N \rightarrow \infty} \mathcal{K}_N[I], \quad (6)$$

$$\mathcal{T}[[0, \infty)] = \left\{ f(x) \mid f(x) = \int_0^x \varphi(t) dt < \infty, f(0) = 0, \varphi(t) \in \mathcal{C}[(0, \infty)] \right\}. \quad (7)$$

These classes of functions have the following inclusion relations for  $N \in \mathbb{N} \cup \{\infty\}$ :

$$\mathcal{D}_1^\alpha[I] = \mathcal{L}_1[I] \subset \mathcal{C}_1[I] = \mathcal{C}_1^1[I], \quad \alpha < 0, \quad (8)$$

$$\mathcal{C}_N^\alpha[I] \subset \mathcal{C}_N^{n\alpha}[I], \quad \alpha > 0, \quad n \in \mathbb{N}, \quad (9)$$

$$\mathcal{T}[[0, \infty)] \neq \mathcal{K}[[0, \infty)], \quad (10)$$

$$\mathcal{D}_N^\alpha[I] \subset \mathcal{D}_N^\beta[I], \quad \alpha < \beta < 0, \quad (11)$$

$$\mathcal{D}_N^\alpha[I] \subset \mathcal{C}_N^\beta[I], \quad \alpha < 0, \quad \beta > 0, \quad (12)$$

$$\mathcal{D}_N^{-\alpha}[I] \subset \mathcal{L}_N[I] \subset \mathcal{C}_N^\alpha[I], \quad \alpha > 0, \quad (13)$$

$$\mathcal{S} \subset \mathcal{D}^{-1}[(0, \infty)] \subset \mathcal{L}[(0, \infty)] \subset \mathcal{C}[(0, \infty)], \quad (14)$$

$$\mathcal{C}_{N+1}^\alpha[I] \subset \mathcal{C}_N^\alpha[I], \quad \alpha > 0, \quad (15)$$

$$\mathcal{D}_{N+1}^\alpha[I] \subset \mathcal{D}_N^\alpha[I], \quad \alpha < 0, \quad (16)$$

$$\mathcal{L}_{N+1}[I] \subset \mathcal{L}_N[I]. \quad (17)$$

Many basic properties of the classes of functions mentioned above were reproved, extended, collected, corrected and established in [22], among other things.

In Section 2 of this paper, we will prove the following results about the class  $\mathcal{L}_N[I]$  of logarithmically  $N$ -alternating monotonic functions, analogies of them have recently been found for the class  $\mathcal{L}[I]$  in [15, 16].

**Theorem 1.** *For  $N \in \mathbb{N} \cup \{\infty\}$ , if  $h(x) \in \mathcal{K}_N[I]$  and  $f \in \mathcal{L}_N[h(I)]$ , then  $f(h(x)) \in \mathcal{L}_N[I]$ .*

**Theorem 2.** *Let  $N \in \mathbb{N} \cup \{\infty\}$  and  $f_i(x) \in \mathcal{L}_N[I]$  and  $\alpha_i \geq 0$  for  $1 \leq i \leq n$  with  $n \in \mathbb{N}$ . Then  $\prod_{i=1}^n [f_i(x)]^{\alpha_i} \in \mathcal{L}_N[I]$ .*

**Theorem 3.** *Let  $N \in \mathbb{N}$  and  $f(x) \in \mathcal{L}_N[I]$ . Then  $f(x)/f(x + \alpha) \in \mathcal{L}_{N-1}[J]$  if and only if  $\alpha > 0$ , where  $J = I \cap \{x + \alpha \in I\}$ .*

Let  $r \geq 2$  be an integer. Canfield proved in [4] that the sequence  $\binom{r}{m} \sqrt{m} / c_1 c_2^m$  is increasing with  $m \geq 1$ , where  $c_1 = \sqrt{r/2\pi(r-1)}$ ,  $c_2 = r^r / (r-1)^{r-1}$ , and the quantity  $c_1 c_2^m / \sqrt{m}$  is the asymptotic value of  $\binom{r}{m}$ . Motivated by Canfield's problem, Clark and Ismail obtained in [5] that the function

$$G(x) = \frac{\prod_{k=1}^n \Gamma(a_k x + 1)}{\Gamma(sx + 1)(2\pi x)^{(n-1)/2}} \frac{s^{sx+1/2}}{\prod_{k=1}^n a_k^{a_k x + 1/2}} \quad (18)$$

is decreasing in  $(0, \infty)$ , where  $a_i > 0$  for  $1 \leq i \leq n$ ,  $s = \sum_{k=1}^n a_k$ , and  $\Gamma(x)$  denotes the classical Euler gamma function defined by  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$  for  $\operatorname{Re} z > 0$ . The gamma function  $\Gamma(x)$ , the psi or digamma function  $\psi(x) = [\ln \Gamma(x)]' = \Gamma'(x)/\Gamma(x)$  and the polygamma functions  $\psi^{(i)}(x)$  for  $i \in \mathbb{N}$  are a class of the most important special functions [1, 24, 25] and have much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

As a generalization of monotonicity result for the function  $G(x)$ , we shall show in Section 2 the following

**Theorem 4.** *Let  $a_i > 0$  for  $1 \leq i \leq n \in \mathbb{N}$  and  $s = \sum_{k=1}^n a_k$ . If  $\sum_{i=1}^n a_i^k \geq s^k$  holds for some  $k \in \mathbb{N}$ , then  $G(x) \in \mathcal{L}_k[(0, \infty)]$ , that is, the function  $G(x)$  defined by (18) is logarithmically  $k$ -alternating monotonic in the interval  $(0, \infty)$ .*

In [5] it is shown that the function

$$F_{a,b}(x) = \frac{[\Gamma(x+b)]^m}{x^{m/2}\Gamma(mx+a)} \quad (19)$$

is decreasing for  $x \geq \max\{0, b-2, (b-2a)/(2m-3)\}$ , where  $b > a > 0$  and  $m \geq 2$  is a positive integer.

As a generalization of the monotonicity result for the function  $F_{a,b}(x)$ , we shall show in Section 2 the following

**Theorem 5.** *Let  $a$  and  $b$  be positive numbers and*

$$\tau(a,b) = \inf_{u \in (0,1]} \{u^{-a} - u^{1-a} + 2u^{b-a}\}. \quad (20)$$

Further, let  $m \geq 2$  and  $k \in \mathbb{N}$  be positive integers and

$$\lambda(a,b,k,m) = \frac{(k-1)\ln m + \ln 2 - \ln \tau(a,b)}{m-1}. \quad (21)$$

Then  $F_{a,b}(x) \in \mathcal{L}_k[(\lambda(a,b,k,m), \infty)] \cap \mathcal{L}_k[(0, \infty)]$ . In particular,

$$F_{a,1}(x) \in \begin{cases} \mathcal{L}_k \left[ \left[ \frac{(k-1)\ln m}{m-1}, \infty \right) \right] & \text{for } a > \frac{1}{2}, \\ \mathcal{L}_k \left[ \left[ \frac{(k-1)\ln m + \ln 2 + \ln[a^a(1-a)^{1-a}]}{m-1}, \infty \right) \right] & \\ \text{for } 0 < a \leq \frac{1}{2}; \end{cases} \quad (22)$$

and

$$F_{a,b}(x) \in \begin{cases} \mathcal{L}_k \left[ \left( \frac{(k-1)\ln m + \ln 2}{m-1}, \infty \right) \right] & \text{for } 0 < b < 1, \\ \mathcal{L}_k \left[ \left[ \frac{(k-1)\ln m + \ln 2 - \ln \left[ 1 + (1-b)(2b^b)^{1/(1-b)} \right]}{m-1}, \infty \right) \right] & \\ \text{for } b > 1. \end{cases} \quad (23)$$

Recall the notation

$$(a; q)_m = \prod_{k=1}^m (1 - aq^{k-1}) \quad (24)$$

for  $m \in \mathbb{N} \cup \{\infty\}$  and that, when  $0 < q < 1$ , the  $q$ -gamma function is defined [2, 7] by

$$\Gamma_q(z) = (1 - q)^{1-z} \prod_{i=0}^{\infty} \frac{1 - q^{i+1}}{1 - q^{z+i}}. \quad (25)$$

It is well known that  $q$ -gamma function is the  $q$ -analogue of the gamma function, that is,  $\lim_{q \rightarrow 1^-} \Gamma_q(z) = \Gamma(z)$ .

Let  $a_k > 0$  for  $1 \leq k \leq n$  and  $s = \sum_{i=1}^n a_k$ . Define

$$H(x) = \frac{\prod_{k=1}^n \Gamma_q(a_k x + 1)}{\Gamma_q(sx + 1) [(q; q)_\infty]^{n-1}} \quad (26)$$

for  $x \in (0, \infty)$ . In [5] it was proved that the function  $H(x)$  decreases to 1 on  $(0, \infty)$ .

As a generalization of this result, the following logarithmically  $N$ -alternating monotonic property for the function  $H(x)$  defined by (26) is obtained.

**Theorem 6.** *Let  $a_k > 0$  for  $1 \leq k \leq n$  and  $s = \sum_{i=1}^n a_k$ . If  $\sum_{i=1}^n a_i^k \geq s^k$  holds for some  $k \in \mathbb{N}$ , then  $H(x) \in \mathcal{L}_k[(0, \infty)]$ .*

## 2. PROOFS OF THEOREMS

*Proof of Theorem 1.* Since  $f \in \mathcal{L}_N[h(I)]$  is equivalent to  $-f'/f \in \mathcal{C}_{N-1}[h(I)]$ , where  $\mathcal{C}_0[I]$  denote the class of positive functions on the interval  $I$ . From the condition  $h(x) \in \mathcal{K}_N[I]$  which means  $(-1)^i h^{(i+1)} \geq 0$  for  $0 \leq i \leq N-1$ ,  $\mathcal{K}_N[I] \subset \mathcal{K}_{N-1}[I]$  which can be deduced readily from (4) and (15), and [22, Theorem A] which states that if  $h \in \mathcal{K}_N[I]$  and  $f \in \mathcal{C}_N[h(I)]$  then  $f(h) \in \mathcal{C}_N[I]$  for  $N \in \mathbb{N} \cup \{0, \infty\}$ , it is easy to see that  $-f'(h)/f(h) \in \mathcal{C}_{N-1}[I]$ , that is,  $(-1)^i [-f'(h)/f(h)]^{(i)} \geq 0$  for  $0 \leq i \leq N-1$ . Therefore, directly calculating gives

$$\begin{aligned} (-1)^k [\ln f(h(x))]^{(k)} &= (-1)^k \left[ \frac{f'(h(x))}{f(h(x))} h'(x) \right]^{(k-1)} \\ &= (-1)^k \sum_{i=0}^{k-1} \binom{k-1}{i} \left[ \frac{f'(h(x))}{f(h(x))} \right]^{(i)} h^{(k-i)}(x) \\ &= \sum_{i=0}^{k-1} \binom{k-1}{i} \left\{ (-1)^i \left[ -\frac{f'(h(x))}{f(h(x))} \right]^{(i)} \right\} [(-1)^{k-i-1} h^{(k-i)}(x)] \\ &\geq 0 \end{aligned} \quad (27)$$

for  $0 \leq k \leq N$ . The proof is complete.  $\square$

*Proof of Theorem 2.* By standard arguments, it follows that

$$(-1)^k \left[ \ln \prod_{i=1}^n (f_i(x))^{\alpha_i} \right]^{(k)} = \sum_{i=1}^n \alpha_i \{ (-1)^k [\ln f_i(x)]^{(k)} \} \geq 0 \quad (28)$$

for  $1 \leq k \leq N$ , since  $f_i(x) \in \mathcal{L}_N[I]$ , that is,  $(-1)^k [\ln f_i(x)]^{(k)} \geq 0$  hold for  $1 \leq k \leq N$  and  $1 \leq i \leq n$ , and  $\alpha_i \geq 0$  for  $1 \leq i \leq n$ . The proof is complete.  $\square$

*Proof of Theorem 3.* From  $f(x) \in \mathcal{L}_N[I]$ , it follows that  $(-1)^k [\ln f(x)]^{(k)} \geq 0$  for  $1 \leq k \leq N$ . This is equivalent to  $[\ln f(x)]^{(2i)} \geq 0$  for  $1 \leq 2i \leq N$  and  $[\ln f(x)]^{(2i-1)} \leq 0$  for  $1 \leq 2i-1 \leq N$ , and then  $[\ln f(x)]^{(2i)}$  is decreasing for  $1 \leq 2i \leq N-1$  and  $[\ln f(x)]^{(2i-1)}$  is increasing for  $1 \leq 2i-1 \leq N-1$ . Therefore, from  $\alpha > 0$  it follows that  $\{\ln[f(x)/f(x+\alpha)]\}^{(2i)} = [\ln f(x)]^{(2i)} - [\ln f(x+\alpha)]^{(2i)} \geq 0$  for  $1 \leq 2i \leq N-1$  and  $\{\ln[f(x)/f(x+\alpha)]\}^{(2i-1)} \leq 0$  for  $1 \leq 2i-1 \leq N-1$ , that is,  $(-1)^i \{\ln[f(x)/f(x+\alpha)]\}^{(i)} \geq 0$  for  $1 \leq i \leq N-1$ . The proof is complete.  $\square$

*Proof of Theorem 4.* Taking the logarithm of  $G(x)$ , using the first Binet's formula for  $\ln \Gamma(x)$

$$\ln \Gamma(x+1) = \left(x + \frac{1}{2}\right) \ln x - x + \frac{\ln(2\pi)}{2} + \int_0^\infty \left[ \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] \frac{e^{-xt}}{t} dt \quad (29)$$

which can be found in [24] and [25, p. 106], and differentiating successively gives

$$(-1)^\ell [\ln G(x)]^{(\ell)} = \int_0^\infty t^{\ell-1} \left[ \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] \left[ \sum_{i=1}^n a_i^\ell e^{-a_i x t} - s^\ell e^{-s x t} \right] dt \quad (30)$$

for any nonnegative integer  $\ell$ .

Since the derivative  $\delta'(t)$  of the function

$$\delta(t) = \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \quad (31)$$

is decreasing and positive in  $(0, \infty)$ , see [13], thus it is easy to obtain that  $\delta(t)$  is increasing and positive in  $(0, \infty)$ , see also [5]. Therefore, it is sufficient to prove

$$\sum_{i=1}^n a_i^k e^{-a_i u} \geq s^k e^{-s u} \quad (32)$$

for all  $u = xt \geq 0$ , which is equivalent to

$$\sum_{i=1}^n a_i^k \exp \left[ \left( \sum_{j \neq i} a_j \right) u \right] \geq s^k. \quad (33)$$



It is obvious that inequality (33) holds if

$$\sum_{i=1}^n a_i^k \geq s^k = \left( \sum_{i=1}^n a_i \right)^k, \quad (34)$$

which can be rewritten as

$$\sum_{i=1}^n \left( \frac{a_i}{\sum_{j=1}^n a_j} \right)^k \geq 1. \quad (35)$$

Since  $a_i / \sum_{j=1}^n a_j < 1$ , then for all  $1 \leq p < k$

$$\sum_{i=1}^n \left( \frac{a_i}{\sum_{j=1}^n a_j} \right)^p > \sum_{i=1}^n \left( \frac{a_i}{\sum_{j=1}^n a_j} \right)^k \geq 1. \quad (36)$$

This implies  $(-1)^q [\ln G(x)]^{(q)} \geq 0$  for all  $1 \leq q \leq k$ . The proof is complete.  $\square$

*Proof of Theorem 5.* It is well known [1, 24, 25] that for  $x > 0$  and  $r > 0$

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt. \quad (37)$$

The psi and polygamma functions can be expressed [1, 24, 25] as

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt \quad (38)$$

and

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} dt, \quad k \in \mathbb{N}. \quad (39)$$

Taking the logarithm of  $F(x)$ , differentiating with respect to  $x$ , utilizing formulas (37), (38) and (39), and simplifying yields

$$\begin{aligned} [\ln F(x)]^{(k)} &= m \left[ \psi^{(k-1)}(x+b) - m^{k-1} \psi^{(k-1)}(mx+a) + \frac{(-1)^k (k-1)!}{2x^k} \right] \\ &= (-1)^k m \int_0^\infty t^{k-1} \left( \frac{1}{2} - \frac{m^{k-1} e^{-[(m-1)x+a]t} - e^{-bt}}{1 - e^{-t}} \right) e^{-xt} dt \end{aligned} \quad (40)$$

for  $k \in \mathbb{N}$ .

In order that  $(-1)^k [\ln F(x)]^{(k)} \geq 0$ , it is sufficient to show

$$\frac{1}{2} - \frac{m^{k-1} e^{-[(m-1)x+a]t} - e^{-bt}}{1 - e^{-t}} \geq 0 \quad (41)$$

for all  $t \geq 0$ , which is equivalent to

$$\begin{aligned} x &\geq \frac{1}{m-1} \left[ (k-1) \ln m - \ln \frac{1 - e^{-t} + 2e^{-bt}}{2e^{-at}} \right] \\ &= \frac{1}{m-1} \left[ (k-1) \ln m + \ln 2 - \ln \frac{1 - u + 2u^b}{u^a} \right] \\ &= \frac{(k-1) \ln m + \ln 2 + a \ln u - \ln (1 - u + 2u^b)}{m-1} \end{aligned} \quad (42)$$

for all  $t \geq 0$  and  $0 < u = e^{-t} \leq 1$ . The first conclusion follows.

If  $b = 1$  and  $a > 1/2$ , the function  $a \ln u - \ln(1+u) \leq -\ln 2$  is increasing in  $(0, 1]$ ; if  $b = 1$  and  $0 < a \leq 1/2$ , the function  $a \ln u - \ln(1+u)$  for  $u \in (0, 1]$  has a maximum  $\ln [a^a(1-a)^{1-a}]$ . By calculus, it is easy to show that the function  $\ln [x^x(1-x)^{1-x}]$  is decreasing in  $x \in (0, 1/2]$ , and then  $0 > \ln [a^a(1-a)^{1-a}] \geq -\ln 2$  for  $0 < a \leq 1/2$ . This implies the second conclusion (22).

If  $b \neq 1$ , then inequality (42) is valid if

$$x \geq \frac{(k-1) \ln m + \ln 2 - \ln (1 - u + 2u^b)}{m-1} \quad (43)$$

for  $u \in (0, 1]$ . It is easy to obtain that the function  $2u^b - u$  has a unique critical point which is a minimum point  $(2b)^{1/(1-b)}$  in  $(0, 1]$  if  $b > 1$ , has an unique critical point which is a maximum point  $(2b)^{1/(1-b)}$  in  $(0, 1]$  if  $0 < b \leq 1/2$ , and is increasing in  $(0, 1]$  if  $1 > b > 1/2$ . Therefore, if  $0 < b < 1$  then  $\ln (1 - u + 2u^b) > 0$ , if  $b > 1$  then  $\ln (1 - u + 2u^b) \geq \ln [1 + 2(2b)^{b/(1-b)} - (2b)^{1/(1-b)}]$  in  $(0, 1]$ . This means that  $(-1)^k [\ln F(x)]^{(k)} \geq 0$  holds if

$$x \begin{cases} > \frac{(k-1) \ln m + \ln 2}{m-1} & \text{for } 0 < b < 1, \\ \geq \frac{(k-1) \ln m + \ln 2 - \ln [1 + (1-b)(2b)^{1/(1-b)}]}{m-1} & \text{for } b > 1. \end{cases} \quad (44)$$

The proof is complete.  $\square$

*Proof of Theorem 6.* Straightforward computation yields

$$\begin{aligned} [\ln H(x)]' &= (\ln q) \sum_{i=1}^{\infty} \left[ \sum_{j=1}^n \frac{a_j q^{i+a_j x}}{1 - q^{i+a_j x}} - \frac{sq^{i+sx}}{1 - q^{i+sx}} \right] \\ &= (\ln q) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[ \sum_{k=1}^n a_k q^{(j+a_k x)i} - sq^{(j+sx)i} \right] \end{aligned} \quad (45)$$

and

$$[\ln H(x)]^{(\ell)} = (\ln q)^\ell \sum_{i=1}^{\infty} i^{\ell-1} \sum_{j=1}^{\infty} \left[ \sum_{k=1}^n a_k^\ell q^{(j+a_k x)i} - s^\ell q^{(j+sx)i} \right] \quad (46)$$

for  $\ell \in \mathbb{N}$ . Thus it suffices to show that

$$\sum_{k=1}^n a_k^\ell q^{(j+a_k x)i} \geq s^\ell q^{(j+sx)i} \quad (47)$$

which is equivalent to

$$\sum_{k=1}^n a_k^\ell q^{a_k i x} \geq s^\ell q^{s i x}. \quad (48)$$

Furthermore, it is clear that inequality (48) is equivalent to (32), which has already been established in the proof of Theorem 4. The proof is complete.  $\square$

#### REFERENCES

- [1] M. Abramowitz and I.A. Stegun (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, with corrections, Washington, 1965.
- [2] G. E. Andrews, R. A. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [3] C. Berg, *Integral representation of some functions related to the gamma function*, *Mediterr. J. Math.* **1** (2004), no. 4, 433–439.
- [4] R. Canfield, *Problem 10310*, *Amer. Math. Monthly* **100** (1993), 499. Solution, *Amer. Math. Monthly* **103** (1996), 431–432.
- [5] W. E. Clark and M. E. H. Ismail, *Inequalities involving gamma and psi functions*, *Anal. Appl. (Singap.)* **1** (2003), no. 1, 129–140.
- [6] A. M. Fink, *Kolmogorov-Landau inequalities for monotone functions*, *J. Math. Anal. Appl.* **90** (1982), 251–258.
- [7] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.
- [8] A. Z. Grinshpan and M. E. H. Ismail, *Completely monotonic functions involving the Gamma and  $q$ -Gamma functions*, *Proc. Amer. Math. Soc.*, to appear.
- [9] R. A. Horn, *On infinitely divisible matrices, kernels and functions*, *Z. Wahrscheinlichkeitstheorie und Verw. Geb* **8** (1967), 219–230.
- [10] J.-Ch. Kuang, *Chángyòng Bùděngshì (Applied Inequalities)*, 2nd ed., Hunan Education Press, Changsha, China, 1993. (Chinese)
- [11] L. Lorch and D. J. Newman, *On the composition of completely monotonic functions and completely monotonic sequences and related questions*, *J. London Math. Soc.* **28** (1983), no. 2, 31–45.

- [12] L. Lorch, M. E. Muldoon, and P. Szegő, *Higher monotonicity properties of certain Sturm-Liouville functions, III*, *Canad. J. Math.* **22** (1970), 1238–1265.
- [13] F. Qi, *A monotonicity result of a function involving the exponential function and an application*, *RGMIA Res. Rep. Coll.* **7** (2004), no. 3, Art. 16. Available online at <http://rgmia.vu.edu.au/v7n3.html>.
- [14] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, *J. Math. Anal. Appl.* **296** (2004), no. 2, 603–607.
- [15] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, *RGMIA Res. Rep. Coll.* **7** (2004), no. 1, Art. 8, 63–72. Available online at <http://rgmia.vu.edu.au/v7n1.html>.
- [16] F. Qi and B.-N. Guo, *Some classes of logarithmically completely monotonic functions involving gamma function*, (2005), submitted.
- [17] F. Qi, B.-N. Guo, and Ch.-P. Chen, *The best bounds in Gautschi-Kershaw inequalities*, *Math. Inequal. Appl.* (2005), accepted.
- [18] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, *J. Austral. Math. Soc.* **79** (2006), no. 1, in press.
- [19] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, *RGMIA Res. Rep. Coll.* **7** (2004), no. 1, Art. 5, 31–36. Available online at <http://rgmia.vu.edu.au/v7n1.html>.
- [20] I. J. Schoenberg, *Metric spaces and completely monotone functions*, *Ann. Math.* **39** (1938), 811–841.
- [21] H. van Haeringen, *Completely monotonic and related functions*, *J. Math. Anal. Appl.* **204** (1996), no. 2, 389–408.
- [22] H. van Haeringen, *Completely Monotonic and Related Functions*, Report 93-108, Faculty of Technical Mathematics and Informatics, Delft University of Technology, Delft, The Netherlands, 1993.
- [23] H. van Haeringen, *Inequalities for real powers of completely monotonic functions*, *J. Math. Anal. Appl.* **210** (1997), no. 1, 102–113.
- [24] Zh.-X. Wang and D.-R. Guo, *Special Functions*, Translated from the Chinese by D.-R. Guo and X.-J. Xia, World Scientific Publishing, Singapore, 1989.
- [25] Zh.-X. Wang and D.-R. Guo, *Tèshū Hánshù Gàilùn (A Panorama of Special Functions)*, The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)

(F. Qi) RESEARCH INSTITUTE OF MATHEMATICAL INEQUALITY THEORY, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN PROVINCE, 454010, CHINA

*E-mail address:* [qifeng@hpu.edu.cn](mailto:qifeng@hpu.edu.cn), [fengqi618@member.ams.org](mailto:fengqi618@member.ams.org)

*URL:* <http://rgmia.vu.edu.au/qi.html>