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SOME MONOTONICITY PROPERTIES OF THE q -GAMMA FUNCTION

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ABSTRACT. We prove some properties of completely monotonic functions and apply them to obtain new results on gamma and q -gamma functions.

1. INTRODUCTION

The q -gamma function is defined for positive real numbers x and $q \neq 1$ by

$$\begin{aligned}\Gamma_q(x) &= (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \quad 0 < q < 1; \\ \Gamma_q(x) &= (q-1)^{1-x} q^{\frac{1}{2}x(x-1)} \prod_{n=0}^{\infty} \frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \quad q > 1.\end{aligned}$$

We note here[11]

$$\lim_{q \rightarrow 1^-} \Gamma_q(x) = \Gamma(x) = \int_0^{\infty} t^x e^{-t} \frac{dt}{t},$$

the well-known Euler's gamma function. From the definition, for x positive and $0 < q < 1$,

$$\Gamma_{1/q}(x) = q^{(x-1)(1-x/2)} \Gamma_q(x),$$

we see that $\lim_{q \rightarrow 1} \Gamma_q(x) = \Gamma(x)$. For historical remarks on gamma and q -gamma functions, we refer the reader to [1], [2] and [11].

There exists an extensive and rich literature on inequalities for the gamma and q -gamma functions. For the recent developments in this area, we refer the reader to the articles [1]-[3], [8], [15] and the references therein. Many of these inequalities follow from the monotonicity properties of functions which are closely related to Γ (resp. Γ_q) and its logarithmic derivative ψ (resp. ψ_q). Here we recall that a function $f(x)$ is said to be absolutely monotonic on (a, b) if it has derivatives of all orders and $f^{(k)}(x) \geq 0, x \in (a, b), k \in \mathbb{N}$. A function $f(x)$ is said to be completely monotonic on (a, b) if it has derivatives of all orders and $(-1)^k f^{(k)}(x) \geq 0, x \in (a, b), k \in \mathbb{N}$.

We note here that $\lim_{q \rightarrow 1} \psi_q(x) = \psi(x)$ (see [12]) and that ψ' and ψ'_q are completely monotonic functions on $(0, \infty)$ (see [3], [9]). Thus, one expects to deduce results on gamma and q -gamma functions from properties of completely monotonic functions, by applying them to functions related to ψ' or ψ'_q . It is our goal in this paper to obtain some results on gamma and q -gamma functions via this approach. Our key tool is Lemma 2.2 below and we first illustrate here three examples which only use the fact that ψ' (resp. ψ'_q) is positive and decreasing on $(0, \infty)$. For instance, for positive numbers a, x, y , Lemma 2.2 implies

$$\psi(a) + \psi(a+x+y) \leq \psi(a+x) + \psi(a+y),$$

which is discussed in [4, p. 59]. Similarly, one checks easily that $\Gamma^\alpha(x)$ is convex on $(0, \infty)$ for $\alpha \geq 0$. Hence it follows from Lemma 2.2 that for positive numbers $x, y, z, \alpha \geq 0$,

$$\Gamma^\alpha(x+y) + \Gamma^\alpha(x+z) \leq \Gamma^\alpha(x) + \Gamma^\alpha(x+y+z),$$

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which is (4.3) in [5]. As another example, we note Alzer[3, Lemma 2.4] has shown that $\psi(e^x)$ is strictly concave on \mathbb{R} . It follows from this and Lemma 2.2 that for positive numbers x, y and real numbers r, s with $r + s \neq 0$,

$$(1.1) \quad \begin{aligned} \psi(x) + \psi(y) &\leq \psi(E(r, s; x, y)) + \psi(E(-r, -s; x, y)) \leq 2\psi(\sqrt{xy}), \\ \psi(x) + \psi(y) &\leq \psi(G(r, s; x, y)) + \psi(G(-r, -s; x, y)) \leq 2\psi(\sqrt{xy}). \end{aligned}$$

Other than the cases of equalities, the above is Theorem 3.7 in [3]. We shall only need to use the inequality $\psi(x) + \psi(y) \leq 2\psi(\sqrt{xy})$ in our subsequent discussions, so we will omit the definitions of $E(r, s; x, y)$ and $G(r, s; x, y)$ here and refer the reader to [3].

2. LEMMAS

Lemma 2.1. ([1, Lemma 1]) *If $f'(x)$ is completely monotonic on $(0, \infty)$, then $\exp(-f(x))$ is also completely monotonic on $(0, \infty)$.*

Lemma 2.2. *Let a_i and b_i ($i = 1, \dots, n$) be real numbers such that $0 < a_1 \leq \dots \leq a_n$, $0 < b_1 \leq \dots \leq b_n$, and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1, \dots, n$. If the function $f(x)$ is decreasing and convex on $(0, \infty)$, then*

$$\sum_{i=1}^n f(b_i) \leq \sum_{i=1}^n f(a_i).$$

If $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, then one only needs $f(x)$ being convex for the above inequality to hold.

The above lemma is similar to Lemma 2 in [1], except here we only assume a_i, b_i 's to be positive and $f(x)$ defined on $(0, \infty)$. We leave the proof to the reader by pointing out that it follows from the theory of majorization, for example, see the discussions in Chap. 1, §28 – §30 of [6].

Lemma 2.3. (Hadamard's inequality) *Let $f(x)$ be a convex function on $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

Lemma 2.4. ([2, Lemma 2.1]) *Let $a > 0, b > 0$ and r be real numbers with $a \neq b$, and let*

$$\begin{aligned} L_r(a, b) &= \left(\frac{a^r - b^r}{r(a-b)}\right)^{1/(r-1)} \quad (r \neq 0, 1), \\ L_0(a, b) &= \frac{a-b}{\log a - \log b}, \\ L_1(a, b) &= \frac{1}{e} \left(\frac{a^a}{b^b}\right)^{1/(a-b)}. \end{aligned}$$

The function $r \mapsto L_r(a, b)$ is strictly increasing on \mathbb{R} .

3. MAIN RESULTS

Theorem 3.1. *Let a_i and b_i ($i = 1, \dots, n$) be real numbers such that $0 \leq a_1 \leq \dots \leq a_n$, $0 \leq b_1 \leq \dots \leq b_n$, and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1, \dots, n$. If $f''(x)$ is completely monotonic on $(0, \infty)$, then*

$$\exp\left(\sum_{i=1}^n (f(x+a_i) - f(x+b_i))\right)$$

is completely monotonic on $(0, \infty)$.

Proof. By Lemma 2.1, it suffices to show that

$$-\sum_{i=1}^n (f'(x+a_i) - f'(x+b_i))$$

is completely monotonic on $(0, \infty)$ or for $k \geq 1$,

$$(-1)^k \sum_{i=1}^n f^{(k)}(x+a_i) \geq (-1)^k \sum_{i=1}^n f^{(k)}(x+b_i).$$

By Lemma 2.2, it suffices to show that $(-1)^k f^{(k)}(x)$ is decreasing and convex on $(0, \infty)$ or equivalently, $(-1)^k f^{(k+1)}(x) \leq 0$ and $(-1)^k f^{(k+2)}(x) \geq 0$ for $k \geq 1$. The last two inequalities hold since we assume that $f''(x)$ is completely monotonic on $(0, \infty)$. This completes the proof. \square

As a direct consequence of Theorem 3.1, we now generalize a result of Alzer[1, Theorem 10], we note here one can also put our next result into a form similar to that of Theorem 4.1 in [8], we leave this to the reader.

Corollary 3.1. *Let a_i and b_i ($i = 1, \dots, n$) be real numbers such that $0 \leq a_1 \leq \dots \leq a_n$, $0 \leq b_1 \leq \dots \leq b_n$, and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1, \dots, n$. Then,*

$$x \mapsto \prod_{i=1}^n \frac{\Gamma_q(x+a_i)}{\Gamma_q(x+b_i)}$$

is completely monotonic on $(0, \infty)$.

Proof. Apply Theorem 3.1 to $f(x) = \log \Gamma_q(x)$ and note that $f''(x) = \psi'_q(x)$ is completely monotonic on $(0, \infty)$ and this completes the proof. \square

Theorem 3.2. *Let $f''(x)$ be completely monotonic on $(0, \infty)$, then for $0 \leq s \leq 1$, the functions*

$$\begin{aligned} x &\mapsto \exp\left(-\left(f(x+1) - f(x+s) - (1-s)f'\left(x + \frac{1+s}{2}\right)\right)\right), \\ x &\mapsto \exp\left(f(x+1) - f(x+s) - \frac{(1-s)}{2}(f'(x+1) + f'(x+s))\right) \end{aligned}$$

are completely monotonic on $(0, \infty)$.

Proof. We may assume $0 \leq s < 1$. We will prove the first assertion and the second one can be shown similarly. By Lemma 2.1, it suffices to show that

$$f'(x+1) - f'(x+s) - (1-s)f''\left(x + \frac{1+s}{2}\right)$$

is completely monotonic on $(0, \infty)$ or for $k \geq 1$,

$$\frac{1}{1-s} \int_{x+s}^{x+1} (-1)^{k+1} f^{(k+1)}(t) dt \geq (-1)^{k+1} f^{(k+1)}\left(x + \frac{1+s}{2}\right).$$

The last inequality holds by Lemma 2.3 and our assumption that $f''(x)$ is completely monotonic on $(0, \infty)$. This completes the proof. \square

Corollary 3.2. *For $0 \leq s \leq 1$, the functions*

$$\begin{aligned} x &\mapsto \frac{\Gamma_q(x+s)}{\Gamma_q(x+1)} \exp\left((1-s)\psi_q\left(x + \frac{1+s}{2}\right)\right), \\ x &\mapsto \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \exp\left(-\frac{(1-s)}{2}(\psi_q(x+1) + \psi_q(x+s))\right) \end{aligned}$$

are completely monotonic on $(0, \infty)$.

Proof. Apply Theorem 3.2 to $f(x) = \log \Gamma_q(x)$ and note that $f''(x) = \psi'_q(x)$ is completely monotonic on $(0, \infty)$ and this completes the proof. \square

By applying Lemma 2.3 to $f(x) = -\psi_q(x)$, we obtain

Theorem 3.3. *For positive x and $0 \leq s \leq 1$,*

$$\exp\left(\frac{(1-s)}{2}(\psi_q(x+1) + \psi_q(x+s))\right) \leq \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \leq \exp\left((1-s)\psi_q\left(x + \frac{1+s}{2}\right)\right).$$

The upper bound in Theorem 3.3 is due to Ismail and Muldoon[8]. Our proof here is similar to that of Corollary 3 in [13]. We further note the following integral analogue of Theorem 3.15 in [3]:

$$\psi(L_0(b, a)) \leq \frac{1}{b-a} \int_a^b \psi(x) dx \leq \psi(L_1(b, a)), \quad b > a > 0.$$

It follows from this that for positive x and $0 \leq s \leq 1$,

$$\exp\left((1-s)\psi(L_0(x+1, x+s))\right) \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\left((1-s)\psi(L_1(x+1, x+s))\right).$$

By Lemma 2.4, observing that $L_{-1}(x+1, x+s) = \sqrt{(x+1)(x+s)}$ and $L_2(x+1, x+s) = x+(1+s)/2$, we obtain

$$(3.1) \quad \exp\left((1-s)\psi(\sqrt{(x+1)(x+s)})\right) \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\left((1-s)\psi\left(x + \frac{1+s}{2}\right)\right).$$

Note by (1.1),

$$\psi(x+1) + \psi(x+s) \leq 2\psi(\sqrt{(x+1)(x+s)}),$$

also note that $\psi(x)$ is an increasing function on $(0, \infty)$ and $\sqrt{(x+1)(x+s)} \geq x + s^{1/2}$, we see that the inequalities in (3.1) refine the case $q \rightarrow 1$ in Theorem 3.3 and the following result of Kershaw[10], which states that for positive x and $0 \leq s \leq 1$,

$$\exp\left(\frac{(1-s)}{2}\psi(x + s^{1/2})\right) \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\left((1-s)\psi\left(x + \frac{1+s}{2}\right)\right).$$

We now show the lower bound above and the corresponding one of the case $q \rightarrow 1$ in Theorem 3.3 are not comparable in general(see p. 856, [7] for a similar discussion). In fact, on letting $x \rightarrow 0$ and by Theorem 3.7 of [3], we have

$$\psi(1) + \psi(s) < 2\psi(s^{1/2}), \quad 0 < s < 1.$$

On the other hand, using the well-known series representation(see, for example, [8, (1.8)]):

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{x+n} \right)$$

with $\gamma = 0.57721 \dots$ denoting Euler's constant, we obtain for $x > 1$,

$$\psi(x+1) + \psi(x+s) - 2\psi(x + s^{1/2}) = \sum_{n=0}^{\infty} \frac{(1-s^{1/2})^2(x+n-s^{1/2})}{(x+n+1)(x+n+s)(x+n+s^{1/2})} > 0.$$

We end our paper by answering a question of Qi in [14], stated as: If $f(x)$ is an absolutely or completely monotonic function on the interval $(-\infty, +\infty)$, then the following inequality holds for $0 \leq x < +\infty$ or reverses for $-\infty < x \leq 0$:

$$E(x; f) := f^2(x)f'''(x) - 3f(x)f'(x)f''(x) + 2(f'(x))^3 \leq 0.$$

We point out here in general the above assertion is not true. As one can check easily that for any constant $a > 0$, $g(x) = e^x + a$ is an absolutely monotonic function on $(-\infty, +\infty)$ while $h(x) = e^{-x} + a$ is a completely monotonic function on $(-\infty, +\infty)$. However, $E(0; g) = -E(0; h) = a(a-1)$ which shows the falsity of the above assertion.

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