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TRAPEZOIDAL TYPE INEQUALITIES FOR n -TIME DIFFERENTIABLE FUNCTIONS

S.S. DRAGOMIR AND A. SOFO

ABSTRACT. In this paper, by utilising a result given by Fink [2], we obtain some new results relating to the trapezoidal inequality and some of its generalisations for n -time differentiable functions.

1. INTRODUCTION

The following result is known in the literature as Ostrowski's inequality [3, p. 468].

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M.$$

The Ostrowski inequality has been generalised in a number of different ways, see [3] and [1].

Fink [2] also obtained the following result for n -time differentiable functions.

Theorem 2. *Let $f^{(n-1)}(t)$ be absolutely continuous on $[a, b]$ with $f^{(n)}(t) \in L_p(a, b)$ and let*

$$(1.1) \quad F_k(x) := \frac{n-k}{k!} \left[\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} \right],$$

$k = 1, 2, \dots, n-1;$

then

$$(1.2) \quad \left| \frac{1}{n} \left[f(x) + \sum_{k=1}^{n-1} F_k(x) \right] - \frac{1}{b-a} \int_a^b f(y) dy \right|$$

$$\leq \begin{cases} \left[\frac{[(x-a)^{nq+1} + (b-x)^{nq+1}]^{\frac{1}{q}}}{n!(b-a)} \right] B^{\frac{1}{q}}(q+1, (n-1)q+1) \|f^{(n)}\|_p, & \text{for } f \in L_p[a, b]; \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(n-1)^{n-1}}{n^n n!} \cdot \frac{\max\{(x-a)^n, (b-x)^n\}}{b-a} \|f^{(n)}\|_1; & \text{for } f \in L_1[a, b]; \\ \left[\frac{(x-a)^{n+1} + (b-x)^{n+1}}{n(n+1)!(b-a)} \right] \|f^{(n)}\|_\infty; & \text{for } f \in L_\infty[a, b]; \end{cases}$$

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where $B(\alpha, \beta)$ is Euler's Beta function,

$$\begin{aligned} \|f^{(n)}\|_p &:= \left(\int_a^b |f^{(n)}(t)|^p dt \right)^{\frac{1}{p}}, \quad p \geq 1 \quad \text{and} \\ (1.3) \quad \|f^{(n)}\|_\infty &:= \operatorname{ess\,sup} |f^{(n)}(t)| < \infty. \end{aligned}$$

Remark 1. The result above on the infinite norm was given by Milovanović and Pečarić in 1976 (see [3, p. 468]).

Note that for $n = 1$ and the infinite norm, Theorem 2 reduces to Theorem 1.

In the next section we develop an integral equality that will permit us to obtain bounds for the error estimate in a generalised trapezoid formula. The new results complement some of the earlier inequalities related to the trapezoidal rule reported in [1]

2. THE RESULTS

The following generalisation of the trapezoid formula holds:

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping such that its $(n-1)^{th}$ derivative $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then we have the equality

$$\begin{aligned} (2.1) \quad & \frac{1}{n} \left[\frac{f(a) + f(b)}{2} + \sum_{k=1}^{n-1} \frac{(n-k)(b-a)^{k-1}}{k!} \right. \\ & \quad \times \left. \left\{ \frac{f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)}{2} \right\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ &= \frac{1}{2n!(b-a)} \int_a^b (t-a)(b-t) \left[(b-t)^{n-2} + (-1)^{n-2} (t-a)^{n-2} \right] f^{(n)}(t) dt. \end{aligned}$$

Proof. We may use Fink's identity [2] which states

$$\begin{aligned} (2.2) \quad & \frac{1}{n} \left[f(x) + \sum_{k=1}^{n-1} F_k(x) \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ &= \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1} K(t, x) f^{(n)}(t) dt, \end{aligned}$$

where

$$K(t, x) := \begin{cases} t-a & \text{if } a \leq t \leq x \leq b \\ t-b & \text{if } a \leq x \leq t \leq b \end{cases}$$

and $F_k(x)$ is defined by (1.1).

If in (2.2) we put $x = a$, then we obtain

$$\begin{aligned} (2.3) \quad & \frac{1}{n} \left[f(a) + \sum_{k=1}^{n-1} F_k(a) \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ &= \frac{(-1)^n}{n!(b-a)} \int_a^b (t-a)^{n-1} (b-t) f^{(n)}(t) dt, \end{aligned}$$

where

$$F_k(a) = \frac{(-1)^{k-1} (n-k) f^{(k-1)}(b) (b-a)^{k-1}}{k!}.$$

Similarly, if in (2.2) we put $x = b$, we get

$$(2.4) \quad \begin{aligned} \frac{1}{n} \left[f(b) + \sum_{k=1}^{n-1} F_k(b) \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{1}{n! (b-a)} \int_a^b (b-t)^{n-1} (t-a) f^{(n)}(t) dt, \end{aligned}$$

where

$$F_k(b) = \frac{(n-k) f^{(k-1)}(a) (b-a)^{k-1}}{k!}.$$

Adding (2.3) to (2.4) and dividing by 2, we have

$$\begin{aligned} \frac{1}{2n} \left[f(a) + f(b) + \sum_{k=1}^{n-1} \{F_k(a) + F_k(b)\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{1}{2n! (b-a)} \int_a^b (t-a)(b-t) \left[(b-t)^{n-2} + (-1)^{n-2} (t-a)^{n-2} \right] f^{(n)}(t) dt, \end{aligned}$$

replacing $F_k(a)$ and $F_k(b)$ we obtain identity (2.1), hence the theorem is proved. ■

Remark 2. (a) For $n = 1$, we recapture the known identity,

$$(2.5) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(y) dy = \frac{1}{b-a} \int_a^b \left[t - \left(\frac{a+b}{2} \right) \right] f'(t) dt.$$

(b) For $n = 2$, we deduce the equality below, which is also well known in the literature,

$$(2.6) \quad \begin{aligned} \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(y) dy &= \frac{1}{2(b-a)} \int_a^b (t-a)(b-t) f''(t) \\ &= \frac{1}{2(b-a)} \int_a^b \left[\left(\frac{b-a}{2} \right)^2 - \left(t - \left(\frac{a+b}{2} \right) \right)^2 \right] f''(t) dt. \end{aligned}$$

(c) For $n = 3$, we have some extra terms involving the first derivative at the end points, namely:

$$(2.7) \quad \begin{aligned} \frac{f(a) + f(b)}{2} + \frac{b-a}{12} [f'(a) - f'(b)] - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{1}{6(b-a)} \int_a^b (t-a)(b-t) \left(\frac{a+b}{2} - t \right) f'''(t) dt. \end{aligned}$$

(d) Finally, for $n = 4$, we have the following

$$\begin{aligned}
 (2.8) \quad & \frac{f(a) + f(b)}{2} + \frac{b-a}{8} [f'(a) - f'(b)] \\
 & + \frac{(b-a)^2}{48} [f''(a) + f''(b)] - \frac{1}{b-a} \int_a^b f(y) dy \\
 & = \frac{1}{48(b-a)} \int_a^b (t-a)(b-t) [(b-t)^2 + (t-a)^2] f^{(4)}(t) dt \\
 & = \frac{1}{24(b-a)} \int_a^b (t-a)(b-t) \left[\left(t - \frac{a+b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2 \right] f^{(4)}(t) dt.
 \end{aligned}$$

The following inequalities can now be stated.

Theorem 4. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Define

$$\begin{aligned}
 T(a, b, n) \\
 := \frac{1}{n} \left[\frac{f(a) + f(b)}{2} + \sum_{k=1}^{n-1} \frac{(n-k)(b-a)^{k-1}}{k!} \left\{ \frac{f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)}{2} \right\} \right] \\
 - \frac{1}{b-a} \int_a^b f(y) dy,
 \end{aligned}$$

then

$$\begin{aligned}
 (2.9) \quad & |T(a, b, n)| \\
 & \leq \begin{cases} \frac{(b-a)^{n-1+\frac{1}{q}}}{n!} B^{\frac{1}{q}}(q+1, (n-1)q+1) \|f^{(n)}\|_p & \text{for } \frac{1}{p} + \frac{1}{q} = 1; \\ & p > 1, q > 1; \\ \frac{(b-a)^{n-\frac{1}{2}}}{\sqrt{2(2n+1)!n!}} \left[2(2n-2)! + (-1)^n (n!)^2 \right]^{\frac{1}{2}} \|f^{(n)}\|_2 & \text{for } p = 2, q = 2; \\ \frac{(b-a)^n}{n(n+1)!} \|f^{(n)}\|_\infty, & \end{cases}
 \end{aligned}$$

where $B(x, y)$ is the Beta function and $\|f^{(n)}\|_p, \|f^{(n)}\|_\infty$ are as defined in (1.3).

Proof. From (2.1) and the definition of $T(a, b, n)$ we have

$$(2.10) \quad |T(a, b, n)| = \left| \frac{1}{2n!(b-a)} \int_a^b Y(t) f^{(n)}(t) dt \right|,$$

where

$$(2.11) \quad Y(t) := (t-a)(b-t)^{n-1} + (-1)^n (t-a)^{n-1}(b-t).$$

By Hölder's inequality

$$\begin{aligned}
 (2.12) \quad |T(a, b, n)| & \leq \frac{1}{2n!(b-a)} \left(\int_a^b |Y(t)|^q dt \right)^{\frac{1}{q}} \left(\int_a^b |f^{(n)}(t)|^p dt \right)^{\frac{1}{p}}, \\
 & \frac{1}{p} + \frac{1}{q} = 1, \quad q > 1, \quad p > 1.
 \end{aligned}$$

Now, let us provide some upper bounds for the integral: $\int_a^b |Y(t)|^q dt$. For real α and β and $q > 1$, we have the elementary inequality

$$(2.13) \quad |\alpha + \beta|^q \leq 2^{q-1} (|\alpha|^q + |\beta|^q).$$

Utilising (2.13), we obtain

$$\begin{aligned} \int_a^b |Y(t)|^q dt &\leq 2^{q-1} \int_a^b \left[(t-a)^q (b-t)^{(n-1)q} + (t-a)^{(n-1)q} (b-t)^q \right] dt \\ &= 2^q (b-a)^{nq+1} B(q+1, (n-1)q+1) \end{aligned}$$

by the substitution $w = \frac{t-a}{b-a}$ and the symmetry of the Beta function,

$$B(\alpha, \beta) = B(\beta, \alpha) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt; \quad \alpha, \beta > 0.$$

Also

$$\left(\int_a^b |Y(t)|^q dt \right)^{\frac{1}{q}} \leq 2 (b-a)^{n+\frac{1}{q}} B^{\frac{1}{q}}(q+1, (n-1)q+1)$$

so from (2.12) we deduce

$$|T(a, b, x)| \leq \frac{(b-a)^{n-1+\frac{1}{q}}}{n!} B^{\frac{1}{q}}(q+1, (n-1)q+1) \|f^{(n)}\|_p$$

and the first part of the theorem is proved.

For the Euclidean norm ($p = 2, q = 2$), we can compute exactly

$$\begin{aligned} &\int_a^b |Y(t)|^2 dt \\ &= \int_a^b \left| (t-a)(b-t)^{n-1} + (-1)^n (t-a)^{n-1}(b-t) \right|^2 dt \\ &= \int_a^b \left((t-a)^2 (b-t)^{2n-2} + 2(-1)^n (t-a)^n (b-t)^n + (t-a)^{2n-2} (b-t)^2 \right) dt \\ &= 2(b-a)^{2n+1} [B(3, 2n-1) + (-1)^n B(n+1, n+1)] \\ &= \frac{2(b-a)^{2n+1}}{(2n+1)!} [2(2n-2)! + (-1)^n (n!)^2]. \end{aligned}$$

From (2.12)

$$|T(a, b, n)| \leq \frac{(b-a)^{n-\frac{1}{2}}}{\sqrt{2(2n+1)!} n!} [2(2n-2)! + (-1)^n (n!)^2]^{\frac{1}{2}} \|f^{(n)}\|_2$$

and the second part of the theorem is proved.

For the third part of the theorem, observe that

$$\begin{aligned} \int_a^b |Y(t)| dt &= \int_a^b \left| (t-a)(b-t)^{n-1} + (-1)^n (t-a)^{n-1}(b-t) \right| dt \\ &\leq \int_a^b (t-a)(b-t)^{n-1} dt + \int_a^b (t-a)^{n-1}(b-t) dt \\ &= \frac{2(b-a)^{n+1}}{n(n+1)}. \end{aligned}$$

From (2.12)

$$|T(a, b, n)| \leq \frac{(b-a)^n}{n(n+1)!} \|f^{(n)}\|_\infty,$$

the third part of the theorem is proved, hence Theorem 4 is proved. ■

For the 1-norm we can also delineate the following theorem.

Theorem 5. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$.*

(i) *For n even, let $n = 2k$, $k \geq 1$, then*

$$|T(a, b, 2k)| \leq \begin{cases} \max_{t^*} \left\{ \frac{(t^*-a)(b-t^*)^{2k-1} + (t^*-a)^{2k-1}(b-t^*)}{2(2k)! (b-a)} \right\} \|f^{(2k)}\|_1, & \text{for } k > 2; \\ \frac{b-a}{8} \|f''\|_1 & \text{for } k = 1; \\ \frac{(b-a)^3}{384} \|f^{(4)}\|_1 & \text{for } k = 2; \end{cases}$$

where $t^* \in (a, \frac{a+b}{2}]$ is a solution of the polynomial equation

$$(b-t)^{2k-1} - (t-a)^{2k-1} + (2k-1) \left[(t-a)^{2k-2} (b-t) - (t-a)(b-t)^{2k-2} \right] = 0.$$

(ii) *For n odd, let $n = 2k+1$, $k \geq 0$*

$$|T(a, b, 2k+1)| \leq \begin{cases} \max_{\tau} \left\{ \frac{(\tau-a)(b-\tau)^{2k} - (\tau-a)^{2k}(b-\tau)}{2(2k+1)! (b-a)} \right\} \|f^{(2k)}\|_1 & \text{for } k \geq 1 \\ \frac{1}{2} \|f'\|_1 & \text{for } k = 0, n = 1 \end{cases},$$

where $\tau \in (a, \frac{a+b}{2}]$ is a solution of the polynomial equation

$$(t-a)^{2k} + (b-t)^{2k} - 2k \left[(t-a)(b-t)^{2k-1} + (t-a)^{2k-1}(b-t) \right] = 0.$$

The following two lemmas will be useful in the proof of Theorem 5.

Lemma 1. *Let $b > a$, k is an integer and $t \in [a, b]$. Define $M(t, k) := (t-a)(b-t)^{2k-1} + (t-a)^{2k-1}(b-t)$, then $M(t, k)$ has exactly two zeros as a function of t on $[a, b]$.*

Proof. Observe that

$$M(t, k) = -(t-a)(t-b) \left[(t-a)^{2k-2} + (t-b)^{2k-2} \right].$$

Since for $t \in (a, b)$, $\left[(t-a)^{k-1} \right]^2 + \left[(t-b)^{k-1} \right]^2 > 0$, hence $M(t, k) = 0$ has the only real solutions $t = a$ and $t = b$. ■

Lemma 2. *Let $b > a$, k is an integer and $t \in [a, b]$. Define*

$$P(t, k) = (t-a)(b-t)^{2k} + (t-a)^{2k}(b-t),$$

then $P(t, k)$ has exactly three zeros in $[a, b]$.

Proof. We have

$$P(t, k) = (t - a)(t - b) \left[(t - b)^{2k-1} + (t - a)^{2k-1} \right].$$

Observe that

$$P(a, k) = P(b, k) = 0.$$

Since

$$(t - a)^{2k-1} + (t - b)^{2k-1} = (2t - a + b) \left[(t - a)^{2k-2} + \dots + (b - t)^{2k-2} \right]$$

and

$$(t - a)^{2k-2} + (t - a)^{2k-3}(b - t) \dots + (t - a)(b - t)^{2k-3} + (b - t)^{2k-2} > 0$$

for $t \in [a, b]$, hence the third solution of $P(t, k) = 0$ for $t \in [a, b]$ is $t = \frac{a+b}{2}$. ■

Proof of Theorem 5. (i) Consider the case of n even, let $n = 2k$, $k \geq 1$ and denote

$$Y := \sup_{t \in [a, b]} \left| (t - a)(b - t)^{2k-1} + (t - a)^{2k-1}(b - t) \right|.$$

For $M(t, k)$ defined above, simple calculations show that

$$\begin{aligned} M'(t, k) &= (b - t)^{2k-1} - (t - a)^{2k-1} \\ &\quad + (2k - 1) \left[(t - a)^{2k-2}(b - t) - (t - a)(b - t)^{2k-2} \right], \end{aligned}$$

$$\begin{aligned} M''(t, k) &= -2(2k - 1) \left[(t - a)^{2k-2} + (b - t)^{2k-2} \right] \\ &\quad + (2k - 2)(2k - 1) \left[(t - a)(b - t)^{2k-3} + (t - a)^{2k-3}(b - t) \right], \end{aligned}$$

and

$$M(a, k) = M(b, k) = 0; \quad M\left(\frac{a+b}{2}, k\right) = 2\left(\frac{b-a}{2}\right)^{2k-2} > 0$$

$$M'(a, k) = (b - a)^{2k-1} > 0; \quad M'(b, k) = -(b - a)^{2k-1} < 0, \quad M'\left(\frac{a+b}{2}, k\right) = 0,$$

$$M''\left(\frac{a+b}{2}, k\right) = 2(2k - 1)(2k - 2)\left(\frac{b-a}{2}\right)^{2k-2} > 0 \quad \text{for } k > 2.$$

The local extrema for the function $M(\cdot, k)$ are the real numbers $t^* \in [a, \frac{a+b}{2}]$ that are solutions of the polynomial equation

$$\begin{aligned} (b - t)^{2k-1} - (t - a)^{2k-1} \\ + (2k - 1) \left[(t - a)^{2k-2}(b - t) - (t - a)(b - t)^{2k-2} \right] = 0. \end{aligned}$$

Therefore, by Lemma 1

$$Y = \max_{t^*} \left\{ (t^* - a)(b - t^*)^{2k-1} + (t^* - a)^{2k-1}(b - t^*) \right\} > 0$$

hence

$$|T(a, b, 2k)| \leq \max_{t^*} \left\{ \frac{(t^* - a)(b - t^*)^{2k-1} + (t^* - a)^{2k-1}(b - t^*)}{2(2k)!(b - a)} \right\} \|f^{(2k)}\|_1$$

for $k > 2$.

For the two special cases $k = 1, (n = 2)$, we have

$$|T(a, b, 2)| \leq \frac{b-a}{8} \|f^{(2)}\|_1$$

and for $k = 2, (n = 4)$

$$|T(a, b, 4)| \leq \frac{(b-a)^3}{384} \|f^{(4)}\|_1.$$

(ii) When n is odd, let $n = 2k + 1, k \geq 0$, and denote

$$Z := \sup_{t \in [a, b]} \left| (t-a)(b-t)^{2k} - (t-a)^{2k}(b-t) \right|.$$

With $P(t, k)$ defined above in Lemma 2, we have

$$P'(t, k) = (t-a)^{2k} + (b-t)^{2k} - 2k \left[(t-a)(b-t)^{2k-1} + (t-a)^{2k-1}(b-t) \right],$$

$$P(a, k) = P(b, k) = P\left(\frac{a+b}{2}, k\right) = 0,$$

$$P'(a, k) = P'(b, k) = (b-a)^{2k} > 0,$$

$$P'\left(\frac{a+b}{2}, k\right) = 2(1-2k) \left(\frac{b-a}{2}\right)^{2k} < 0, \quad \text{for } k \geq 1,$$

so there exists at least one point $\tau \in (a, \frac{a+b}{2})$ such that

$$P(\tau, k) := (\tau-a)(b-\tau)^{2k} - (\tau-a)^{2k}(b-\tau) > 0.$$

One can realise that the local extrema for $P(\cdot, k)$ are the real numbers $\tau \in (a, \frac{a+b}{2}]$ that are solutions of the polynomial equation

$$(t-a)^{2k} + (b-t)^{2k} - 2k \left[(t-a)(b-t)^{2k-1} + (t-a)^{2k-1}(b-t) \right] = 0.$$

Now, by Lemma 2,

$$Z = \max_{\tau} \left\{ (\tau-a)(b-\tau)^{2k-1} - (\tau-a)^{2k-1}(b-\tau) \right\} > 0$$

hence

$$|T(a, b, 2k)| \leq \max_{\tau} \left\{ \frac{(\tau-a)(b-\tau)^{2k-1} - (\tau-a)^{2k-1}(b-\tau)}{2(2k+1)!(b-a)} \right\} \|f^{(2k+1)}\|_1.$$

For the trivial case $k = 0 (n = 1)$, we have

$$|T(a, b, 1)| \leq \frac{1}{2} \|f'\|_1,$$

hence Theorem 5 is proved. ■

3. SOME EXAMPLES

In this section we give some examples that highlight Theorems 4 and 5.

(i) For $n = 4$, let

$$\begin{aligned} T(a, b, 4) &:= \frac{f(a) + f(b)}{2} + \frac{b-a}{8} [f'(a) - f'(b)] \\ &\quad + \frac{(b-a)^2}{48} [f''(a) + f''(b)] - \frac{1}{b-a} \int_a^b f(y) dy \end{aligned}$$

then

$$|T(a, b, 4)| \leq \begin{cases} \frac{(b-a)^{3+\frac{1}{q}}}{24} B^{\frac{1}{q}} (q+1, 3q+1) \|f^{(4)}\|_p, & \text{for } f \in L_p[a, b], \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^{\frac{7}{2}}}{144\sqrt{10}} \|f^{(4)}\|_2, & \text{for } f \in L_2[a, b]; \\ \frac{(b-a)^4}{480} \|f^{(4)}\|_\infty, & \text{for } f \in L_\infty[a, b]; \\ \frac{(b-a)^3}{384} \|f^{(4)}\|_1, & \text{for } f \in L_1[a, b]. \end{cases}$$

(ii) For $n = 5$, let

$$\begin{aligned} T(a, b, 5) := & \frac{f(a) + f(b)}{2} + \frac{3(b-a)(f'(a) - f'(b))}{20} \\ & + \frac{(b-a)^2(f''(a) + f''(b))}{30} + \frac{(b-a)^3(f'''(a) - f'''(b))}{240} \\ & - \frac{1}{b-a} \int_a^b f(y) dy, \end{aligned}$$

then

$$|T(a, b, 5)| \leq \begin{cases} \frac{(b-a)^{4+\frac{1}{q}}}{120} B^{\frac{1}{q}} (q+1, 4q+1) \|f^{(5)}\|_p, & \text{for } f \in L_p[a, b], \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^{\frac{9}{2}}}{720} \left(\frac{23}{770}\right)^{\frac{1}{2}} \|f^{(5)}\|_2, & \text{for } f \in L_2[a, b]; \\ \frac{(b-a)^5}{3600} \|f^{(5)}\|_\infty, & \text{for } f \in L_\infty[a, b]; \\ \frac{(9-\sqrt{6})\sqrt{10\sqrt{6}-15}(b-a)^4}{60000} \|f^{(5)}\|_1, & \text{for } f \in L_1[a, b]; \end{cases}$$

For $n = 5$, $k = 2$, solving

$$(\tau - a)^4 + (\tau - b)^4 + 4[(\tau - a)(\tau - b)^3 + (\tau - a)^3(\tau - b)] = 0$$

we obtain

$$\tau = \frac{a+b}{2} - \frac{\sqrt{10\sqrt{6}-15}}{10} (b-a), \quad \tau \in \left[a, \frac{a+b}{2}\right],$$

for which

$$P(\tau, 2) = \frac{(9-\sqrt{6})\sqrt{10\sqrt{6}-15}(b-a)^5}{250}$$

is the required maximum in Theorem 5, (ii). Therefore

$$|T(a, b, 5)| \leq \frac{(9-\sqrt{6})\sqrt{10\sqrt{6}-15}(b-a)^4}{60000} \|f^{(5)}\|_1.$$

(iii) For $n = 6$, let

$$\begin{aligned} T(a, b, 6) := & \frac{f(a) + f(b)}{2} + \frac{(b-a)}{6} (f'(a) - f'(b)) \\ & + \frac{(b-a)^2}{24} (f''(a) + f''(b)) + \frac{(b-a)^3}{144} (f'''(a) - f'''(b)) \\ & + \frac{(b-a)^4}{1440} (f^{(4)}(a) + f^{(4)}(b)) - \frac{1}{b-a} \int_a^b f(y) dy, \end{aligned}$$

then

$$|T(a, b, 6)| \leq \begin{cases} \frac{(b-a)^{5+\frac{1}{q}}}{720} B^{\frac{1}{q}}(q+1, 5q+1) \|f^{(6)}\|_p, & \text{for } f \in L_p[a, b], \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^{\frac{11}{2}}}{1440} \left(\frac{5}{2002}\right)^{\frac{1}{2}} \|f^{(6)}\|_2, & \text{for } f \in L_2[a, b]; \\ \frac{(b-a)^6}{30240} \|f^{(6)}\|_\infty, & \text{for } f \in L_\infty[a, b]; \\ \frac{(5\sqrt{10}-14)(b-a)^5}{38880} \|f^{(6)}\|_1, & \text{for } f \in L_1[a, b]. \end{cases}$$

For $n = 6$, $k = 3$,

$$t^* = \frac{a+b}{2} - \frac{\sqrt{6\sqrt{10}-15}(b-a)}{6} \in \left[a, \frac{a+b}{2}\right]$$

satisfies

$$(b-t^*)^5 - (t^*-a)^5 + 5 \left[(t^*-a)^4 (b-t^*) - (t^*-a)(b-t^*)^4 \right] = 0$$

and

$$M(t^*, 3) = \frac{(5\sqrt{10}-14)(b-a)^6}{27},$$

which provides the required maximum in Theorem 5, (i). Therefore

$$|T(a, b, 6)| \leq \frac{(5\sqrt{10}-14)(b-a)^5}{38880} \|f^{(6)}\|_1.$$

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SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY,
PO BOX 14428, MELBOURNE CITY, VICTORIA 8001, AUSTRALIA.

E-mail address: sever.dragomir@vu.edu.au

URL: <http://rgmia.vu.edu.au/dragomir>

E-mail address: anthony.sofa@vu.edu.au

URL: <http://rgmia.vu.edu.au/sofo>