Generalizations of Weighted Ostrowski Type Inequalities for Mappings of Bounded Variation and their Applications

GENERALIZATIONS OF WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR MAPPINGS OF BOUNDED VARIATION AND THEIR APPLICATIONS

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Abstract. In this paper, we establish some generalizations of weighted Ostrowski type Inequalities, and give several applications for $r-$moments, expectation of a continuous random variable and the Beta mapping.

1. Introduction

Throughout this section, let $a < b$ in $\mathbb{R}$, $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,1,\ldots,n-1)$, $l_i := x_{i+1} - x_i$ $(i = 0,1,\ldots,n-1)$ and $\nu(l) = \max_{i=0,1,\ldots,n-1} l_i$.

The *Ostrowski's inequality* [10, p. 469], states that if f' exists and is bounded on (a,b), then, for all $x \in [a,b]$, we have the inequality

$$\left| \int_a^b f(t)dt - f(x)(b-a) \right| \leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty, \tag{1.1}$$

where

$$\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty.$$

Now if f is as above, then we can approximate the integral $\int_a^b f(t)\,dt$ by the *Ostrowski quadrature formula* $A_O(f, I_n, \xi)$, having an error given by $R_O(f, I_n, \xi)$, where

$$A_O(f, I_n, \xi) := \sum_{i=1}^{n} f(\xi_i)\, l_i,$$

and the remainder satisfies the estimation

$$|R_O(f, I_n, \xi)| \leq \sum_{i=0}^{n-1} \left[\frac{1}{4} l_i^2 + \left(\xi_i - \frac{x_{i-1} + x_i}{2} \right)^2 \right] \|f'\|_\infty.$$

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [2, 3, 8, 9].

Recently, Dragomir [2] proved the following two Ostrowski type inequalities for mappings of bounded variation:

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Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation. Then*

$$\left|\int_a^b f(t)dt - f(x)\,(b-a)\right| \le \left[\frac{1}{2}\,(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

for all $x\in[a,b]$, where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

Theorem 2. *Let $A_O(f,I_n,\xi)$ and $R_O(f,I_n,\xi)$ be as above and let f and $\bigvee_a^b(f)$ be defined as in Theorem 1, then we have*

$$\int_a^b f(t)dt = A_O(f,I_n,\xi)+R_O(f,I_n,\xi),$$

and the remainder term $R_O(f,I_n,\xi)$ satisfies the estimation

$$\begin{aligned}|R_O(f,I_n,\xi)| &\le \max_{i=0,1,\ldots,n-1}\left[\frac{1}{2}l_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b(f)\\ &\le \left[\frac{1}{2}\nu(l)+\max_{i=0,1,\ldots,n-1}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b(f)\\ &\le \nu(l)\bigvee_a^b(f). \qquad (1.3)\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in (1.3).

The *Simpson's inequality*, states that if $f^{(4)}$ exists and is bounded on (a,b), then

$$\left|\int_a^b f(t)dt-\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\le\frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_\infty, \tag{1.4}$$

where

$$\left\|f^{(4)}\right\|_\infty := \sup_{t\in(a,b)}\left|f^{(4)}(t)\right|<\infty.$$

Let f be as above, then we can approximate the integral $\int_a^b f(t)\,dt$ by the *Simpson's quadrature formula* $A_S(f,I_n)$, having an error given by $R_S(f,I_n)$, where

$$A_S(f,I_n):=\sum_{i=0}^{n-1}\frac{l_i}{3}\left[\frac{f(x_i)+f(x_{i+1})}{2}+2f\left(\frac{x_i+x_{i+1}}{2}\right)\right],$$

and the remainder satisfies the estimation

$$|R_S(f,I_n)|\le\frac{1}{2880}\left\|f^{(4)}\right\|_\infty\sum_{i=0}^{n-1}l_i^5.$$

For some recent results which generalize, improve and extend this classic inequality (1.4), see the papers [4] – [7], [12] – [14].

Recently, Dragomir [6] proved the following two Simpson type inequalities for mappings of bounded variation:

Theorem 3. *Let f and $\bigvee_a^b(f)$ be defined as in Theorem 2. Then*

$$\left|\int_a^b f(t)dt-\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\le\frac{1}{3}(b-a)\bigvee_a^b(f). \tag{1.5}$$

The constant $\frac{1}{3}$ is the best possible.

Theorem 4. *Let $A_S(f, I_n)$ and $R_S(f, I_n)$ be as above and let f and $\bigvee_a^b(f)$ be defined as in Theorem 3, then we have*

$$\int_a^b f(t)dt = A_S(f, I_n) + R_S(f, I_n)$$

and the remainder term $R_S(f, I_n)$ satisfies the estimation

$$|R_S(f, I_n)| \le \frac{1}{3}\nu(l)\bigvee_a^b(f). \tag{1.6}$$

The constant $\frac{1}{3}$ is the best possible.

In this paper, we establish weighted generalizations of Theorems 1 – 4, and give several applications for $r-$*moments*, expectation of a continuous random variable and the Beta mapping.

2. Some Integral Inequalities

Theorem 5. *Let $0 \le \alpha \le 1$, $g : [a, b] \to [0, \infty)$ be continuous and positive on (a, b) and let $h : [a, b] \to \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a, b]$. Let $c = h^{-1}\left(\left(1 - \frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)$ and $d = h^{-1}\left(\frac{\alpha}{2}h(a) + \left(1 - \frac{\alpha}{2}\right)h(b)\right)$. Suppose that f and $\bigvee_a^b(f)$ are defined as in Theorem 4. Then, for all $x \in [c, d]$, we have*

$$\left|\int_a^b f(t)g(t)\,dt - \left[(1-\alpha)f(x) + \alpha\cdot\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\,dt\right| \le K\cdot\bigvee_a^b(f), \tag{2.1}$$

where

$$K := \begin{cases} \frac{1-\alpha}{2}\int_a^b g(t)\,dt + \left|h(x) - \frac{h(a)+h(b)}{2}\right|, & \text{if } 0 \le \alpha \le \frac{1}{2} \\ \max\left\{\frac{1-\alpha}{2}\int_a^b g(t)\,dt + \left|h(x) - \frac{h(a)+h(b)}{2}\right|, \frac{\alpha}{2}\int_a^b g(t)\,dt\right\}, & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}\int_a^b g(t)\,dt, & \text{if } \frac{2}{3} \le \alpha \le 1 \end{cases}$$

and $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$. In (2.1), the constant $\frac{\alpha}{2}$ as $0 \le \alpha \le \frac{1}{2}$ and the constant $\frac{1-\alpha}{2}$ as $\frac{2}{3} \le \alpha \le 1$ are the best possible.

Proof. Let $x \in [c, d]$. Define

$$s(t) := \begin{cases} h(t) - \left[\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right], & t \in [a, x) \\ h(t) - \left[\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right], & t \in [x, b] \end{cases}.$$

Using integration by parts, we have the following identity

$$\begin{aligned}
&\int_a^b s(t)\,df(t)\\
&= \left[h(t) - \left[\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right]\right]\cdot f(t)\Big|_{t=a}^{t=x} - \int_a^x f(t)g(t)\,dt\\
&\quad + \left[h(t) - \left[\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right]\right]\cdot f(t)\Big|_{t=x}^{t=b} - \int_x^b f(t)g(t)\,dt\\
&= \left[(1-\alpha)f(x) + \alpha\cdot\frac{f(a)+f(b)}{2}\right][h(b)-h(a)] - \int_a^b f(t)g(t)\,dt
\end{aligned}$$

$$= \left[(1-\alpha)f(x) + \alpha\cdot\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt. \tag{2.2}$$

It is well known [1, p. 159] that if $\mu,\nu : [a,b] \to \mathbb{R}$ are such that μ is continuous on $[a,b]$ and ν is of bounded variation on $[a,b]$, then $\int_a^b \mu(t)\,d\nu(t)$ exists and [1, p. 177]

$$\left|\int_a^b \mu(t)\,d\nu(t)\right| \le \sup_{x\in[a,b]}|\mu(t)|\bigvee_a^b(\nu). \tag{2.3}$$

Now, using (2.2) and (2.3), we have

$$\left|\int_a^b f(t)g(t)\,dt - \left[(1-\alpha)f(x) + \alpha\cdot\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\,dt\right| \le \sup_{t\in[a,b]}|s(t)|\bigvee_a^b(f). \tag{2.4}$$

Since $h(t) - \left[\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right]$ is increasing on the interval $[a,x)$, $h(t) - \left[\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right]$ is increasing on the interval $[x,b]$, $\max\{\sigma,\rho\} = \frac{\sigma+\rho}{2} + \frac{1}{2}|\sigma-\rho|$ for $\sigma,\rho\in\mathbb{R}$ and

$$\left|h(x) - \frac{h(a)+h(b)}{2}\right| \le \frac{1-\alpha}{2}(h(b)-h(a)) = \frac{1-\alpha}{2}\int_a^b g(t)\,dt,$$

we have

$$\begin{aligned}
&\sup_{t\in[a,b]}|s(t)|\\
&= \max\left\{h(x) - \left[\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right],\right.\\
&\qquad\left.\left[\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right] - h(x), \frac{\alpha}{2}[h(b)-h(a)]\right\}\\
&= \max\left\{\frac{1-\alpha}{2}[h(b)-h(a)] + \left|h(x) - \frac{h(a)+h(b)}{2}\right|, \frac{\alpha}{2}[h(b)-h(a)]\right\}\\
&= \max\left\{\frac{1-\alpha}{2}\int_a^b g(t)\,dt + \left|h(x) - \frac{h(a)+h(b)}{2}\right|, \frac{\alpha}{2}\int_a^b g(t)\,dt\right\}
\end{aligned}$$

$$= K. \tag{2.5}$$

Thus, by (2.4) and (2.5), we obtain (2.1).

Suppose $0 \leq \alpha \leq \frac{1}{2}$. We assume that the inequality (2.1) holds with a constant $C_1 > 0$, i.e.,

$$\left| \int_a^b f(t)g(t)\,dt - \left[(1-\alpha) f(x) + \alpha \cdot \frac{f(a)+f(b)}{2} \right] \int_a^b g(t)\,dt \right| \leq \left[C_1 \int_a^b g(t)\,dt + \left| h(x) - \frac{h(a)+h(b)}{2} \right| \right] \cdot \bigvee_a^b (f).$$

Let

$$f(t) = \begin{cases} 0 & \text{as } t \in [a,b] \Big\backslash \left\{ h^{-1}\left(\frac{h(a)+h(b)}{2}\right) \right\} \\ \frac{1}{2} & \text{as } t = h^{-1}\left(\frac{h(a)+h(b)}{2}\right) \end{cases}.$$

Then f is with bounded variation on $[a,b]$, and

$$\int_a^b f(t)g(t)\,dt = 0, \qquad \bigvee_a^b (f) = 1$$

and for $x = h^{-1}\left(\frac{h(a)+h(b)}{2}\right)$, we get in (2.1)

$$\frac{1-\alpha}{2} \leq C_1,$$

which implies the constant $\frac{1-\alpha}{2}$ is the best possible.

Suppose $\frac{2}{3} \leq \alpha \leq 1$. We assume that the inequality (2.1) holds with a constant $C_2 > 0$, i.e.,

$$\left| \int_a^b f(t)g(t)\,dt - \left[(1-\alpha) f(x) + \alpha \cdot \frac{f(a)+f(b)}{2} \right] \int_a^b g(t)\,dt \right| \leq C_2 \int_a^b g(t)\,dt \cdot \bigvee_a^b (f).$$

Let

$$f(t) = \begin{cases} 0 & \text{as } t \in [a,b) \\ 1 & \text{as } t = b \end{cases}.$$

Then f is with bounded variation on $[a,b]$ and

$$\int_a^b f(t)g(t)\,dt = 0, \qquad \bigvee_a^b (f) = 1,$$

we get in (2.1)

$$\frac{\alpha}{2} \leq C_2$$

which implies the constant $\frac{\alpha}{2}$ is the best possible.

This completes the proof. ∎

Under the conditions of Theorem 5, we have the following remarks and corollaries.

Remark 1.

(1) *If we choose* $\alpha = 0$ *and* $g(t) \equiv 1, h(t) = t$ *on* $[a, b]$*, then the inequality (2.1) reduces to (1.2).*

(2) *If we choose* $\alpha = \frac{1}{3}$*,* $g(t) \equiv 1, h(t) = t$ *on* $[a, b]$ *and* $x = \frac{a+b}{2}$*, then the inequality (2.1) reduces to (1.5).*

(3) *If we choose* $\alpha = 0$*, then for all* $x \in [a, b]$ *the inequality (2.1) reduces to the following inequality*

$$\left| \int_a^b f(t)g(t)\,dt - f(x) \cdot \int_a^b g(t)\,dt \right| \leq \left[\frac{1}{2} \int_a^b g(t)\,dt + \left| h(x) - \frac{h(a)+h(b)}{2} \right| \right] \cdot \bigvee_a^b (f),$$

which is the "weighted Ostrowski" inequality.

(4) *If we choose* $\alpha = 1$*, then the inequality (2.1) reduces to the following inequality*

$$\left| \int_a^b f(t)g(t)\,dt - \frac{f(a)+f(b)}{2} \int_a^b g(t)\,dt \right| \leq \frac{1}{2} \int_a^b g(t)\,dt \cdot \bigvee_a^b (f)$$

which is the "weighted trapezoid" inequality.

(5) *If we choose* $\alpha = \frac{1}{3}$ *and* $x = h^{-1}\left(\frac{h(a)+h(b)}{2}\right)$*, then the inequality (2.1) reduces to the following inequality*

$$\left| \int_a^b f(t)g(t)\,dt - \left[\frac{2}{3} f(x) + \frac{1}{3} \cdot \frac{f(a)+f(b)}{2} \right] \int_a^b g(t)\,dt \right| \leq \frac{1}{3} \int_a^b g(t)\,dt \cdot \bigvee_a^b (f)$$

which is the "weighted Simpson" inequality.

Corollary 1. *Let* $0 \leq \alpha \leq 1$*,* $f \in C^{(1)}[a, b]$*. Then we have the inequality*

$$\left| \int_a^b f(t)g(t)\,dt - \left[(1-\alpha) f(x) + \alpha \cdot \frac{f(a)+f(b)}{2} \right] \int_a^b g(t)\,dt \right| \leq K \cdot \|f'\|_1$$

for all $x \in [c, d]$*, where* $\|\cdot\|_1$ *is the* L_1-*norm, namely*

$$\|f'\|_1 := \int_a^b |f'(t)|\,dt.$$

Corollary 2. *Let* $0 \leq \alpha \leq 1$*,* $f : [a, b] \to \mathbb{R}$ *be a Lipschitzian mapping with the constant* $L > 0$*. Then we have the inequality*

$$\left| \int_a^b f(t)g(t)\,dt - \left[(1-\alpha) f(x) + \alpha \cdot \frac{f(a)+f(b)}{2} \right] \int_a^b g(t)\,dt \right| \leq KL(b-a)$$

for all $x \in [c, d]$*.*

Corollary 3. *Let* $f : [a, b] \to \mathbb{R}$ *be a monotonic mapping. Then we have the inequality*

$$\left| \int_a^b f(t)g(t)\,dt - \left[(1-\alpha) f(x) + \alpha \cdot \frac{f(a)+f(b)}{2} \right] \int_a^b g(t)\,dt \right| \leq K \cdot |f(b) - f(a)|$$

for all $x \in [c,d]$.

Remark 2. *The following inequality is well-known in the literature as the Bullen's inequality* [11, p. 141]*:*

$$\int_a^b f(t)dt \leq \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq (b-a)\frac{f(a)+f(b)}{2}, \tag{2.6}$$

where $f:[a,b]\to\mathbb{R}$ *is convex. Using the above results and (2.1), letting* $\alpha=\frac{1}{2}$, $g(t)\equiv 1$ *on* $[a,b]$, $h(t)=t$ *on* $[a,b]$, $x=\frac{a+b}{2}$, *we obtain the following error bound of the first inequality in (2.6),*

$$0 \leq \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\int_a^b f(t)dt \leq \frac{1}{4}(b-a)\bigvee_a^b(f),$$

provided that f *is of bounded variation on* $[a,b]$.

3. Applications for Quadrature Formula

Throughout this section, let $a<b$ in $\mathbb{R}$ and let α, g and h be defined as in Theorem 5. Let $f:[a,b]\to\mathbb{R}$, and let $I_n: a=x_0<x_1<\cdots<x_n=b$ be a partition of $[a,b]$ and $c_i=h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(x_i)+\frac{\alpha}{2}h(x_{i+1})\right)$, $d_i=h^{-1}\left(\frac{\alpha}{2}h(x_i)+\left(1-\frac{\alpha}{2}\right)h(x_{i+1})\right)$ and $\zeta_i\in[c_i,d_i]$ $(i=0,1,\ldots,n-1)$. Put $L_i:=h(x_{i+1})-h(x_i)=\int_{x_i}^{x_{i+1}}g(t)\,dt$ and define the sum

$$A_O(f,g,h,I_n,\zeta):=\sum_{i=0}^{n-1}\left[(1-\alpha)f(\zeta_i)+\alpha\cdot\frac{f(x_i)+f(x_{i+1})}{2}\right]L_i$$

and

$$R_O(f,g,h,I_n,\zeta)=\int_a^b f(t)g(t)dt-A_O(f,g,h,I_n,\zeta).$$

We have the following approximation of the integral $\int_a^b f(t)g(t)\,dt$.

Theorem 6. *Let* f *be defined as in Theorem 5 and let*

$$\int_a^b f(t)g(t)dt=A_O(f,g,h,I_n,\zeta)+R_O(f,g,h,I_n,\zeta),$$

then, the remainder term $R_O(f,g,h,I_n,\zeta)$ *satisfies the estimation*

$$\begin{aligned}|R_O(f,g,h,I_n,\zeta)| &\leq \sum_{i=0}^{n-1}K_i\bigvee_{x_i}^{x_{i+1}}(f)\\ &\leq M_1\cdot\bigvee_a^b(f)\\ &\leq M_2\cdot\bigvee_a^b(f)\\ &\leq M_3\cdot\bigvee_a^b(f),\end{aligned} \tag{3.1}$$

where

$$K_i := \begin{cases} \frac{1-\alpha}{2} L_i + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, & \text{if } 0 \le \alpha \le \frac{1}{2} \\ \max\left\{ \frac{1-\alpha}{2} L_i + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, \frac{\alpha}{2} L_i \right\}, & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2} L_i, & \text{if } \frac{2}{3} \le \alpha \le 1 \end{cases}$$

$$(i = 0, 1, \dots, n-1),$$

$$M_1 := \begin{cases} \max_{i=0,1,\dots,n-1} \left\{ \frac{1-\alpha}{2} L_i + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right| \right\}, & \text{if } 0 \le \alpha \le \frac{1}{2} \\ \max_{i=0,1,\dots,n-1} \left\{ \max\left\{ \frac{1-\alpha}{2} \nu(L) + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, \frac{\alpha}{2} \nu(L) \right\} \right\}, & \\ & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2} \nu(L), & \text{if } \frac{2}{3} \le \alpha \le 1 \end{cases},$$

$$M_2 := \begin{cases} \frac{1-\alpha}{2} \nu(L) + \max_{i=0,1,\dots,n-1} \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, & \text{if } 0 \le \alpha \le \frac{1}{2} \\ \max_{i=0,1,\dots,n-1} \left\{ \max\left\{ \frac{1-\alpha}{2} \nu(L) + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, \frac{\alpha}{2} \nu(L) \right\} \right\}, & \\ & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2} \nu(L), & \text{if } \frac{2}{3} \le \alpha \le 1 \end{cases},$$

$$M_3 := \begin{cases} (1-\alpha) \nu(L), & \text{if } 0 \le \alpha \le \frac{2}{3} \\ \frac{\alpha}{2} \nu(L), & \text{if } \frac{2}{3} \le \alpha \le 1 \end{cases},$$

and $\nu(L) := \max\{L_i \,|i = 0, 1, \dots, n-1\}$. *In the third inequality of* (3.1), *the constant* $\frac{\alpha}{2}$ *as* $0 \le \alpha \le \frac{1}{2}$ *and the constant* $\frac{1-\alpha}{2}$ *as* $\frac{2}{3} \le \alpha \le 1$ *are the best possible.*

Proof. Apply Theorem 5 on the intervals $[x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$ to get

$$\left| \int_{x_i}^{x_{i+1}} f(t) g(t)\, dt - \left[(1-\alpha) f(\zeta_i) + \alpha \cdot \frac{f(x_i) + f(x_{i+1})}{2} \right] L_i \right| \le K_i \bigvee_{x_i}^{x_{i+1}} (f),$$

for all $i = 0, 1, \dots, n-1$.

Using this and the generalized triangle inequality, we have

$$\begin{aligned} & |R_O(f, g, h, I_n, \zeta)| \\ & \le \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(t) g(t)\, dt - \left[(1-\alpha) f(\zeta_i) + \alpha \cdot \frac{f(x_i) + f(x_{i+1})}{2} \right] L_i \right| \\ & \le \sum_{i=0}^{n-1} K_i \bigvee_{x_i}^{x_{i+1}} (f) \\ & \le \left(\max_{i=0,1,\dots,n-1} K_i \right) \cdot \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) = M_1 \cdot \bigvee_a^b (f) \le M_2 \cdot \bigvee_a^b (f) \end{aligned}$$

and the first inequality, second inequality and third inequality in (3.1) are proved.

For the fourth inequality in (3.1), we observe that

$$\left| h(\zeta_i) - \frac{h(x_i) + h(x_{i+1})}{2} \right| \le \frac{1-\alpha}{2} \cdot L_i \qquad (i = 0, 1, \dots, n-1);$$

and then

$$\max_{i=0,1,\dots,n-1}\left|h\left(\zeta_i\right)-\frac{h(x_i)+h(x_{i+1})}{2}\right|\le\frac{1-\alpha}{2}\nu\left(L\right)$$

and $M_2\le M_3$. Thus the theorem is proved. ∎

Under the conditions of Theorem 6, we have the following remarks and corollaries.

Remark 3.

(1) *If we choose $\alpha=0$ and $g\left(t\right)\equiv1,h\left(t\right)=t$ on $[a,b]$ and $\xi_i=\zeta_i$ $(i=0,1,\dots,n-1)$, then the inequality (3.1) reduces to (1.3).*

(2) *If we choose $\alpha=\frac{1}{3}$, $g\left(t\right)\equiv1,h\left(t\right)=t$ on $[a,b]$ and $\zeta_i=\frac{x_i+x_{i+1}}{2}$ $(i=0,1,\dots,n-1)$, then the third inequality in (3.1) reduces to (1.6).*

Corollary 4. *In Theorem 6, let $f:[a,b]\to\mathbb{R}$ be a Lipschitzian mapping with the constant $L>0$ and choose $\zeta_i:=h^{-1}\left(\frac{h(x_i)+h(x_{i+1})}{2}\right)$ $(i=0,1,\dots,n-1)$. Then*

$$M_1:=\begin{cases}\frac{(1-\alpha)}{2}\nu\left(L\right), & \text{if } 0\le\alpha\le\frac{1}{2}\\[2mm] \frac{\alpha}{2}\nu\left(L\right), & \text{if } \frac{1}{2}\le\alpha\le1\end{cases}$$

and we have the formula

$$\begin{aligned}\int_a^b f(t)g(t)dt&=A_O\left(f,g,h,I_n,\zeta\right)+R_O\left(f,g,h,I_n,\zeta\right)\\&=\sum_{i=0}^{n-1}\left[\left(1-\alpha\right)f\left(\zeta_i\right)+\alpha\cdot\frac{f(x_i)+f(x_{i+1})}{2}\right]L_i+R_O\left(f,g,h,I_n,\zeta\right)\end{aligned}$$

and the remainder satisfies the estimation

$$\left|R_O\left(f,g,h,I_n,\zeta\right)\right|\le M_1L\left(b-a\right).$$

Corollary 5. *In Theorem 6, let $f:[a,b]\to\mathbb{R}$ be a monotonic mapping and let ζ_i $(i=0,1,\dots,n-1)$ and M_1 be defined as in Corollary 4. Then the remainder $R_O\left(f,g,h,I_n,\zeta\right)$ satisfies the estimation*

$$\left|R_O\left(f,g,h,I_n,\zeta\right)\right|\le M_1\cdot\left|f\left(b\right)-f\left(a\right)\right|.$$

The case of equidistant divisions is embodied in the following corollary and remark:

Corollary 6. *Suppose that*

$$x_i:=h^{-1}\left[h\left(a\right)+\frac{i\left(h(b)-h(a)\right)}{n}\right]\qquad(i=0,1,\dots,n)$$

and

$$\begin{aligned}L_i&:=h(x_{i+1})-h(x_i)\\&=\frac{h(b)-h(a)}{n}=\frac{1}{n}\int_a^b g\left(t\right)dt\qquad(i=0,1,\dots,n-1).\end{aligned}$$

In Theorem 6, let $\zeta_i=h^{-1}\left(\frac{h(x_i)+h(x_{i+1})}{2}\right)$ $(i=0,1,\dots,n-1)$, then

$$M_1:=\begin{cases}\frac{(1-\alpha)}{2n}\int_a^b g\left(t\right)dt, & \text{if } 0\le\alpha\le\frac{1}{2}\\[2mm] \frac{\alpha}{2n}\int_a^b g\left(t\right)dt, & \text{if } \frac{1}{2}\le\alpha\le1\end{cases}$$

and we have the formula

$$\begin{aligned}\int_a^b f(t)g(t)dt &= A_O(f,g,h,I_n,\zeta) + R_O(f,g,h,I_n,\zeta) \\ &= \frac{1}{n}\int_a^b g(t)\,dt \cdot \sum_{i=0}^{n-1}\left[(1-\alpha) f(\zeta_i) + \alpha \cdot \frac{f(x_i)+f(x_{i+1})}{2}\right] L_i \\ &\quad + R_O(f,g,h,I_n,\zeta)\end{aligned}$$

and the remainder satisfies the estimate

$$|R_O(f,g,h,I_n,\zeta)| \le M_1 \cdot \bigvee_a^b (f).$$

Remark 4. *If we want to approximate the integral* $\int_a^b f(t)\,g(t)\,dt$ *by* $A_O(f,g,h,I_n,\zeta)$ *with an accuracy less than* $\varepsilon > 0$, *we need at least* $n_\varepsilon \in \mathbb{N}$ *points for the partition* I_n, *where*

$$K_\varepsilon := \begin{cases} \frac{(1-\alpha)}{2\varepsilon}\int_a^b g(t)\,dt, & \text{if } 0 \le \alpha \le \frac{1}{2} \\ \frac{\alpha}{2\varepsilon}\int_a^b g(t)\,dt, & \text{if } \frac{1}{2} \le \alpha \le 1 \end{cases}, \qquad n_\varepsilon := \left[K_\varepsilon \cdot \bigvee_a^b (f)\right] + 1$$

and $[r]$ *denotes the Gaussian integer of* $r \in \mathbb{R}$.

4. Some Inequalities for Random Variables

Throughout this section, let $0 < a < b$ in $\mathbb{R}$, $r \in \mathbb{R}$, and let X be a continuous random variable having the continuous probability density function $g : [a,b] \to [0,\infty)$ which is positive on (a,b) and assume that the $r-$*moment*

$$E_r(X) := \int_a^b t^r g(t)\,dt,$$

is finite.

Theorem 7. *The inequality*

$$\left| E_r(X) - \left[(1-\alpha)\cdot\left(h^{-1}\left(\frac{1}{2}\right)\right)^r + \alpha \cdot \frac{a^r+b^r}{2}\right]\right| \le \overline{K} \cdot |b^r - a^r| \tag{4.1}$$

holds where $h(t) = \int_a^t g(x)\,dx$ $(t \in [a,b])$ *and*

$$\overline{K} := \begin{cases} \frac{(1-\alpha)}{2}, & \text{if } 0 \le \alpha \le \frac{1}{2} \\ \frac{\alpha}{2}, & \text{if } \frac{1}{2} \le \alpha \le 1 \end{cases}.$$

Proof. If we put $f(t) = t^r$, and $x = h^{-1}\left(\frac{h(a)+h(b)}{2}\right)$ in Corollary 3, then

$$\overline{K} = K = \begin{cases} \frac{(1-\alpha)}{2}, & \text{if } 0 \le \alpha \le \frac{1}{2} \\ \frac{\alpha}{2}, & \text{if } \frac{1}{2} \le \alpha \le 1 \end{cases}$$

and we obtain the inequality

$$\begin{aligned}(4.2)\quad & \left|\int_a^b f(t)g(t)\,dt - \left[(1-\alpha)\cdot f\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right)\right.\right. \\ & \left.\left. +\alpha\cdot\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\,dt\right| \le \overline{K}\cdot|f(b)-f(a)|\,.\end{aligned}$$

Since

$$\int_a^b f(t)g(t)\,dt = E_r(X)\,,\quad h(a)=0,\quad h(b)=\int_a^b g(t)\,dt = 1,$$
$$\frac{f(a)+f(b)}{2} = \frac{a^r+b^r}{2},\quad \text{and}\quad |f(b)-f(a)| = |b^r-a^r|\,,$$

(4.1) follows from (4.2). ∎

If we choose $r=1$ in Theorem 7, then we have the following remark:

Remark 5. *If $E(X)$ is the expectation of the random variable X, then*

$$\left|E(X) - \left[(1-\alpha)\cdot h^{-1}\left(\frac{1}{2}\right)+\alpha\cdot\frac{a+b}{2}\right]\right| \le \overline{K}\cdot(b-a)\,.$$

5. An Inequality for the Beta Mapping

The following mapping is well-known in the literature as the *Beta mapping*:

$$\beta(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt,\quad p>0,\ q>0.$$

Theorem 8. *Let $p>0$, $q>1$ and n be a positive integer. Then the inequality*

$$\begin{aligned}(5.1)\quad & \left|\beta(p,q) - \frac{1}{np}\sum_{i=0}^{n-1}\left\{\frac{\alpha}{2}\left(\left[1-\left(\frac{i}{n}\right)^{\frac{1}{p}}\right]^{q-1}+\left[1-\left(\frac{i+1}{n}\right)^{\frac{1}{p}}\right]^{q-1}\right)\right.\right. \\ & \left.\left. +(1-\alpha)\left[1-\left(\frac{2i+1}{2n}\right)^{\frac{1}{p}}\right]^{q-1}\right\}\right| \le \overline{M}\end{aligned}$$

holds where

$$\overline{M} := \begin{cases}\frac{(1-\alpha)}{2np}, & \text{if } 0\le\alpha\le\frac{1}{2}\\ \frac{\alpha}{2np}, & \text{if } \frac{1}{2}\le\alpha\le 1\end{cases}.$$

Proof. If we put $a=0$, $b=1$, $f(t)=(1-t)^{q-1}$, $g(t)=t^{p-1}$ and $h(t)=\frac{t^p}{p}$ $(t\in[0,1])$ in Corollary 6, then, $\int_a^b g(t)dt=\frac{1}{p}$, $h^{-1}(t)=(pt)^{\frac{1}{p}}$ $(t\in[0,1])$, $x_i=\left(\frac{i}{n}\right)^{\frac{1}{p}}$ $(i=0,1,\dots,n)$, $\zeta_i=\frac{2i+1}{2np}$ $(i=0,1,\dots,n-1)$, $\bigvee_a^b(f)=1$ and $\overline{M}=M_1$, so that the inequality (5.1) holds. ∎

References

[1] T. M. Apostol, *Mathematical Analysis,* Second Edition, Addision-Wesley Publishing Company , 1975.

[2] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Inequal. Applics.*, **4**(1) (2001), 59–66.

[3] S. S. Dragomir, On the Ostrowski's integral inequality for Lipschitzian mappings and applications, *Studia Univ. Babes-Bolyai Math.*, **46**(1) (2001), 33–40.

[4] S. S. Dragomir, On Simpson's quadrature formula and applications, *Mathematica,* **43**(66)(2) (2001), 185–194.

[5] S. S. Dragomir, On Simpson's quadrature formula for Lipschitzian mappings and applications, *Soochow J. Math.*, **25**(2) (1999), 175–180.

[6] S. S. Dragomir, On Simpson's quadrature formula for mappings of bounded variation and applications, *Tamkang J. of Math.*, **30**(1) (1999), 53–58.

[7] S. S. Dragomir, R. P. Agarwal and P. Cerone, On Simpson's inequality and applications, *J. Inequal. Appl.*, **5**(6) (2000), 533–579.

[8] S. S. Dragomir, A generalization of Ostrowski's integral inequality for mappings with bounded variation and applications in numerical integration, *Bull. Australian Math. Soc.*, accepted.

[9] S. S. Dragomir, A new generalization of Ostrowski's integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means, *Appl. Math. Lett.*, **11**(1) (1998), 105–109.

[10] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, 1994.

[11] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press Inc., 1992.

[12] J. Pečarić and S. Varošanec, A note on Simpson's inequality for Lipschitzian functions, *Soochow J. Math.*, **27**(1) (2001), 53–57.

[13] J. Pečarić and S. Varošanec, A note on Simpson's inequality for functions of bounded variation, *Tamkang J. of Math.*, **31**(3) (2000), 239–242.

[14] N. Ujević, New bounds for Simpson's inequality, *Tamkang J. of Math.*, **33**(2) (2002), 129–138.

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