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ON A CONJECTURE ON THE SYMMETRIC MEANS

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Abstract. In this paper, we study some inequalities involving the symmetric means. The main result is a proof of a conjecture of Alzer et al..

1. Introduction

Let $M_{n,r}(\mathbf{x};\mathbf{q})$ be the generalized weighted means: $M_{n,r}(\mathbf{x};\mathbf{q})=(\sum_{i=1}^n q_ix_i^r)^{\frac{1}{r}}$, where $M_{n,0}(\mathbf{x};\mathbf{q})$ denotes the limit of $M_{n,r}(\mathbf{x};\mathbf{q})$ as $r\rightarrow 0^+$, $\mathbf{x}=(x_1,\ldots,x_n)$, $\mathbf{q}=(q_1,\ldots,q_n)$ with $q_i>0$ $(1\leq i\leq n)$ satisfying $\sum_{i=1}^n q_i=1$. In this paper, we let $q=\min q_i$ and always assume $0<x_1\leq x_2\leq\cdots\leq x_n$.

To any given $\mathbf{x}$ and $t\geq 0$, we set $\mathbf{x}'=(1-x_1,\ldots,1-x_n), \mathbf{x}_t=(x_1+t,\ldots,x_n+t)$ and $\mathbf{x}^{-1}=(1/x_1,\ldots,1/x_n)$.

Let $k\in\{0,1,\ldots,n\}$, the k-th symmetric function $E_{n,k}$ of $\mathbf{x}$ and its mean $P_{n,k}$ are defined by

$$E_{n,k}(\mathbf{x})=\sum_{1\leq i_1<\cdots<i_k\leq n}\prod_{j=1}^k x_{i_j}, 1\leq k\leq n;\quad E_{n,0}=1;\quad P_{n,k}^k(\mathbf{x})=\frac{E_{n,k}(\mathbf{x})}{\binom{n}{k}}.$$

We define $A_n(\mathbf{x};\mathbf{q})=M_{n,1}(\mathbf{x};\mathbf{q}), G_n(\mathbf{x})=M_{n,0}(\mathbf{x};\mathbf{q}), H_n(\mathbf{x};\mathbf{q})=M_{n,-1}(\mathbf{x};\mathbf{q})$ and we shall write $M_{n,r}$ for $M_{n,r}(\mathbf{x};\mathbf{q})$, $M_{n,r,t}$ for $M_{n,r}(\mathbf{x}_t;\mathbf{q})$ and $M'_{n,r}$ for $M_{n,r}(\mathbf{x}')$ if $x_n<1$ and similarly for other means when there is no risk of confusion. We further denote $\sigma_n=\sum_{i=1}^n q_i(x_i-A_n)^2$.

When $x_n<1$, we define

$$\Delta'_{r,s}=\frac{M'_{n,r}-M'_{n,s}}{M_{n,r}-M_{n,s}}.$$

In order to include the case of equality for various inequalities in our discussions, for any given inequality, we define 0/0 to be the number which makes the inequality an equality. The author [7, Theorem 2.1] has shown that

Theorem 1.1. *For $r>s$, the following inequalities are equivalent:*

$$\frac{r-s}{2x_1}\sigma_n\geq\quad M_{n,r}-M_{n,s}\quad\geq\frac{r-s}{2x_n}\sigma_n, \tag{1.1}$$

$$\frac{x_n}{1-x_n}\geq\qquad \Delta'_{r,s}\qquad\geq\frac{x_1}{1-x_1}, \tag{1.2}$$

where in (1.2) *we require $x_n<1$.*

In fact, one can further show that (see [9]) the two inequalities in Theorem 1.1 are equivalent to

$$\frac{x_n}{t+x_1}\geq\frac{M_{n,r,t}-M_{n,s,t}}{M_{n,r}-M_{n,s}}\geq\frac{x_1}{t+x_n} \tag{1.3}$$

being valid for all $t\geq 0$.

We note that inequality (1.1) doesn't hold for all pairs r,s (see [7]). Cartwright and Field [5] first proved the validity of (1.1) for $r=1,s=0$. For other extensions and refinements of (1.1), see [3], [11], [12], [8], [13] and [9]. Inequality (1.2) is commonly referred as the additive Ky Fan's

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inequality. We refer the reader to the survey article [2] and the references therein for an account of Ky Fan's inequality.

When inequality (1.1) holds for some r, s, one can often expect for a better result than (1.3), namely

$$\frac{x_n}{t+x_n} \geq \frac{M_{n,r,t}-M_{n,s,t}}{M_{n,r}-M_{n,s}} \geq \frac{x_1}{t+x_1}.$$

In this paper, we study some Ky Fan-type inequalities involving the symmetric means. Much of our study was motivated by the following well-known Newton's inequalities (see [10], Theorem 52)

Theorem 1.2. *For $0<k<n$,*

$$P^{k-1}_{n,k-1}P^{k+1}_{n,k+1} \leq P^{2k}_{n,k}.$$

Alzer et al. [4] conjectured that:

Conjecture 1.1. *For $x_i \in (0,1/2]$, $2 \leq k \leq n$,*

$$\frac{E^{1/k}_{n,k}(\mathbf{x}^{-1})}{E^{1/k}_{n,k}(\mathbf{x}'^{-1})} \leq \frac{E^{1/(k-1)}_{n,k-1}(\mathbf{x}^{-1})}{E^{1/(k-1)}_{n,k-1}(\mathbf{x}'^{-1})}. \tag{1.4}$$

In view of the above discussion on the analogues between Ky Fan-type inequalities involving $\mathbf{x}'$'s and those involving $\mathbf{x}_t$'s, it is natural to expect that if Conjecture 1.1 is true, it should also be true if one replaces $\mathbf{x}'$ there with $\mathbf{x}_t$. Note that

$$P^j_{n,j}(\mathbf{x}^{-1}) = P^{n-j}_{n,n-j}(\mathbf{x})/P^n_{n,n}(\mathbf{x}).$$

Using this, we can recast the analogue of (1.4) with $\mathbf{x}'$ replaced by $\mathbf{x}_t$ as

$$\begin{aligned} & \frac{n-k+1}{k-1}\ln P_{n,n-k+1} - \frac{n-k}{k}\ln P_{n,n-k} - \frac{n}{k(k-1)}\ln G_n \\ \geq\ & \frac{n-k+1}{k-1}\ln P_{n,n-k+1,t} - \frac{n-k}{k}\ln P_{n,n-k,t} - \frac{n}{k(k-1)}\ln G_{n,t}. \end{aligned}$$

It is easy to deduce the above inequality from the following analogues for Newton's inequalities. Namely, for $2 \leq k \leq n-1$,

$$\ln P^{2k}_{n,k} - \ln P^{k+1}_{n,k+1} - \ln P^{k-1}_{n,k-1} \geq \ln P^{2k}_{n,k,t} - \ln P^{k+1}_{n,k+1,t} - \ln P^{k-1}_{n,k-1,t}. \tag{1.5}$$

In fact, in view of the similar results obtained in [9], one expect stronger inequalities to hold and we will prove in this paper the following result:

Theorem 1.3. *For $2 \leq k \leq n-1$, $t \geq 0$,*

$$\begin{aligned} x^2_n(\ln P^{2k}_{n,k} - \ln P^{k+1}_{n,k+1} - \ln P^{k-1}_{n,k-1}) &\geq (x_n+t)^2(\ln P^{2k}_{n,k,t} - \ln P^{k+1}_{n,k+1,t} - \ln P^{k-1}_{n,k-1,t}), \\ x^2_1(\ln P^{2k}_{n,k} - \ln P^{k+1}_{n,k+1} - \ln P^{k-1}_{n,k-1}) &\leq (x_1+t)^2(\ln P^{2k}_{n,k,t} - \ln P^{k+1}_{n,k+1,t} - \ln P^{k-1}_{n,k-1,t}). \end{aligned}$$

By using a similar method as in the proof of Theorem 2.1 in [7], one can deduce from the above theorem the following corollaries and we shall omit the proofs here.

Corollary 1.1. *For $2 \leq k \leq n-1$,*

$$\frac{\sigma_n}{(n-1)x^2_1} \geq \ln P^{2k}_{n,k} - \ln P^{k+1}_{n,k+1} - \ln P^{k-1}_{n,k-1} \geq \frac{\sigma_n}{(n-1)x^2_n}.$$

Corollary 1.2. *For $2 \leq k \leq n-1$, $x_n < 1/2$,*

$$\begin{aligned} x^2_n(\ln P^{2k}_{n,k} - \ln P^{k+1}_{n,k+1} - \ln P^{k-1}_{n,k-1}) &\geq (1-x_n)^2(\ln P'^{2k}_{n,k} - \ln P'^{k+1}_{n,k+1} - \ln P'^{k-1}_{n,k-1}), \\ x^2_1(\ln P^{2k}_{n,k} - \ln P^{k+1}_{n,k+1} - \ln P^{k-1}_{n,k-1}) &\leq (1-x_1)^2(\ln P'^{2k}_{n,k} - \ln P'^{k+1}_{n,k+1} - \ln P'^{k-1}_{n,k-1}). \end{aligned}$$

It is also easy to deduce from Corollary 1.2 the following stronger version of (1.4):

Corollary 1.3. *For $2 \leq k \leq n-1$, $x_n \leq 1/2$,*

$$\begin{aligned} & x_n^2\Big(\frac{n-k+1}{k-1}\ln P_{n,n-k+1}-\frac{n-k}{k}\ln P_{n,n-k}-\frac{n}{k(k-1)}\ln G_n\Big) \\ \geq \ & (1-x_n)^2\Big(\frac{n-k+1}{k-1}\ln P'_{n,n-k+1}-\frac{n-k}{k}\ln P'_{n,n-k}-\frac{n}{k(k-1)}\ln G'_n\Big), \end{aligned}$$

with the above inequality reversed if one replaces x_n by x_1 above.

In order to prove Conjecture 1.1, one can in fact use results which are weaker than Theorem 1.3. For example, it's easy to check that Corollary 1.1 will imply Conjecture 1.1. In Section 3, we will give another proof of Conjecture 1.1 by giving a direct proof of Corollary 1.1.

It is conjectured by Alzer [1] that for $x_i \in (0, 1/2], q_i = 1/n$, the following inequality holds:

$$(1-\frac{1}{n})A_n+\frac{1}{n}H_n-G_n \geq (1-\frac{1}{n})A'_n+\frac{1}{n}H'_n-G'_n.$$

Once again one may ask whether the analogue of the above inequality holds with $\mathbf{x}'$ replaced by $\mathbf{x}_t$ throughout. We will show that this is indeed true in Section 4.

2. Proof of Theorem 1.3

We first state a few lemmas:

Lemma 2.1. *Let $2 \leq r \leq n$, $\mathbf{x}=(x_1,\ldots,x_n)$, $x_1 \leq x_2 \leq \cdots \leq x_n$. There exists $\mathbf{y}=(y_1,\ldots,y_r)$ with $x_1 \leq y_1 \leq \cdots \leq y_r \leq x_n$ such that $P_{n,i}(\mathbf{x}) = P_{r,i}(\mathbf{y}), 0 \leq i \leq r$. Moreover, if $x_1,\ldots,x_n$ are not all equal, then $y_1,\ldots,y_r$ are also not all equal.*

The above lemma is due to Wu, Wang and Fu [14] (see also [2, p. 317-318]), it will play a key role in our proof of Theorem 1.3.

Lemma 2.2. *Theorem 1.3 holds for the case $k = n-1$.*

Proof. Since the proofs are similar, we will only prove the first inequality in Theorem 1.3. In this case we need to show that for $t \geq 0$,

$$g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}_t) \geq 0,$$

where

$$f(\mathbf{x}) = x_n^2\left(2\ln\left(n^{-1}(\sum_{i=1}^n\frac{1}{x_i})\right)-\ln\left(\binom{n}{2}^{-2}(\sum_{\substack{i,j=1\\ i\neq j}}^n\frac{1}{x_ix_j})\right)\right).$$

If $x_1 = x_n$, then there is nothing to prove. Otherwise let $0 < x = x_1 = \cdots = x_k < x_{k+1}$ for some $1 \leq k < n$, then

$$\frac{\partial g}{\partial x} = \sum_{i=1}^k \frac{\partial g}{\partial x_i}.$$

We want to show that the right-hand side above is non-positive. It suffices to show each single term in the sum is non-negative. Without loss of generality, we now show that $\partial g/\partial x_1 \leq 0$. Calculation yields that

$$\frac{\partial g}{\partial x_1} = h(\mathbf{x}_t) - h(\mathbf{x}),$$

where

$$h(\mathbf{x}) = \frac{x_n^2\sum_{i=1}^n\frac{x_i-x_1}{x_i^2}}{x_1^3(\sum_{i=1}^n\frac{1}{x_i})(\sum_{\substack{i,j=1\\ i\neq j}}^n\frac{1}{x_ix_j})}.$$

It is easy to check that

$$\frac{x_n}{x_i} \geq \frac{x_n+t}{x_i+t}, \quad \frac{x_1}{x_i} \leq \frac{x_1+t}{x_i+t}.$$

We then deduce that $\partial g/\partial x_1 \leq 0$. By letting $x \to x_{k+1}$ and repeating the above argument, we conclude that $g(\mathbf{x}) \geq g(x_n, x_n, \ldots, x_n) = 0$ which completes the proof. □

Lemma 2.3. *For $t \geq 0$, inequality (1.5) holds for $2 \leq k \leq n-1$. Equivalently,*

$$2k\frac{P^{k-1}_{n,k-1}}{P^k_{n,k}} - (k+1)\frac{P^k_{n,k}}{P^{k+1}_{n,k+1}} - (k-1)\frac{P^{k-2}_{n,k-2}}{P^{k-1}_{n,k-1}} \leq 0. \tag{2.1}$$

Proof. Let

$$f(\mathbf{x}, t) = \ln P^{2k}_{n,k,t} - \ln P^{k+1}_{n,k+1,t} - \ln P^{k-1}_{n,k-1,t}.$$

Since $\mathbf{x}, t$ are arbitrary, (1.5) is equivalent to

$$\frac{\partial f}{\partial t}\Big|_{t=0} \leq 0. \tag{2.2}$$

Using the relation that for $1 \leq k \leq n$,

$$\frac{\partial \ln P_{n,k,t}}{\partial t}\Big|_{t=0} = \frac{P^{k-1}_{n,k-1}}{P^k_{n,k}},$$

we see that (2.2) is just (2.1). By Lemma 2.1, there exists $\mathbf{y} = (y_1, \ldots, y_{k+1})$ with $x_1 \leq y_1 \leq \cdots \leq y_{k+1} \leq x_n$ such that $P_{n,i}(\mathbf{x}) = P_{k+1,i}(\mathbf{y}), 0 \leq i \leq k+1$. It follows from this and Theorem 1.2 that

$$\begin{aligned} & 2k\frac{P^{k-1}_{n,k-1}(\mathbf{x})}{P^k_{n,k}(\mathbf{x})} - (k+1)\frac{P^k_{n,k}(\mathbf{x})}{P^{k+1}_{n,k+1}(\mathbf{x})} - (k-1)\frac{P^{k-2}_{n,k-2}(\mathbf{x})}{P^{k-1}_{n,k-1}(\mathbf{x})} \\ = \quad & 2k\frac{P^{k-1}_{k+1,k-1}(\mathbf{y})}{P^k_{k+1,k}(\mathbf{y})} - (k+1)\frac{P^k_{k+1,k}(\mathbf{y})}{P^{k+1}_{k+1,k+1}(\mathbf{y})} - (k-1)\frac{P^{k-2}_{k+1,k-2}(\mathbf{y})}{P^{k-1}_{k+1,k-1}(\mathbf{y})}. \end{aligned}$$

So it suffices to show the right-hand side expression above is non-positive. In this case note that Lemma 2.2 implies (for the same reason as (1.5) being equivalent to (2.2))

$$2(\ln P^{2k}_{k+1,k} - \ln P^{k+1}_{k+1,k+1} - \ln P^{k-1}_{k+1,k-1}) + x_n\Big(2k\frac{P^{k-1}_{k+1,k-1}}{P^k_{n,k}} - (k+1)\frac{P^k_{k+1,k}}{P^{k+1}_{k+1,k+1}} - (k-1)\frac{P^{k-2}_{k+1,k-2}}{P^{k-1}_{k+1,k-1}}\Big) \leq 0.$$

Now the lemma follows from this and Theorem 1.2. □

Now we are ready to prove Theorem 1.3. As the proofs are similar, we will only prove the first inequality here. As in the proof of Lemma 2.3, it suffices to show that for $2 \leq k \leq n-1$,

$$2(\ln P^{2k}_{n,k} - \ln P^{k+1}_{n,k+1} - \ln P^{k-1}_{n,k-1}) + x_n\Big(2k\frac{P^{k-1}_{n,k-1}}{P^k_{n,k}} - (k+1)\frac{P^k_{n,k}}{P^{k+1}_{n,k+1}} - (k-1)\frac{P^{k-2}_{n,k-2}}{P^{k-1}_{n,k-1}}\Big) \leq 0. \tag{2.3}$$

By Lemma 2.1, there exists $\mathbf{y} = (y_1, \ldots, y_{k+1})$ with $x_1 \leq y_1 \leq \cdots \leq y_{k+1} \leq x_n$ such that $P_{n,i}(\mathbf{x}) = P_{k+1,i}(\mathbf{y}), 0 \leq i \leq k+1$. It follows from this and Lemma 2.3 that

$$\begin{aligned} & x_n\Big(2k\frac{P^{k-1}_{n,k-1}(\mathbf{x})}{P^k_{n,k}(\mathbf{x})} - (k+1)\frac{P^k_{n,k}(\mathbf{x})}{P^{k+1}_{n,k+1}(\mathbf{x})} - (k-1)\frac{P^{k-2}_{n,k-2}(\mathbf{x})}{P^{k-1}_{n,k-1}(\mathbf{x})}\Big) \\ \leq \quad & y_{k+1}\Big(2k\frac{P^{k-1}_{k+1,k-1}(\mathbf{y})}{P^k_{k+1,k}(\mathbf{y})} - (k+1)\frac{P^k_{k+1,k}(\mathbf{y})}{P^{k+1}_{k+1,k+1}(\mathbf{y})} - (k-1)\frac{P^{k-2}_{k+1,k-2}(\mathbf{y})}{P^{k-1}_{k+1,k-1}(\mathbf{y})}\Big). \end{aligned}$$

Thus it suffices to prove (2.3) for the case $n = k+1$ and this case is just Lemma 2.2 and this completes our proof of Theorem 1.3.

3. Another Proof of Conjecture 1.1

As mentioned in the introduction, it suffices to prove Corollary 1.1 in order to establish Conjecture 1.1. In this section, we will make use of A_n, H_n and $M_{n,2}$ and we will assume throughout this section that the weights associated to them are always $q_i = 1/n$.

Before we give a direct proof of Corollary 1.1, we would like to give an account of a motivation of this by pointing out that we may regard $P^k_{n,k}/P^{k-1}_{n,k-1}$, $1 \leq k \leq n$ as certain "means", in the sense that for any constant $c > 0$,

$$\frac{P^k_{n,k}(c\mathbf{x})}{P^{k-1}_{n,k-1}(c\mathbf{x})} = c\frac{P^k_{n,k}(\mathbf{x})}{P^{k-1}_{n,k-1}(\mathbf{x})},$$

and that by Theorem 1.2,

$$x_1 \leq P^n_{n,n}/P^{n-1}_{n,n-1} \leq P^{n-1}_{n,n-1}/P^{n-2}_{n,n-2} \leq \cdots \leq P^2_{n,2}/A_n \leq A_n \leq x_n.$$

From this point of view, one may regard Theorem 1.3 as Ky Fan-type inequalities concerning $\ln(P^k_{n,k}/P^{k-1}_{n,k-1})$'s. It is then natural to ask whether such inequalities hold with $\ln(P^k_{n,k}/P^{k-1}_{n,k-1})$'s replaced by $(P^k_{n,k}/P^{k-1}_{n,k-1})$'s. We now show that in general this is not true by considering the case $k = n$ here. First,

$$\frac{P^{n-1}_{n,n-1}}{P^{n-2}_{n,n-2}} - \frac{P^n_{n,n}}{P^{n-1}_{n,n-1}} \geq \frac{P^{n-1}_{n,n-1,t}}{P^{n-2}_{n,n-2,t}} - \frac{P^n_{n,n,t}}{P^{n-1}_{n,n-1,t}} \tag{3.1}$$

doesn't hold in general. The left-hand side above can be rewritten as

$$\frac{P^{n-1}_{n,n-1}(\mathbf{x})}{P^{n-2}_{n,n-2}(\mathbf{x})} - \frac{P^n_{n,n}(\mathbf{x})}{P^{n-1}_{n,n-1}(\mathbf{x})} = \frac{A_n(\mathbf{x}^{-1})}{P^2_{n,2}(\mathbf{x}^{-1})} - \frac{1}{A_n(\mathbf{x}^{-1})} = \frac{A^2_n(\mathbf{x}^{-1}) - P^2_{n,2}(\mathbf{x}^{-1})}{P^2_{n,2}(\mathbf{x}^{-1})A_n(\mathbf{x}^{-1})}. \tag{3.2}$$

Using the relation

$$A^2_n - P^2_{n,2} = \frac{1}{2(n-1)n^2}\sum^n_{i,j=1}(x_i - x_j)^2,$$

we see that in the case $x_1 = \ldots = x_m = x, x_{m+1} = \ldots = x_n = y$, the last expression in (3.2) becomes

$$\frac{C(x-y)^2}{x^2y^2\big((\frac{m}{x} + \frac{n-m}{y})^2 - \frac{m}{x^2} - \frac{n-m}{y^2}\big)(\frac{m}{x} + \frac{n-m}{y})} \tag{3.3}$$

for some positive constant C. For (3.1) to hold in this case, the expression in (3.3) must be greater than an analogue one with x, y replaced by $x+t, y+t$ respectively and it is easy to see this is not true when $x \to 0$. Thus (3.1) doesn't hold in general which further implies that

$$x_n\Big(\frac{P^{n-1}_{n,n-1}}{P^{n-2}_{n,n-2}} - \frac{P^n_{n,n}}{P^{n-1}_{n,n-1}}\Big) \geq (x_n + t)\Big(\frac{P^{n-1}_{n,n-1,t}}{P^{n-2}_{n,n-2,t}} - \frac{P^n_{n,n,t}}{P^{n-1}_{n,n-1,t}}\Big)$$

doesn't hold in general.

Next, we show

$$x_1\Big(\frac{P^{n-1}_{n,n-1}}{P^{n-2}_{n,n-2}} - \frac{P^n_{n,n}}{P^{n-1}_{n,n-1}}\Big) \leq (x_1 + t)\Big(\frac{P^{n-1}_{n,n-1,t}}{P^{n-2}_{n,n-2,t}} - \frac{P^n_{n,n,t}}{P^{n-1}_{n,n-1,t}}\Big) \tag{3.4}$$

doesn't hold in general. We proceed as in the previous case above and similar to (3.3), the left-hand side expression in (3.4) in the case $x_1 = \ldots = x_m = x, x_{m+1} = \ldots = x_n = y$ becomes

$$C(x-y)^2/f(\frac{y}{x}),$$

where

$$f(z)=z^{-1}\Big((mz+n-m)^2-mz^2-n+m\Big)(mz+n-m).$$

It is then easy to see that in order for (3.4) to hold, $f(z)$ needs to be an increasing function of $z\geq 1$ but one checks that this is not always true.

However, a special analogue of Theorem 1.3 holds:

Theorem 3.1. *For $1\leq k\leq n$,*

$$\begin{aligned} x_n\Big(A_n-\frac{P^k_{k,n}}{P^{k-1}_{n,k-1}}\Big) &\geq (x_n+t)\Big(A_{n,t}-\frac{P^k_{n,k,t}}{P^{k-1}_{n,k-1,t}}\Big), \\ x_1\Big(A_n-\frac{P^k_{k,n}}{P^{k-1}_{n,k-1}}\Big) &\leq (x_1+t)\Big(A_{n,t}-\frac{P^k_{n,k,t}}{P^{k-1}_{n,k-1,t}}\Big). \end{aligned} \tag{3.5}$$

The proof of the above theorem is similar to that of Theorem 1.3, one can first reduce to the case $k=n$ and then note this case follows from case (v) of Theorem 3.1 in [9] by taking $s=-1$ there. We will leave the details to the reader.

Similar to Lemma 2.3, one sees that the case $k=n$ of (3.5) is equivalent to

$$A_n-H_n\leq (n-1)x_n\Big(1-\frac{P^n_{n,n}P^{n-2}_{n,n-2}}{P^{2(n-1)}_{n,n-1}}\Big). \tag{3.6}$$

Apply the inequality $1+x\leq e^x$ with $x=\ln P^n_{n,n}+\ln P^{n-2}_{n,n-1}-\ln P^{2(n-1)}_{n,n-1}$, we obtain

$$\Big(1-\frac{P^n_{n,n}P^{n-2}_{n,n-1}}{P^{2(n-1)}_{n,n-1}}\Big)\leq 2(n-1)\ln P_{n,n-1}-n\ln P_{n,n}-(n-2)\ln P_{n,n-2}. \tag{3.7}$$

Note that (see, for example, Theorem 4.2 of [9])

$$\frac{\sigma_n}{x_1}\geq A_n-H_n\geq\frac{\sigma_n}{x_n}. \tag{3.8}$$

We then deduce form (3.6), (3.7) and the right-hand side inequality of (3.8) the right-hand side inequality in Corollary 1.1 for the case $k=n-1$. One can then deduce from this the right-hand side inequality in Corollary 1.1 for general k's by following the method in the proof of Theorem 1.3.

What about the left-hand side inequality in Corollary 1.1? As in the discussion above, it suffices to prove the case $k=n-1$. In this case, we may attempt to prove, similar to (3.6),

$$\frac{A_n-H_n}{n-1}\geq x_1\Big(2(n-1)\ln P_{n,n-1}-n\ln P_{n,n}-(n-2)\ln P_{n,n-2}\Big),$$

since this combined with the left-hand side inequality of (3.8) will yield the desired result. By a change of variables: $x_i\to 1/x_{n-i+1}$, we can rewrite the above as

$$\frac{A_n-H_n}{(n-1)A_nH_n}\geq\frac{1}{x_n}(\ln A_n^2-\ln P^2_{n,2}). \tag{3.9}$$

Note by Corollary 5.3 in [9], we have

$$(n-1)A_nH_n\leq nA_n^2-M^2_{n,2}=(n-1)P^2_{n,2}.$$

It follows from this and (3.8) that

$$\frac{A_n-H_n}{(n-1)A_nH_n}\geq\frac{A_n-H_n}{(n-1)P^2_{n,2}}\geq\frac{1}{x_n}(\frac{A_n^2}{P^2_{n,2}}-1)\geq\frac{1}{x_n}\ln\frac{A_n^2}{P^2_{n,2}},$$

where the last inequality above follows from the inequality $1+x\leq e^x$ by taking $x=\ln A_n^2-\ln P^2_{n,2}$. This established (3.9) which in turn completes our proof of Corollary 1.1.

4. More Ky Fan-type Inequalities

Our goal in this section is to prove some Ky-Fan type inequalities that are related to those mentioned in the introduction of the paper.

First, we give some refinements of certain inequalities of Ky Fan-type. This is motivated by a recent result of Mercer [13], which refines (1.1) for the case $r = 1, s = 0$ by using Hadamard's inequality. A version of his result is given by Theorem 4.3 of [9] (note there is a typo in the original statement though):

Theorem 4.1 ([9, Theorem 4.3])**.**

$$\sum_{i=1}^{n} \frac{q_i(x_i - A_n)^2}{x_i + \min(x_i, A_n)} \geq A_n - G_n \geq \sum_{i=1}^{n} \frac{q_i(x_i - G_n)^2}{x_i + \max(x_i, G_n)},$$

with equality holding if and only if $x_1 = \cdots = x_n$.

We now improve the lower bound of (1.1) for the case $r = 1, -1 \leq s < 0$ in a similar way. First, we need a lemma:

Lemma 4.1. *Let* $t > 0$*. For* $-1 \leq \alpha < 0$ *or* $\alpha \geq 1$,

$$t^{\alpha} - \alpha t + \alpha - 1 \geq \frac{\alpha(\alpha - 1)}{2\max(1, t)}(t - 1)^2.$$

The above inequality reverses when $0 < \alpha \leq 1$.

Proof. We will prove the assertion for $t \geq 1$ and $-1 \leq \alpha < 0$ or $\alpha \geq 1$. The other cases can be shown similarly. It suffices to show that

$$f(t) = t^{\alpha+1} - \alpha t^2 + (\alpha - 1)t - \frac{\alpha(\alpha - 1)}{2}(t - 1)^2 \geq 0, \quad t \geq 1.$$

One checks easily that $f''(t) \geq 0$ for $t \geq 1$ so that $f'(t) \geq f'(1) = 0$ for $t \geq 1$. This implies that $f(t) \geq f(1) = 0$ for $t \geq 1$ which completes the proof. □

Theorem 4.2. *For* $-1 \leq s < 0$,

$$A_n - P_{n,s} \geq \frac{1 - s}{2} \sum_{i=1}^{n} \frac{q_i}{\max(x_i, P_{n,s})}(x_i - P_{n,s})^2.$$

Proof. We apply Lemma 4.1 with $\alpha = s$ and $t = x_i/P_{n,s}$ to get

$$\frac{x_i^s}{P_{n,s}^s} - s\frac{x_i}{P_{n,s}} + s - 1 \geq \frac{s(s - 1)}{2P_{n,s}\max(x_i, P_{n,s})}(x_i - P_{n,s})^2.$$

Our assertion then follows by multiplying the above inequality by q_i and summing over i from 1 to n. □

Next, we prove an analogue of a conjecture of Alzer mentioned at the end of Section 1:

Theorem 4.3. *For* $x_i > 0, q_i = 1/n, t \geq 0$,

$$(1 - \frac{1}{n})A_n + \frac{1}{n}H_n - G_n \geq (1 - \frac{1}{n})A_{n,t} + \frac{1}{n}H_{n,t} - G_{n,t}.$$

Proof. We may assume $n \geq 2$ here. Let

$$f(\mathbf{x}, t) = (1 - \frac{1}{n})A_{n,t} + \frac{1}{n}H_{n,t} - G_{n,t}.$$

Again it suffices to show that

$$\frac{\partial f}{\partial t}\Big|_{t=0} \leq 0.$$

Direct calculations show that this is equivalent to

$$(1-\frac{1}{n})+\frac{1}{n}\Big(n-(n-1)\frac{P^{n}_{n,n}P^{n-2}_{n,n-2}}{P^{2(n-1)}_{n,n-1}}\Big)-\frac{P^{n-1}_{n,n-1}}{P^{n-1}_{n,n}}\leq 0.$$

We can recast the above as

$$(n-1)\Big(1-\frac{P^{n}_{n,n}P^{n-2}_{n,n-2}}{P^{2(n-1)}_{n,n-1}}\Big)\leq n\Big(\frac{P^{n-1}_{n,n-1}}{P^{n-1}_{n,n}}-1\Big). \tag{4.1}$$

Using the inequality $1+x\leq e^x$ for $x=\ln P^{n-1}_{n,n-1}-\ln P^{n-1}_{n,n}$, we get

$$\ln P^{n-1}_{n,n-1}-\ln P^{n-1}_{n,n}\leq \frac{P^{n-1}_{n,n-1}}{P^{n-1}_{n,n}}-1.$$

Using (3.7), we see that (4.1) follows from the following inequality:

$$(n-1)\Big(2\ln P^{n-1}_{n,n-1}-\ln P^{n}_{n,n}-\ln P^{n-2}_{n,n-2}\Big)\leq n\Big(\ln P^{n-1}_{n,n-1}-\ln P^{n-1}_{n,n}\Big).$$

One checks easily that the above inequality is equivalent to

$$P_{n,n-1}\leq P_{n,n-2},$$

which is a consequence of Newton's inequalities, see [10, Theorem 51]. This completes the proof. □

We remark here that one may expect to have (with $q_i=1/n$ here)

$$x_n^2\Big((1-\frac{1}{n})A_n+\frac{1}{n}H_n-G_n\Big)\geq (x_n+t)^2\Big((1-\frac{1}{n})A_{n,t}+\frac{1}{n}H_{n,t}-G_{n,t}\Big). \tag{4.2}$$

However, a dual form of the above inequality

$$x_1^2\Big((1-\frac{1}{n})A_n+\frac{1}{n}H_n-G_n\Big)\leq (x_1+t)^2\Big((1-\frac{1}{n})A_{n,t}+\frac{1}{n}H_{n,t}-G_{n,t}\Big)$$

does not hold because one can deduce from the above inequality by a similar argument as used in [9] that

$$(1-\frac{1}{n})A_n+\frac{1}{n}H_n-G_n\leq \frac{1-2/n}{x_1^2}\sigma_n$$

and the above inequality fails to hold in the case of $n=2$.

As an evidence for (4.2), we note that the case of $r=n-1$ of the right-hand side inequality of (7.4) in [9] implies that (by using a similar argument as in [6], again we take $q_i=1/n$ here)

$$A_n-\frac{G_n^{\frac{n}{n-1}}}{H_n^{\frac{1}{n-1}}}=A_n-P_{n,n-1}\geq \frac{(n-2)\sigma_n}{2(n-1)x_n}.$$

Now the arithmetic-geometric mean inequality implies that

$$\frac{G_n^{\frac{n}{n-1}}}{H_n^{\frac{1}{n-1}}}\leq \frac{n}{n-1}G_n-\frac{1}{n-1}H_n.$$

We then deduce that

$$(1-\frac{1}{n})A_n+\frac{1}{n}H_n-G_n\geq \frac{1-2/n}{x_n^2}\sigma_n,$$

which is consistent with (4.2) on taking $t\to+\infty$ there.

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