On Ostrowski Type Inequalities for Stieltjes Integrals with Absolutely Continuous Integrands and Integrators of Bounded Variation

ON OSTROWSKI TYPE INEQUALITIES FOR STIELTJES INTEGRALS WITH ABSOLUTELY CONTINUOUS INTEGRANDS AND INTEGRATORS OF BOUNDED VARIATION

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Abstract. Some Ostrowski type inequalities are given for the Stieltjes integral where the integrand is absolutely continuous while the integrator is of bounded variation. The case when $|f'|$ is convex is explored. Applications for the midpoint rule and a generalised trapezoid type rule are also presented.

1. Introduction

The following result is known in the literature as Ostrowski's inequality:

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \leq M$ for all $t \in (a,b)$. Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right](b-a)M \tag{1.1}$$

for all $x \in (a,b)$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The above result has been naturally extended for absolutely continuous functions and Lebesgue $p-$norms of the derivative f' in [11] – [13] and can be stated as:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$ we have:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.2}$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right](b-a)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\[2ex] \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1} \right](b-a)^{\frac{1}{q}}\|f'\|_q \\ \qquad\qquad \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\[2ex] \left[\frac{1}{2} + \left| \frac{x-\frac{a+b}{2}}{b-a} \right| \right]\|f'\|_1, \end{cases}$$

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where $\|\cdot\|_r$ $(r \in [1,\infty])$ *are the usual Lebesgue norms on* $L_r[a,b]$, *i.e.,*

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)| \quad and \quad \|g\|_r := \left(\int_a^b |g(t)|^r dt\right)^{\frac{1}{r}}, \quad r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{1/p}}$ *and* $\frac{1}{2}$ *respectively are sharp in the sense mentioned above.*

They can also be obtained, in a slightly different form, as particular cases of some results established by A.M. Fink in [14] for $n-$time differentiable functions.

For other Ostrowski type inequalities concerning Lipschitzian and $r-H-$Hölder type functions, see [8] and [10].

The cases of bounded variation functions and monotonic functions were considered in [4] and [7] while the case of convex functions was studied in [3].

In an effort to obtain an Ostrowski type inequality for the Stieltjes integral, which obviously contains the weighted integrals case, S.S. Dragomir established in [5] the following result:

Theorem 2. *Let* $f : [a,b] \to \mathbb{R}$ *be a function of bounded variation and* $u : [a,b] \to \mathbb{R}$ *a function of* $r-H-$*Hölder type, i.e.,*

$$|u(x) - u(y)| \le H|x-y|^r \quad \text{for any } x,y \in [a,b], \tag{1.3}$$

where $r \in (0,1]$ *and* $H > 0$ *are given. Then, for any* $x \in [a,b]$,

$$\begin{aligned} &\left|[u(b)-u(x)]f(x) - \int_a^b f(t)\,du(t)\right| \\ &\le H\left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\right] \\ &\le H \times \begin{cases} [(x-a)^r + (b-x)^r]\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right]; \\ [(x-a)^{qr} + (b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x (f)\right)^p + \left(\bigvee_x^b (f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad if \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b (f), \end{cases} \end{aligned} \tag{1.4}$$

where $\bigvee_c^d (f)$ *denotes the total variation of* f *on the interval* $[c,d]$.

The dual case was considered in [6] and can be stated as follows:

Theorem 3. *Let* $u : [a,b] \to \mathbb{R}$ *be a function of bounded variation on* $[a,b]$ *and* $f : [a,b] \to \mathbb{R}$ *a function of* $r-H-$*Hölder type. Then*

$$\begin{aligned} &\left|[u(b)-u(a)]f(x) - \int_a^b f(t)\,du(t)\right| \\ &\qquad\qquad \le H\left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b (u) \end{aligned} \tag{1.5}$$

for any $x \in [a,b]$.

For other results concerning inequalities for Stieltjes integrals, see [1], [15] and [16].

The aim of the present paper is to continue the study of Ostrowski type inequalities for Stieltjes integrals $\int_a^b f(t)\,du(t)$ where the function f, the *integrand*, is assumed to be absolutely continuous while the *integrator* u, is of bounded variation. Applications to the midpoint rule and for a generalised trapezoid rule are also pointed out.

2. General Bounds for Absolutely Continuous Functions

The following representation result is of interest:

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ such that the Stieltjes integrals*

$$\int_a^b f(t)\,du(t) \qquad \text{and} \qquad \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t)$$

exist for each $x \in [a,b]$. Then

$$f(x)[u(b)-u(a)] - \int_a^b f(t)\,du(t) = \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t) \tag{2.1}$$

or, equivalently,

$$\int_a^b u(t)\,df(t) - u(b)[f(b)-f(x)] - u(a)[f(x)-f(a)] = \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t) \tag{2.2}$$

for each $x \in [a,b]$.

Proof. Since f is absolutely continuous on $[a,b]$, hence, for any $x,t \in [a,b]$ with $x \neq t$, one has

$$\frac{f(x)-f(t)}{x-t} = \frac{\int_t^x f'(u)\,du}{x-t} = \int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda$$

giving the equality (see also [9]):

$$f(x) = f(t) + (x-t)\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda \tag{2.3}$$

for any $x,t \in [a,b]$.

Integrating the identity (2.3) we deduce

$$f(x)\int_a^b du(t) = \int_a^b f(t)\,du(t) + \int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda\right) du(t),$$

which is exactly the desired inequality (2.1).

Now, on utilising the integration by parts formula for the Stieltjes integral, we have

$$\begin{aligned} & f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t) \\ &= f(x)[u(b)-u(a)]-\left[f(b)u(b)-f(a)u(a)-\int_a^b u(t)\,df(t)\right] \\ &= \int_a^b u(t)\,df(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)] \end{aligned}$$

and the representation (2.2) is also obtained. □

For an absolutely continuous function $f:[a,b]\to\mathbb{R}$, let us denote by $\mu(f;x,t):=\left|\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right|$, where $(t,x)\in[a,b]^2$. It is obvious that, by the Hölder inequality, we have

$$\mu(f;x,t)\le\begin{cases}\|f'\|_{[t,x],\infty} & \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_{[t,x],p} & \text{if } f'\in L_p[a,b],\ p\ge 1,\end{cases} \tag{2.4}$$

where

$$\|f'\|_{[t,x],\infty}:=\sup_{\substack{u\in[t,x]\\(u\in[x,t])}}|f'(u)|,$$

$$\|f'\|_{[t,x],p}:=\left|\int_t^x |f'(u)|^p\,du\right|^{\frac{1}{p}},\quad p\ge 1$$

and $t,x\in[a,b]$.

We can also state the following result of Ostrowski type for the Stieltjes integral:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function and $u:[a,b]\to\mathbb{R}$ a function of bounded variation on $[a,b]$. Then*

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right|\le M(x), \tag{2.5}$$

and, equivalently

$$\left|\int_a^b u(t)\,df(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)]\right|\le M(x), \tag{2.6}$$

where $M(x)=M_1(x)+M_2(x)$ and

$$M_1(x):=\bigvee_a^x(u)\sup_{t\in[a,x]}[(x-t)\,\mu(f;x,t)],$$

$$M_2(x):=\bigvee_x^b(u)\sup_{t\in[x,b]}[(t-x)\,\mu(f;x,t)],$$

for $x\in[a,b]$.

Remark 1. *Using the notations in Theorem 4, we have*

$$\begin{aligned} M_1(x) &\leq (x-a)\bigvee_a^x(u)\sup_{t\in[a,x]}\mu(f;x,t) \\ &\leq (x-a)\bigvee_a^x(u)\cdot\begin{cases}\|f'\|_{[a,x],\infty} & \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_{[a,x],p} & \text{if } f'\in L_p[a,b],\ p\geq 1,\end{cases}\end{aligned}$$

$$\begin{aligned} M_2(x) &\leq (b-x)\bigvee_x^b(u)\sup_{t\in[x,b]}\mu(f;x,t) \\ &\leq (b-x)\bigvee_x^b(u)\cdot\begin{cases}\|f'\|_{[x,b],\infty} & \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_{[x,b],p} & \text{if } f'\in L_p[a,b],\ p\geq 1,\end{cases}\end{aligned}$$

for any $x\in[a,b]$.

Proof. We use the fact that, if $p,v:[c,d]\to\mathbb{R}$ are such that p is continuous and v is of bounded variation, then the Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and

$$\left|\int_c^d p(x)\,dv(x)\right|\leq\sup_{x\in[c,d]}|p(x)|\bigvee_c^d(v).$$

Utilising the representation (2.1) we have

$$\begin{aligned} &\left|f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right| \\ &=\left|\int_a^x(x-t)\left(\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right)du(t)\right. \\ &\qquad\left.+\int_x^b(x-t)\left(\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right)du(t)\right| \\ &\leq\left|\int_a^x(x-t)\left(\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right)du(t)\right| \\ &\qquad+\left|\int_x^b(x-t)\left(\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right)du(t)\right| \\ &\leq\bigvee_a^x(u)\sup_{t\in[a,x]}[(x-t)\mu(f;x,t)]+\bigvee_x^b(u)\sup_{t\in[x,b]}[(t-x)\mu(f;x,t)] \\ &\leq M_1(x)+M_2(x)=:M(x).\end{aligned}$$

The other inequalities for M_1 and M_2 are obvious from the inequality (2.4) and the details are omitted. □

Remark 2. *Hence, if we denote by* $\|f'\|_{[c,d],p}$ *the* p *norm on the interval* $[c,d]$, *where* $1 \leq p \leq \infty$, *then for* $f' \in L_p[a,b]$, *we have*

$$\left| f(x)[u(b)-u(a)] - \int_a^b f(t)\,du(t) \right| \leq (x-a)\bigvee_a^x(u)\,\|f'\|_{[a,x],p} + (b-x)\bigvee_x^b(u)\,\|f'\|_{[x,b],p} =: N(x), \tag{2.7}$$

where $p \in [1,\infty]$ *and* $x \in [a,b]$.

Obviously one can derive many upper bounds for the function $N(x)$ defined above. We intend to present in the following only a few that are simple and perhaps of interest for applications.

Estimate 1:

$$N(x) \leq \left[(x-a)\bigvee_a^x(u) + (b-x)\bigvee_x^b(u)\right]\|f'\|_{[a,b],p} \tag{2.8}$$

$$\leq \|f'\|_{[a,b],p}\cdot\begin{cases} \max\{x-a,b-x\}\left[\bigvee_a^x(u)+\bigvee_x^b(u)\right]; \\ \left[(x-a)^\alpha+(b-x)^\alpha\right]^{\frac{1}{\alpha}}\left[\left(\bigvee_a^x(u)\right)^\beta+\left(\bigvee_x^b(u)\right)^\beta\right]^{\frac{1}{\beta}} \\ \qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (b-a)\max\left\{\bigvee_a^x(u),\bigvee_x^b(u)\right\} \end{cases}$$

$$= \|f'\|_{[a,b],p}\cdot\begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u); \\ \left[(x-a)^\alpha+(b-x)^\alpha\right]^{\frac{1}{\alpha}}\left[\left(\bigvee_a^x(u)\right)^\beta+\left(\bigvee_x^b(u)\right)^\beta\right]^{\frac{1}{\beta}} \\ \qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (b-a)\left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right] \end{cases}$$

for any $x \in [a,b]$.

Estimate 2:

$$N(x) \leq \max\{x-a,b-x\}\left[\bigvee_a^x(u)\,\|f'\|_{[a,x],p} + \bigvee_x^b(u)\,\|f'\|_{[x,b],p}\right]$$

$$= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\bigvee_a^x(u)\,\|f'\|_{[a,x],p} + \bigvee_x^b(u)\,\|f'\|_{[x,b],p}\right]$$

$$\leq \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]$$

$$\times \begin{cases} \max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\} \bigvee_a^b(u); \\ \left[\|f'\|_{[a,x],p}^p + \|f'\|_{[x,b],p}^p\right]^{\frac{1}{p}} \left[\left(\bigvee_a^x(u)\right)^q + \left(\bigvee_x^b(u)\right)^q\right]^{\frac{1}{q}} \\ \qquad\qquad\qquad \text{if } \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}\right] \end{cases}$$

$$= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]$$

$$\times \begin{cases} \max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\} \bigvee_a^b(u); \\ \|f'\|_{[a,b],p} \left[\left(\bigvee_a^x(u)\right)^q + \left(\bigvee_x^b(u)\right)^q\right]^{\frac{1}{q}} \\ \qquad\qquad\qquad \text{if } \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}\right] \end{cases}$$

for any $x \in [a,b]$.

Estimate 3:

$$N(x) \leq \max\left\{\bigvee_a^x(u), \bigvee_x^b(u)\right\}\left[(x-a)\|f'\|_{[a,x],p}+(b-x)\|f'\|_{[x,b],p}\right]$$

$$= \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\left[(x-a)\|f'\|_{[a,x],p}+(b-x)\|f'\|_{[x,b],p}\right]$$

$$\leq \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]$$

$$\times \begin{cases} \max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\}(b-a); \\ \left[(x-a)^q+(b-x)^q\right]^{\frac{1}{q}}\|f'\|_{[a,b],p} \\ \qquad\qquad\qquad \text{if } \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}\right] \end{cases}$$

for each $x \in [a,b]$.

In practical applications, the midpoint rule, that results for $x = \frac{a+b}{2}$, is of obvious interest due to its simpler form.

Corollary 1. *With the assumptions in Theorem 4, we have the inequalities:*

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \tag{2.9}$$
$$\leq \frac{1}{2}(b-a)\left[\bigvee_a^{\frac{a+b}{2}}(u)\,\|f'\|_{[a,\frac{a+b}{2}],p}+\bigvee_{\frac{a+b}{2}}^{b}(u)\,\|f'\|_{[\frac{a+b}{2},b],p}\right]$$
$$\leq \frac{1}{2}(b-a)\begin{cases}\max\left\{\|f'\|_{[a,\frac{a+b}{2}],p},\|f'\|_{[\frac{a+b}{2},b],p}\right\}\bigvee_a^b(u);\\[2ex] \left[\|f'\|^{\alpha}_{[a,\frac{a+b}{2}],p}+\|f'\|^{\alpha}_{[\frac{a+b}{2},b],p}\right]^{\frac{1}{\alpha}}\\ \qquad\times\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^{\beta}+\left(\bigvee_{\frac{a+b}{2}}^{b}(u)\right)^{\beta}\right]^{\frac{1}{\beta}}\\ \qquad\qquad \textit{if}\ \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^{b}(u)\right|\right]\\ \qquad\times\left[\|f'\|_{[a,\frac{a+b}{2}],p}+\|f'\|_{[\frac{a+b}{2},b],p}\right],\end{cases}$$

where $p\in[1,\infty]$.

From the above, it is obvious that we can get some appealing inequalities as follows:

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \tag{2.10}$$
$$\leq \frac{1}{2}(b-a)\begin{cases}\|f'\|_{[a,b],\infty}\bigvee_a^b(u), \qquad \text{if}\ \ f'\in L_\infty[a,b];\\[2ex] \|f'\|_{[a,b],p}\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^{q}+\left(\bigvee_{\frac{a+b}{2}}^{b}(u)\right)^{q}\right]^{\frac{1}{q}}\\ \qquad \text{if}\ \ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p[a,b];\\[2ex] \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^{b}(u)\right|\right]\|f'\|_{[a,b],1}.\end{cases}$$

Remark 3. *Similar inequalities can be obtained for the generalised trapezoid rule. We only state here the following simple results:*

$$\left|\int_a^b u(t)\,df(t)-u(b)\left[f(b)-f\left(\frac{a+b}{2}\right)\right]-u(a)\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\right|$$
$$\leq \frac{1}{2}(b-a)\begin{cases}\|f'\|_{[a,b],\infty}\bigvee_a^b(u), \qquad \textit{if}\ \ f'\in L_\infty[a,b];\\[2ex] \|f'\|_{[a,b],p}\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^{q}+\left(\bigvee_{\frac{a+b}{2}}^{b}(u)\right)^{q}\right]^{\frac{1}{q}}\\ \qquad \textit{if}\ \ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p[a,b];\\[2ex] \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^{b}(u)\right|\right]\|f'\|_{[a,b],1}\end{cases}$$

provided that u is of bounded variation and f is absolutely continuous on $[a,b]$.

3. Bounds in the Case of $|f'|$ a Convex Function

Some of the above results can be improved provided that a convexity assumption for $|f'|$ is in place:

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$, $u : [a,b] \to \mathbb{R}$ a function of bounded variation on $[a,b]$ and $x \in [a,b]$. If $|f'|$ is convex on $[a,x]$ and $[x,b]$ (and the intervals can be reduced at a single point), then*

$$\begin{aligned} (3.1)\qquad & \left| [u(b)-u(a)] f(x) - \int_a^b f(t)\,du(t) \right| \\ & \leq \frac{1}{2}\left[\bigvee_a^x (u) \sup_{t\in[a,x]} \{(x-t)|f'(t)|\} + \bigvee_x^b (u) \sup_{t\in[x,b]} \{(t-x)|f'(t)|\} \right] \\ & \qquad + \frac{1}{2}|f'(x)| \left[(x-a)\bigvee_a^x (u) + (b-x)\bigvee_x^b (u) \right] \\ & \leq \frac{1}{2}\left[(x-a)\bigvee_a^x (u) \|f'\|_{[a,x],\infty} + (b-x)\bigvee_x^b (u)\|f'\|_{[x,b],\infty} \right] \\ & \qquad + \frac{1}{2}|f'(x)| \left[(x-a)\bigvee_a^x (u) + (b-x)\bigvee_x^b (u) \right], \end{aligned}$$

for any $x \in [a,b]$.

Proof. As in the proof of Theorem 4, we have

$$\begin{aligned} & \left| f(x)[u(b)-u(a)] - \int_a^b f(t)\,du(t) \right| \\ & \leq \sup_{t\in[a,x]} \left[(x-t) \left| \int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda \right| \right] \bigvee_a^x (u) \\ & \quad + \sup_{t\in[x,b]} \left[(t-x) \left| \int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda \right| \right] \bigvee_x^b (u) \\ & \leq \sup_{t\in[a,x]} \left[(x-t) \int_0^1 |f'[\lambda t + (1-\lambda)x]|\,d\lambda \right] \bigvee_a^x (u) \\ & \quad + \sup_{t\in[x,b]} \left[(t-x) \int_0^1 |f'[\lambda t + (1-\lambda)x]|\,d\lambda \right] \bigvee_x^b (u) \\ & \leq \sup_{t\in[a,x]} \left[(x-t) \frac{|f'(t)| + |f'(x)|}{2} \right] \bigvee_a^x (u) \\ & \quad + \sup_{t\in[x,b]} \left[(t-x) \frac{|f'(t)| + |f'(x)|}{2} \right] \bigvee_x^b (u) \end{aligned}$$

$$\leq \frac{1}{2}\left[\sup_{t\in[a,x]}\left\{(x-t)\left|f'(t)\right|\right\}\cdot\bigvee_a^x(u)+\sup_{t\in[x,b]}\left\{(t-x)\left|f'(t)\right|\right\}\cdot\bigvee_x^b(u)\right]$$
$$+\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

which proves the first inequality in (3.1).

The second inequality in (3.1) is obvious using properties of sup and the theorem is completely proved. □

The midpoint inequality is of interest in applications and provides a much simpler inequality:

Corollary 2. *If f and u are as above and $|f'|$ is convex on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$, then*

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \tag{3.2}$$
$$\leq\frac{1}{4}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}\bigvee_a^{\frac{a+b}{2}}(u)+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\bigvee_{\frac{a+b}{2}}^b(u)\right]$$
$$+\frac{1}{4}(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|\bigvee_a^b(u)$$
$$\leq\frac{1}{4}(b-a)\bigvee_a^b(u)\left[\|f'\|_{[a,b],\infty}+\left|f'\left(\frac{a+b}{2}\right)\right|\right].$$

Remark 4. *If we denote, from the second inequality in (3.1),*

$$L_1(x):=\frac{1}{2}\left[(x-a)\|f'\|_{[a,x],\infty}\bigvee_a^x(u)+(b-x)\|f'\|_{[x,b],\infty}\bigvee_x^b(u)\right]$$

and

$$L_2(x):=\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

for $x\in[a,b]$, then we can point out various upper bounds for the functions L_1 and L_2 on $[a,b]$.

For instance, we have

$$L_1(x)\leq\frac{1}{2}\|f'\|_{[a,b],\infty}\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

and by (3.1) we can state the following inequality of interest:

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \tag{3.3}$$
$$\le \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$
$$\le \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u)\\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a)\end{cases}$$

for each $x\in[a,b]$.

Remark 5. *A similar result to (3.3) can be stated for the generalised trapezoid rule, out of which we would like to note the following one that is of particular interest:*

$$\left|\int_a^b u(t)\,df(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)]\right| \tag{3.4}$$
$$\le \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$
$$\le \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u)\\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a)\end{cases}$$

for each $x\in[a,b]$.

As in Corollary 2, the case $x=\frac{a+b}{2}$ in (3.4) provides the simple result

$$\left|\int_a^b u(t)\,df(t)-u(b)\left[f(b)-f\left(\frac{a+b}{2}\right)\right]-u(a)\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\right| \tag{3.5}$$
$$\le \frac{1}{4}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}\bigvee_a^{\frac{a+b}{2}}(u)+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\bigvee_{\frac{a+b}{2}}^b(u)\right]$$
$$+\frac{1}{4}(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|\bigvee_a^b(u)$$
$$\le \frac{1}{4}(b-a)\bigvee_a^b(u)\left[\|f'\|_{[a,b],\infty}+\left|f'\left(\frac{a+b}{2}\right)\right|\right].$$

Remark 6. *Similar inequalities may be stated if one assumes either that* $|f'|$ *is quasi-convex or that* $|f'|$ *is log-convex on* $[a,x]$ *and* $[x,b]$. *The details are left to the interested readers.*

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