A Type of Mean Values of Several Positive Numbers with Two Parameters

A TYPE OF MEAN VALUES OF SEVERAL POSITIVE NUMBERS WITH TWO PARAMETERS

ZHEN-GANG XIAO, ZHI-HUA ZHANG, AND FENG QI

Abstract. In this article, a type of mean values of several positive numbers with two parameters are defined by using the generalized Vandermonde determinant, some basic properties of them are given, and several inequalities for some mean values including the generalized symmetric means are established.

1. Introduction and notations

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $\boldsymbol{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}_+^{n+1}$, φ be a function defined in $\mathbb{R}_+$, and

$$V(\boldsymbol{a};\varphi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \varphi(a_0) \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \varphi(a_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & \varphi(a_n) \end{vmatrix}. \tag{1.1}$$

Taking $\varphi(x) = x^{n+r}(\ln x)^k$, then

$$V(\boldsymbol{a};\varphi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^{n+r}(\ln a_0)^k \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^{n+r}(\ln a_1)^k \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & a_n^{n+r}(\ln a_n)^k \end{vmatrix} \triangleq V(\boldsymbol{a}; r, k). \tag{1.2}$$

If letting $r = 0$ and $k = 0$ in (1.2), then

$$V(\boldsymbol{a}; 0, 0) = \sum_{i=0}^{n} (-1)^{n+i} a_i^n V_i(\boldsymbol{a}) = \prod_{0 \le i < j \le n} (a_j - a_i) \tag{1.3}$$

is the determinant of Vandermonde matrix of order $n+1$, where

$$V_i(\boldsymbol{a}) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{i-1} & a_{i-1}^2 & \cdots & a_{i-1}^{n-1} \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix} \tag{1.4}$$

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is the cofactor of the element a_i^n in $V(\boldsymbol{a};0,0)$. So, we call $V(\boldsymbol{a};\varphi)$ the generalized Vandermonde determinant.

Let a_0 and a_1 be two positive numbers. It is well known that the logarithmic mean $L(a_0,a_1)$ and the identric or exponential mean $I(a_0,a_1)$ of a_0 and a_1 are respectively defined by

$$L(a_0,a_1)=\begin{cases}\dfrac{a_1-a_0}{\ln a_1-\ln a_0}, & a_0\neq a_1,\\ a_0, & a_0=a_1\end{cases} \tag{1.5}$$

and

$$I(a_0,a_1)=\begin{cases}\dfrac{1}{e}\left(\dfrac{a_1^{a_1}}{a_0^{a_0}}\right)^{\frac{1}{a_1-a_0}}, & a_0\neq a_1,\\ a_0, & a_0=a_1.\end{cases} \tag{1.6}$$

Let $G(a_0,a_1)=\sqrt{a_0a_1}$ and $A(a_0,a_1)=\frac{a_0+a_1}{2}$ denote the geometric mean and the arithmetic mean of a_0 and a_1, respectively. For $a_0\neq a_1$, the following inequalities are proved in [4]:

$$G(a_0,a_1)<L(a_0,a_1)<I(a_0,a_1)<A(a_0,a_1). \tag{1.7}$$

In [28, 29], Zh.-H. Zhang and Zh.-G. Xiao studied the identric mean and two logarithmic means of $n+1$ positive numbers $a_0,a_1,\dots,a_n$ which are defined as

$$I(\boldsymbol{a})=\exp\left[\frac{V(\boldsymbol{a};0,1)}{V(\boldsymbol{a};0,0)}-\sum_{k=1}^{n}\frac{1}{k}\right], \tag{1.8}$$

$$L(\boldsymbol{a})=\frac{V(\boldsymbol{a};0,0)}{nV(\boldsymbol{a};-1,1)}, \tag{1.9}$$

$$l(\boldsymbol{a})=\frac{n!V_{\ln}(\boldsymbol{a};1,0)}{V_{\ln}(\boldsymbol{a};0,n)}, \tag{1.10}$$

where

$$V_{\ln}(\boldsymbol{a};r,k)\triangleq\begin{vmatrix}1 & \ln a_0 & (\ln a_0)^2 & \cdots & (\ln a_0)^{n-1} & a_0^r(\ln a_0)^k\\ 1 & \ln a_1 & (\ln a_1)^2 & \cdots & (\ln a_1)^{n-1} & a_1^r(\ln a_1)^k\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & \ln a_n & (\ln a_n)^2 & \cdots & (\ln a_n)^{n-1} & a_n^r(\ln a_n)^k\end{vmatrix}. \tag{1.11}$$

They proved for $\boldsymbol{a}$ with $a_i\neq a_j$ for all $i\neq j$ the following inequalities

$$G(\boldsymbol{a})<L(\boldsymbol{a})<I(\boldsymbol{a})<A(\boldsymbol{a}) \tag{1.12}$$

and

$$G(\boldsymbol{a})<l(\boldsymbol{a})<I(\boldsymbol{a})<A(\boldsymbol{a}), \tag{1.13}$$

where $G(\boldsymbol{a})=\prod_{i=0}^{n}a_i^{1/(n+1)}$ and $A(\boldsymbol{a})=\frac{1}{n+1}\sum_{i=0}^{n}a_i$ are the geometric mean and the arithmetic mean of $n+1$ positive numbers $a_0,a_1,\dots,a_n\in\mathbb{R}_+$, respectively.

For $r\in\mathbb{R}\setminus\{-1,0\}$, the extended logarithmic mean $S_r(a_0,a_1)$ and the generalized logarithmic mean $J_r(a_0,a_1)$ of two positive numbers a_0 and a_1 are defined (See [1, 2, 10, 13, 15, 24]) respectively by

$$S_r(a_0,a_1)=\begin{cases}\left(\dfrac{a_1^{r+1}-a_0^{r+1}}{(r+1)(a_1-a_0)}\right)^{1/r}, & a_0\neq a_1,\\ a_0, & a_0=a_1\end{cases} \tag{1.14}$$

and

$$J_r(a_0,a_1)=\begin{cases}\dfrac{r}{r+1}\cdot\dfrac{a_1^{r+1}-a_0^{r+1}}{a_1^r-a_0^r}, & a_0\ne a_1,\\ a_0, & a_0=a_1.\end{cases} \tag{1.15}$$

These two means and other mean values of two positive numbers are special cases of the extended mean values

$$E(p,q;a_0,a_1)=\begin{cases}\left[\dfrac{q(a_1^p-a_0^p)}{p(a_1^q-a_0^q)}\right]^{1/(p-q)} & \text{for } pq(p-q)(a_0-a_1)\ne0,\\ \left[\dfrac{a_1^p-a_0^p}{p(\ln a_1-\ln a_0)}\right]^{1/p} & \text{for } p(a_0-a_1)\ne0 \text{ and } q=0,\\ \exp\left\{-\dfrac{1}{p}+\dfrac{a_1^p\ln a_1-a_0^p\ln a_0}{a_1^p-a_0^p}\right\} \\ \text{for } p(a_0-a_1)\ne0 \text{ and } p=q,\\ \sqrt{a_0a_1} \quad \text{for } a_0-a_1\ne0 \text{ and } p=q=0,\\ a_0 \quad \text{for } p=q \text{ and } a_0=a_1,\end{cases} \tag{1.16}$$

$$=\begin{cases}\left[\dfrac{h(q;a_0,a_1)}{h(p;a_0,a_1)}\right]^{1/(q-p)}, & (p-q)(a_0-a_1)\ne0,\\ \exp\left[\dfrac{\partial h(p;a_0,a_1)/\partial p}{h(p;a_0,a_1)}\right], & a_0-a_1\ne0,\ p=q,\end{cases} \tag{1.17}$$

where

$$h(t;a_0,a_1)=\begin{cases}\dfrac{a_1^t-a_0^t}{t}, & t\ne0\\ \ln a_1-\ln a_0, & t=0\end{cases}=\int_{a_0}^{a_1}u^{t-1}\,\mathrm{d}u. \tag{1.18}$$

There have been a lot of literature about the extended mean values $E(p,q;a_0,a_1)$. For more information, please refer to [5, 6, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 25] and the references therein.

In [26, 27], Zh.-G. Xiao and Zh.-H. Zhang introduced

$$S_r(\boldsymbol{a})=\begin{cases}\left[\dfrac{n!}{\prod_{k=1}^{n}(k+r)}\cdot\dfrac{V(\boldsymbol{a};r,0)}{V(\boldsymbol{a};0,0)}\right]^{1/r}, & r\ne0,-1,\dots,-n,\\ \exp\left[\dfrac{V(\boldsymbol{a};0,1)}{V(\boldsymbol{a};0,0)}-\sum_{k=1}^{n}\dfrac{1}{k}\right], & r=0,\\ \left[\dfrac{n!V(\boldsymbol{a};r,1)}{(-1)^{r+1}(-r-1)!(n+r)!V(\boldsymbol{a};0,0)}\right]^{1/r}, & r=-1,\dots,-n\end{cases} \tag{1.19}$$

and

$$J_r(\boldsymbol{a})=\begin{cases}\dfrac{r}{n+r}\cdot\dfrac{V(\boldsymbol{a},r;0)}{V(\boldsymbol{a};r-1,0)}, & r\ne0,-1,\dots,-n,\\ \dfrac{V(\boldsymbol{a},0;0)}{nV(\boldsymbol{a},-1,1)}, & r=0,\\ \dfrac{r}{n+r}\cdot\dfrac{V(\boldsymbol{a};r,1)}{V(\boldsymbol{a};r-1,1)}, & r=-1,\dots,-(n-1),\\ \dfrac{-nV(\boldsymbol{a};0,1)}{V(\boldsymbol{a};-n-1,0)}, & r=-n\end{cases} \tag{1.20}$$

with $S_0(\boldsymbol{a})=I(\boldsymbol{a})$ and $S_{-1}(\boldsymbol{a})=J_0(\boldsymbol{a})=L(\boldsymbol{a})$.

The main purpose of this article is to define a type of mean values of several positive numbers with two parameters, from which many mean values mentioned above can be deduced, by utilizing the generalized Vandermonde determinant, to research their basic properties, and to establish several inequalities.

2. Lemmas

The following two lemmas are keys for us to research the basic properties of the mean values $E(p,q;\boldsymbol{a})$ defined by Definition 3.1 in next section.

Lemma 2.1. *Let $n\in\mathbb{N}$, $\boldsymbol{a}=(a_0,a_1,\dots,a_n)\in\mathbb{R}_+^{n+1}$ and φ be a n-times differentiable function in $\mathbb{R}_+$. Then*

$$V(\boldsymbol{a};\varphi)=V(\boldsymbol{a};0,0)\int_D\varphi^{(n)}\left(\sum_{i=0}^n a_ix_i\right)\mathrm{d}x, \tag{2.1}$$

where $x_0=1-\sum_{i=1}^n x_i$ and $\mathrm{d}x=\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$ denotes the differential of the volume in

$$D=\left\{(x_1,x_2,\dots,x_n)\,\middle|\,\sum_{i=1}^n x_i\le 1 \text{ with } x_i\ge 0 \text{ for } i=1,2,\dots,n\right\}. \tag{2.2}$$

Proof. Let $\bar{\boldsymbol{a}}=(a_n,a_1,\dots,a_{n-1})\in\mathbb{R}_+^n$ and $\tilde{\boldsymbol{a}}=(a_0,a_1,\dots,a_{n-1})\in\mathbb{R}_+^n$. From (1.3), one finds that

$$\begin{aligned}V(\boldsymbol{a};0,0)&=(-1)^{n-1}\prod_{j=1}^n(a_j-a_0)V(\bar{\boldsymbol{a}};0,0)\\&=(a_n-a_0)\prod_{j=1}^{n-1}(a_n-a_j)V(\tilde{\boldsymbol{a}};0,0),\end{aligned} \tag{2.3}$$

$$V_0(\boldsymbol{a})=\prod_{j=1}^{n-1}(a_n-a_j)V_0(\tilde{\boldsymbol{a}}), \tag{2.4}$$

$$V_i(\boldsymbol{a})=(-1)^n\prod_{j=1}^{n-1}(a_j-a_0)V_i(\bar{\boldsymbol{a}})+\prod_{j=1}^{n-1}(a_n-a_j)V_i(\tilde{\boldsymbol{a}}),\quad 1<i<n, \tag{2.5}$$

$$V_n(\boldsymbol{a})=\prod_{j=1}^{n-1}(a_j-a_0)V_0(\bar{\boldsymbol{a}}). \tag{2.6}$$

The expansion of the generalized Vandermonde determinant (1.1) equals

$$V(\boldsymbol{a};\varphi)=\sum_{i=0}^n(-1)^{n+i}\varphi(a_i)V_i(\boldsymbol{a}). \tag{2.7}$$

It is easy to see that

$$\begin{aligned}&\int_D\varphi^{(n)}\left(\sum_{i=0}^n a_ix_i\right)\mathrm{d}x\\&=\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-\sum_{i=1}^{n-1}x_i}\varphi^{(n)}\left(a_0+\sum_{i=1}^n(a_i-a_0)x_i\right)\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n.\end{aligned} \tag{2.8}$$

For $n=1$, formula (2.1) is true, since

$$(a_1-a_0)\int_0^1 \varphi'(a_0+(a_1-a_0)x_1)\,\mathrm{d}x_1=\varphi(a_1)-\varphi(a_0)=V(\boldsymbol{a};\varphi).$$

Assume that (2.1) holds for $n-1$ with $n\ge 2$. Then

$$\begin{aligned}
V(\bar{\boldsymbol{a}};\varphi)&=(-1)^{n-1}\varphi(a_n)V_0(\bar{\boldsymbol{a}})+\sum_{i=1}^{n-1}(-1)^{n-1+i}\varphi(a_i)V_i(\bar{\boldsymbol{a}})\\
&=V(\bar{\boldsymbol{a}};0,0)\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-\sum_{i=1}^{n-2}x_i}\\
&\qquad\varphi^{(n-1)}\left(a_n+\sum_{i=1}^{n-1}(a_i-a_n)x_i\right)\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{n-1} \qquad (2.9)
\end{aligned}$$

and

$$\begin{aligned}
V(\tilde{\boldsymbol{a}};\varphi)&=\sum_{i=0}^{n-1}(-1)^{n-1+i}\varphi(a_i)V_i(\tilde{\boldsymbol{a}})\\
&=V(\tilde{\boldsymbol{a}};0,0)\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-\sum_{i=1}^{n-2}x_i}\\
&\qquad\varphi^{(n-1)}\left(a_0+\sum_{i=1}^{n-1}(a_i-a_0)x_i\right)\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{n-1}. \qquad (2.10)
\end{aligned}$$

Combination of (2.3) to (2.10) shows that

$$\begin{aligned}
&V(\boldsymbol{a};0,0)\int_D\varphi^{(n)}\left(\sum_{i=0}^{n}a_ix_i\right)\mathrm{d}x\\
&=V(\boldsymbol{a};0,0)\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-\sum_{i=1}^{n-1}x_i}\\
&\quad\varphi^{(n)}\left(a_0+\sum_{i=1}^{n}(a_i-a_0)x_i\right)\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n\\
&=\frac{V(\boldsymbol{a};0,0)}{a_n-a_0}\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-\sum_{i=1}^{n-2}x_i}\left[\varphi^{(n-1)}\left(a_n+\sum_{i=1}^{n-1}(a_i-a_n)x_i\right)\right.\\
&\quad\left.-\varphi^{(n-1)}\left(a_0+\sum_{i=1}^{n-1}(a_i-a_0)x_i\right)\right]\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{n-1}\\
&=(-1)^{n-1}\prod_{j=1}^{n-1}(a_j-a_0)V(\bar{\boldsymbol{a}};0,0)\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-\sum_{i=1}^{n-2}x_i}\\
&\quad\varphi^{(n-1)}\left(a_n+\sum_{i=1}^{n-1}(a_i-a_n)x_i\right)\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\\
&\quad-\prod_{j=1}^{n-1}(a_n-a_j)V(\tilde{\boldsymbol{a}};0,0)\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-\sum_{i=1}^{n-2}x_i}
\end{aligned}$$

$$\varphi^{(n-1)}\left(a_0+\sum_{i=1}^{n-1}(a_i-a_0)x_i\right)\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}$$
$$=(-1)^{n-1}\prod_{j=1}^{n-1}(a_j-a_0)\left[(-1)^{n-1}\varphi(a_n)V_0(\bar{\boldsymbol{a}})+\sum_{i=1}^{n-1}(-1)^{n-1+i}\varphi(a_i)V_i(\bar{\boldsymbol{a}})\right]$$
$$-\prod_{j=1}^{n-1}(a_n-a_j)\sum_{i=1}^{n-1}(-1)^{n-1+i}\varphi(a_i)V_i(\tilde{\boldsymbol{a}})$$
$$=\varphi(a_n)\prod_{j=1}^{n-1}(a_j-a_0)V_0(\bar{\boldsymbol{a}})-(-1)^{n-1}\varphi(a_0)\prod_{j=1}^{n-1}(a_n-a_j)V_0(\tilde{\boldsymbol{a}})$$
$$+\sum_{i=1}^{n-1}(-1)^{n-1+i}\varphi(a_i)\left[(-1)^{n-1}\prod_{j=1}^{n-1}(a_j-a_0)V_i(\bar{\boldsymbol{a}})-\prod_{j=1}^{n-1}(a_n-a_j)V_i(\tilde{\boldsymbol{a}})\right]$$
$$=\varphi(a_n)V_n(\boldsymbol{a})+\sum_{i=1}^{n-1}(-1)^{n+i}\varphi(a_i)V_i(\boldsymbol{a})+(-1)^n\varphi(a_0)V_0(\boldsymbol{a})$$
$$=V(\boldsymbol{a};\varphi).$$

Thus, by induction, the required formula (2.1) is proved. □

Lemma 2.2. *Let r be an integer and $\boldsymbol{a}=(a_0,a_1,\dots,a_n)\in\mathbb{R}_+^{n+1}$. Then*

$$V(\boldsymbol{a};r,0)=\begin{cases}V(\boldsymbol{a};0,0)\displaystyle\sum_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\dots,i_n\ge 0 \text{ are integers}}}\prod_{k=0}^{n}a_k^{i_k} & \text{for } r>0,\\ 0 \quad \text{for } r=0,-1,\dots,-(n-1), & \\ (-1)^nV(\boldsymbol{a};0,0)\displaystyle\sum_{\substack{i_0+i_1+\cdots+i_n=-r\\ i_0,i_1,\dots,i_n\in\mathbb{N}}}\prod_{k=0}^{n}a_k^{-i_k} & \text{for } r\le -n.\end{cases}\tag{2.11}$$

Proof. Taking $V_n(\boldsymbol{a};r,0)\triangleq V(\boldsymbol{a};r,0)$. It is obvious that, if $r=-1,\dots,-n$, then $V_n(\boldsymbol{a};r,0)=V(\boldsymbol{a};r,0)=0$.

In the case of $n\in\mathbb{N}$ and $r\ge 0$, we will verify Lemma 2.2 by mathematical induction. It is clear that identity (2.11) holds trivially for $n=1$, since

$$\begin{aligned}V_2(\boldsymbol{a};r,0)&=a_1^{r+1}-a_0^{r+1}\\&=(a_1-a_0)\left(a_1^r+a_1^{r-1}a_0+\cdots+a_0^r\right)\\&=(a_1-a_0)\sum_{\substack{i_0+i_1=r\\ i_0,i_1\ge 0\text{ are integers}}}a_0^{i_0}a_1^{i_1}.\end{aligned}$$

Suppose identity (2.11) is true for $n-1$ and positive integers t. That is

$$V_{n-1}(\boldsymbol{a};t,0)=V_{n-1}(\boldsymbol{a};0,0)\sum_{\substack{i_0+i_1+\cdots+i_{n-1}=t\\ i_0,i_1,\dots,i_{n-1}\ge 0\text{ are integers}}}\prod_{k=0}^{n-1}a_k^{i_k}.\tag{2.12}$$

By (1.2) and (2.12), one reveals

$$
\begin{aligned}
V_n(\boldsymbol{a};r,0) &= \begin{vmatrix} 1 & a_0-a_n & \cdots & a_0^{n-1}-a_0^{n-2}a_n & a_0^{n+r}-a_0^{n-1}a_n^{r+1} \\ 1 & a_1-a_n & \cdots & a_1^{n-1}-a_1^{n-2}a_n & a_1^{n+r}-a_1^{n-1}a_n^{r+1} \\ \hdotsfor{5} \\ 1 & a_{n-1}-a_n & \cdots & a_{n-1}^{n-1}-a_{n-1}^{n-2}a_n & a_{n-1}^{n+r}-a_{n-1}^{n-1}a_n^{r+1} \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&= (-1)^{n+2} \begin{vmatrix} a_0-a_n & \cdots & a_0^{n-1}-a_0^{n-2}a_n & a_0^{n+r}-a_0^{n-1}a_n^{r+1} \\ a_1-a_n & \cdots & a_1^{n-1}-a_1^{n-2}a_n & a_1^{n+r}-a_1^{n-1}a_n^{r+1} \\ \hdotsfor{4} \\ a_{n-1}-a_n & \cdots & a_{n-1}^{n-1}-a_{n-1}^{n-2}a_n & a_{n-1}^{n+r}-a_{n-1}^{n-1}a_n^{r+1} \end{vmatrix} \\
&= \prod_{i=0}^{n-1}(a_n-a_i) \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \sum_{t+i_n=r} a_0^{n-1+t}a_n^{i_n} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \sum_{t+i_n=r} a_1^{n-1+t}a_n^{i_n} \\ \hdotsfor{6} \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} & \sum_{t+i_n=r} a_{n-1}^{n-1+t}a_n^{i_n} \end{vmatrix} \\
&= \prod_{i=0}^{n-1}(a_n-a_i) \sum_{t+i_n=r} a_n^{i_n} \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^{n-1+t} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^{n-1+t} \\ \hdotsfor{6} \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} & a_{n-1}^{n-1+t} \end{vmatrix} \\
&= \prod_{i=0}^{n-1}(a_n-a_i) \sum_{t+i_n=r} a_n^{i_n} V_{n-1}(\boldsymbol{a};t,0) \\
&= \prod_{i=0}^{n-1}(a_n-a_i)\cdot V_{n-1}(\boldsymbol{a};0,0) \sum_{t+i_n=r} a_n^{i_n} \sum_{\substack{i_0+i_1+\cdots+i_{n-1}=t,\\ i_0,i_1,\dots,i_{n-1}\ge 0 \text{ are integers}}} \prod_{k=0}^{n-1} a_k^{i_k} \\
&= V(\boldsymbol{a};0,0) \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\dots,i_n\ge 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{i_k}.
\end{aligned}
$$

By induction, it is showed that (2.11) holds for n and positive integers r.

From (1.3), it is deduced easily that

$$
\begin{aligned}
V(\boldsymbol{a}^{-1};0,0) &= \sum_{i=0}^{n}(-1)^{n+i}a_i^{-n}V_i(\boldsymbol{a}^{-1}) \\
&= \prod_{0\le i<j\le n}(a_j^{-1}-a_i^{-1}) \\
&= \prod_{0\le i\le n} a_i^{-n} \prod_{0\le i<j\le n}(a_i-a_j) \\
&= (-1)^{n(n+1)/2}\prod_{0\le i\le n} a_i^{-n} V(\boldsymbol{a};0,0),
\end{aligned}
\tag{2.13}
$$

where $\boldsymbol{a}^{-1}=(a_0^{-1},a_1^{-1},\dots,a_n^{-1})\in\mathbb{R}_+^{n+1}$.

In the case of $r < -n$, we have $-(n+r) > 0$. From (1.2) and (2.13), one finds

$$
\begin{aligned}
V(\boldsymbol{a}; r, 0) &= \prod_{0\le i\le n} a_i^{n-1} \begin{vmatrix} a_0^{-(n-1)} & a_0^{-(n-1)} & \cdots & 1 & a_0^{r+1} \\ a_1^{-(n-1)} & a_1^{-(n-2)} & \cdots & 1 & a_1^{r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1}^{-(n-1)} & a_{n-1}^{-(n-2)} & \cdots & 1 & a_{n-1}^{r+1} \end{vmatrix} \\
&= (-1)^{n(n-1)/2} \prod_{0\le i\le n} a_i^{n-1} \begin{vmatrix} 1 & a_0^{-1} & \cdots & a_0^{-(n-1)} & a_0^{-[n-(n+r+1)]} \\ 1 & a_1^{-1} & \cdots & a_1^{-(n-1)} & a_1^{-[n-(n+r+1)]} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1}^{-1} & \cdots & a_{n-1}^{-(n-1)} & a_{n-1}^{-[n-(n+r+1)]} \end{vmatrix} \\
&= (-1)^{n(n-1)/2} \prod_{0\le i\le n} a_i^{n-1} V(\boldsymbol{a}^{-1}; 0, 0) \sum_{\substack{i_0+i_1+\cdots+i_n=-(n+r+1) \\ i_0, i_1, \dots, i_n \ge 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{-i_k} \\
&= (-1)^n \prod_{0\le i\le n} a_i^{-1} V(\boldsymbol{a}; 0, 0) \sum_{\substack{i_0+i_1+\cdots+i_n=-(n+r+1) \\ i_0, i_1, \dots, i_n \ge 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{-i_k} \\
&= (-1)^n V(\boldsymbol{a}; 0, 0) \sum_{\substack{i_0+i_1+\cdots+i_n=-r \\ i_0, i_1, \dots, i_n \ge 1 \text{ are integers}}} \prod_{k=0}^{n} a_k^{-i_k}.
\end{aligned}
$$

The proof of Lemma 2.2 is completed. □

Lemma 2.3 ([19, 21, 23]). *Let f be a continuous function and p a positive continuous weight on I. Then the weighted arithmetic mean of function f with weight p defined by*

$$
F(x, y) = \begin{cases} \dfrac{\int_x^y p(t) f(t) \, \mathrm{d}t}{\int_x^y p(t) \, \mathrm{d}t}, & x \ne y, \\ f(x), & x = y \end{cases} \tag{2.14}
$$

is increasing (decreasing) on I^2 if f is increasing (decreasing) on I.

3. Definition and basic properties

Now we are in a position to give the definition of mean values of several positive numbers with two parameters.

Definition 3.1. Let $\boldsymbol{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}_+^{n+1}$ with $a_i \ne a_j$ for $i \ne j$. The extended mean values $E(p, q; \boldsymbol{a})$ of $(n+1)$-tuple $\boldsymbol{a}$ with two parameters $p \in \mathbb{R}$ and

$q \in \mathbb{R}$ are defined as

$$E(p,q;\boldsymbol{a}) = \begin{cases} \left[\prod_{k=1}^{n} \dfrac{k+q}{k+p} \cdot \dfrac{V(\boldsymbol{a};p,0)}{V(\boldsymbol{a};q,0)}\right]^{1/(p-q)} \\ \text{for } (p-q)\prod_{k=1}^{n}[(k+p)(k+q)] \neq 0, \\ \left[\dfrac{(-1)^{q+1}(-q-1)!(q+n)!}{\prod_{k=1}^{n}(k+p)} \cdot \dfrac{V(\boldsymbol{a};p,0)}{V(\boldsymbol{a};q,1)}\right]^{1/(p-q)} \\ \text{for } p \neq q = -1,-2,\dots,-n, \\ \left[\dfrac{(-1)^{q-p}(-q-1)!(q+n)!}{(-p-1)!(p+n)!} \cdot \dfrac{V(\boldsymbol{a};p,1)}{V(\boldsymbol{a};q,1)}\right]^{1/(p-q)} \\ \text{for } p \neq q \text{ and } p,q = -1,-2,\dots,-n, \\ \exp\left[\dfrac{V(\boldsymbol{a};p,1)}{V(\boldsymbol{a};p,0)} - \sum_{k=1}^{n}\dfrac{1}{k+p}\right] \\ \text{for } p = q \neq -1,-2,\dots,-n, \\ \exp\left[\dfrac{V(\boldsymbol{a};p,2)}{2V(\boldsymbol{a};p,1)} - \sum_{\substack{k=1\\k\neq -p}}^{n}\dfrac{1}{k+p}\right] \\ \text{for } p = q = -1,-2,\dots,-n. \end{cases} \tag{3.1}$$

The mean values $E(p,q;\boldsymbol{a})$ have the same usual properties as many other means listed in [3].

Theorem 3.1. *The following basic properties hold:*

1. $E(p,q;\boldsymbol{a}) = E(q,p;\boldsymbol{a})$,
2. $\lim_{p\to\infty} E(p,q;\boldsymbol{a}) = \lim_{q\to\infty} E(p,q;\boldsymbol{a}) = \max\{a_0,a_1,\dots,a_n\}$,
3. $\lim_{p\to-\infty} E(p,q;\boldsymbol{a}) = \lim_{q\to-\infty} E(p,q;\boldsymbol{a}) = \min\{a_0,a_1,\dots,a_n\}$,
4. $\lim_{p\to q} E(p,q;\boldsymbol{a}) = E(q,q;\boldsymbol{a})$,
5. $\min\{a_0,a_1,\dots,a_n\} \le E(p,q;\boldsymbol{a}) \le \max\{a_0,a_1,\dots,a_n\}$,
6. $E(p,q;\boldsymbol{a}) = a_0$ *if and only if* $a_0 = a_1 = \dots = a_n$,
7. *For* $t>0$, $E(p,q;t\boldsymbol{a}) = tE(p,q;\boldsymbol{a})$,
8. $[E(p,q;\boldsymbol{a})]^{p-q} = [E(p,t;\boldsymbol{a})]^{p-t}[E(t,q;\boldsymbol{a})]^{t-q}$,

where $t\boldsymbol{a} = (ta_0,ta_1,\dots,ta_n)$.

Proof. These follow straightforwardly from Definition 3.1 and Lemma 2.1 by standard arguments. □

The theorem below gives an alternative expression of the mean values $E(p,q;\boldsymbol{a})$.

Theorem 3.2. *Let* $n\in\mathbb{N}$, $\boldsymbol{a} = (a_0,a_1,\dots,a_n)\in\mathbb{R}_+^{n+1}$, *and* $p,q\in\mathbb{R}$. *Then*

$$E(p,q;\boldsymbol{a}) = \begin{cases} \left[\dfrac{\int_D \varphi_1^{(n)}\left(p,\sum_{i=0}^{n}a_ix_i\right)\mathrm{d}x}{\int_D \varphi_1^{(n)}\left(q,\sum_{i=0}^{n}a_ix_i\right)\mathrm{d}x}\right]^{1/(p-q)}, & p\neq q, \\ \exp\left\{\dfrac{\int_D \varphi_2^{(n)}\left(p,\sum_{i=0}^{n}a_ix_i\right)\mathrm{d}x}{\int_D \varphi_1^{(n)}\left(p,\sum_{i=0}^{n}a_ix_i\right)\mathrm{d}x}\right\}, & p=q, \end{cases} \tag{3.2}$$

where D *is the simplex defined by* (2.2), $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$ *is the differential of the volume in* D, $x_0 = 1-\sum_{i=1}^{n}x_i$, $\varphi_1^{(n)}(p,t) = t^p$ *and* $\varphi_2^{(n)}(p,t) = t^p\ln t$.

Proof. This follows from substituting

$$\varphi_1(p,t)=\begin{cases} t^{n+p}\prod\limits_{k=1}^{n}\dfrac{1}{k+p} & \text{for } p\neq -1,\dots,-n,\\ \dfrac{t^{n+p}\ln t}{(-1)^{p+1}(-p-1)!(n+p)!} & \text{for } p=-1,\dots,-n \end{cases} \tag{3.3}$$

and

$$\varphi_2(p,t)=\begin{cases} t^{n+p}\left(\ln t-\sum\limits_{k=1}^{n}\dfrac{1}{k+p}\right)\prod\limits_{k=1}^{n}\dfrac{1}{k+p} \quad \text{for } p\neq -1,\dots,-n,\\ \dfrac{t^{n+p}\ln t}{(-1)^{p+1}(-p-1)!(n+p)!}\left(\dfrac{\ln t}{2}-\sum\limits_{\substack{k=1\\k\neq -p}}^{n}\dfrac{1}{k+p}\right)\\ \text{for } p=-1,\dots,-n \end{cases} \tag{3.4}$$

into Lemma 2.1. □

Corollary 3.1. *Let $n\in\mathbb{N}$ and $\boldsymbol{a}=(a_0,a_1,\dots,a_n)\in\mathbb{R}_+^{n+1}$. Then*

$$S_r(\boldsymbol{a})=E(r,0;\boldsymbol{a})=\begin{cases}\left[n!\int_D\varphi_1^{(n)}\left(r,\sum_{i=0}^n a_ix_i\right)\mathrm{d}x\right]^{1/r}, & r\neq 0,\\ \exp\left\{n!\int_D\varphi_2^{(n)}\left(0,\sum_{i=0}^n a_ix_i\right)\mathrm{d}x\right\}, & r=0,\end{cases} \tag{3.5}$$

$$J_r(\boldsymbol{a})=E(r,r-1;\boldsymbol{a})=\frac{\int_D\varphi_1^{(n)}\left(r,\sum_{i=0}^n a_ix_i\right)\mathrm{d}x}{\int_D\varphi_1^{(n)}\left(r-1,\sum_{i=0}^n a_ix_i\right)\mathrm{d}x}, \tag{3.6}$$

$$E(p,q;\boldsymbol{a})=\left\{\frac{[S_p(\boldsymbol{a})]^p}{[S_q(\boldsymbol{a})]^q}\right\}^{1/(p-q)}, \tag{3.7}$$

where D is defined by (2.2), $\mathrm{d}x=\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$ *denotes the differential of the volume in D, and $x_0=1-\sum_{i=1}^n x_i$.*

Some mean values are special cases of the mean values $E(p,q;\boldsymbol{a})$ as Corollary 3.1 reveals. From Definition 3.1 and Lemma 2.2, we further obtain the following

Theorem 3.3. *Let $n\in\mathbb{N}$ and $\boldsymbol{a}=(a_0,a_1,\dots,a_n)\in\mathbb{R}_+^{n+1}$. Then*

1. $E(1,0;\boldsymbol{a})=S_1(\boldsymbol{a})=J_1(\boldsymbol{a})=A(\boldsymbol{a})$,
2. $E(0,-(n+1);\boldsymbol{a})=S_{-(n+1)}(\boldsymbol{a})=G(\boldsymbol{a})$,
3. $E(-(n+1),-(n+2);\boldsymbol{a})=J_{-(n+1)}(\boldsymbol{a})=H(\boldsymbol{a})$,
4. $E(0,0;\boldsymbol{a})=S_0(\boldsymbol{a})=I(\boldsymbol{a})$,
5. $E(0,-1;\boldsymbol{a})=S_{-1}(\boldsymbol{a})=J_0(\boldsymbol{a})=L(\boldsymbol{a})$,

where $A(\boldsymbol{a})$, $G(\boldsymbol{a})$, $H(\boldsymbol{a})$, $I(\boldsymbol{a})$, and $L(\boldsymbol{a})$ denote respectively the arithmetic, geometric, harmonic, exponential, and logarithmic means of $n+1$ positive numbers $a_0,a_1,\dots,a_n$.

Proof. The proof is straightforward. □

Some special cases of $E(p,q;\boldsymbol{a})$ have relationships with the generalized symmetric means $\sum\limits_{n}^{k}(\boldsymbol{a})$ of order k introduced in [7] by D. M. DeTemple and J. M. Robertson.

Theorem 3.4. *Let r be a positive integer and $\boldsymbol{a}=(a_0,a_1,\dots,a_n)\in\mathbb{R}_+^{n+1}$ for $n\in\mathbb{N}$. Then*

$$E(r,0;\boldsymbol{a})=S_r(\boldsymbol{a})=\sum_{n+1}^{r}(\boldsymbol{a}), \tag{3.8}$$

$$E(r,r-1;\boldsymbol{a}) = J_r(\boldsymbol{a}) = \frac{\sum_{n+1}^{r}(\boldsymbol{a})}{\sum_{n+1}^{r-1}(\boldsymbol{a})}, \tag{3.9}$$

where

$$\sum_{n+1}^{k}(\boldsymbol{a}) = \binom{n+k}{n}^{-1} \sum_{\substack{i_0+i_1+\cdots+i_n=k\\ i_0,i_1,\dots,i_n\ge 0}} a_0^{i_0}a_1^{i_1}\cdots a_n^{i_n} \tag{3.10}$$

denotes the k-th generalized symmetric mean of $n+1$ positive numbers $a_0, a_1, \dots, a_n$.

4. Monotonicity and applications

Theorem 4.1. *The mean values $E(p,q;\boldsymbol{a})$ are increasing strictly with both $p\in\mathbb{R}$ and $q\in\mathbb{R}$.*

Proof. Let $g(x)=\sum_{i=0}^{n} a_i x_i$. If $p=q$, then

$$E(p,p;\boldsymbol{a}) = \exp\left\{\frac{\int_D \varphi_2^{(n)}\big(p,g(x)\big)\,\mathrm{d}x}{\int_D \varphi_1^{(n)}\big(p,g(x)\big)\,\mathrm{d}x}\right\}$$

and

$$\begin{aligned}
&\left[\int_D \varphi_1^{(n)}\big(p,g(x)\big)\,\mathrm{d}x\right]^2 \frac{\mathrm{d}}{\mathrm{d}p}[\ln E(p,p;\boldsymbol{a})]\\
&= \int_D \varphi_2^{(n)}\big(p,g(x)\big)\varphi_2^{(n)}\big(0,g(x)\big)\,\mathrm{d}x \int_D \varphi_1^{(n)}\big(p,g(x)\big)\,\mathrm{d}x\\
&\quad - \left[\int_D \varphi_1^{(n)}\big(p,g(x)\big)\varphi_2^{(n)}\big(0,g(x)\big)\,\mathrm{d}x\right]^2\\
&= \int_D [g(x)]^p[\ln g(x)]^2\,\mathrm{d}x \int_D [g(x)]^p\,\mathrm{d}x - \left[\int_D [g(x)]^p \ln[g(x)]\,\mathrm{d}x\right]^2\\
&> 0
\end{aligned}$$

by using the Cauchy-Schwartz-Buniakowski integral inequality. This implies that the mean values $E(p,p;\boldsymbol{a})$ are strictly increasing with $p\in\mathbb{R}$.

If $p\ne q$, then

$$E(p,q;\boldsymbol{a}) = \left[\frac{\int_D \varphi_1^{(n)}\big(p,g(x)\big)\,\mathrm{d}x}{\int_D \varphi_1^{(n)}\big(q,g(x)\big)\,\mathrm{d}x}\right]^{1/(p-q)}$$

and

$$\begin{aligned}
\ln E(p,q;\boldsymbol{a}) &= \frac{1}{p-q}\int_q^p \frac{\int_D \varphi_2^{(n)}\big(\theta,g(x)\big)\,\mathrm{d}x}{\int_D \varphi_1^{(n)}\big(\theta,g(x)\big)\,\mathrm{d}x}\,\mathrm{d}\theta\\
&= \frac{1}{p-q}\int_q^p \ln E(\theta,\theta;\boldsymbol{a})\,\mathrm{d}\theta.
\end{aligned}$$

From Lemma 2.3 and the fact that $E(p,p;\boldsymbol{a})$ are strictly increasing with $p\in\mathbb{R}$, it is not difficult to see that the mean values $E(p,q;\boldsymbol{a})$ are increasing strictly with both p and q. The proof is complete. □

As applications of the monotonicity of the mean values $E(p,q;\boldsymbol{a})$, the following inequalities about some mean values are obtained readily.

Corollary 4.1. *If $p_1<p_2$ and $q_1<q_2$, then*

$$E_{p_1,q_1}(\boldsymbol{a})<E_{p_2,q_2}(\boldsymbol{a}). \tag{4.1}$$

If $q\gtreqless 0$, then

$$S_q(\boldsymbol{a})\lesseqgtr E(p,q;\boldsymbol{a}). \tag{4.2}$$

If $q\gtreqless p-1$, then

$$J_p(\boldsymbol{a})\lesseqgtr E(p,q;\boldsymbol{a}). \tag{4.3}$$

If $p\lesseqgtr q$, then

$$J_p(\boldsymbol{a})\lesseqgtr J_q(\boldsymbol{a}) \tag{4.4}$$

and

$$S_p(\boldsymbol{a})\lesseqgtr S_q(\boldsymbol{a}). \tag{4.5}$$

If $p\gtreqless 1$, then

$$S_p(\boldsymbol{a})\lesseqgtr J_p(\boldsymbol{a}). \tag{4.6}$$

Corollary 4.2. *Let $G(\boldsymbol{a})$, $L(\boldsymbol{a})$, $I(\boldsymbol{a})$ and $A(\boldsymbol{a})$ denote the geometric, logarithmic, exponential and arithmetic means of $n+1$ positive numbers respectively. Then*

$$G(\boldsymbol{a})<L(\boldsymbol{a})<I(\boldsymbol{a})<A(\boldsymbol{a}). \tag{4.7}$$

Proof. From (4.5) in Corollary 4.1, it follows that

$$S_{-(n+1)}(\boldsymbol{a})<S_{-1}(\boldsymbol{a})<S_0(\boldsymbol{a})<S_1(\boldsymbol{a}).$$

Thus, inequalities in (4.7) is obtained immediately from Theorem 3.3. ☐

Corollary 4.3. *Let $r\in\mathbb{N}$, $n\in\mathbb{N}$, and $\boldsymbol{a}=(a_0,a_1,\dots,a_n)\in\mathbb{R}_+^{n+1}$. Then*

$$\left[\sum_{n+1}^{[r]}(\boldsymbol{a})\right]^2\le\sum_{n+1}^{[r+1]}(\boldsymbol{a})\sum_{n+1}^{[r-1]}(\boldsymbol{a}) \tag{4.8}$$

and

$$\left[\sum_{n+1}^{[r]}(\boldsymbol{a})\right]^{\frac{1}{r}}\le\left[\sum_{n+1}^{[r+1]}(\boldsymbol{a})\right]^{\frac{1}{r+1}}. \tag{4.9}$$

Equalities in (4.8) and (4.9) are valid if and only if $a_0=a_1=\dots=a_n$.

Proof. By using Corollary 4.1, we find

$$J_r(\boldsymbol{a})\le J_{r+1}(\boldsymbol{a}),$$
$$S_r(\boldsymbol{a})\le S_{r+1}(\boldsymbol{a}).$$

Further considering Theorem 3.4 proves the required results. ☐

Corollary 4.4. *If $r\gtreqless 1$, then*

$$S_r(\boldsymbol{a})\lesseqgtr M_r(\boldsymbol{a}), \tag{4.10}$$

where

$$M_r(\boldsymbol{a})=\begin{cases}\left[\dfrac{1}{n+1}\sum\limits_{i=0}^{n}a_i^r\right]^{1/r}, & r\ne 0,\\ G(\boldsymbol{a}), & r=0.\end{cases} \tag{4.11}$$

Proof. Let $g(x) = \sum_{i=0}^{n} a_i x_i$. In the case of $r(r-1) > 0$, using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}\varphi_1^{(n)}\big(r, g(x)\big) &= \left[a_0 + \sum_{i=1}^{n}(a_i - a_0)x_i\right]^r \\ &= \left[\left(1 - \sum_{i=1}^{n} x_i\right)a_0 + \sum_{i=1}^{n} x_i a_i\right]^r \\ &< \left(1 - \sum_{i=1}^{n} x_i\right)a_0^r + \sum_{i=1}^{n} x_i a_i^r \\ &= a_0^r + \sum_{i=1}^{n}(a_i^r - a_0^r)x_i,\end{aligned}$$

which is equivalent to

$$\begin{aligned}S_r^r(\boldsymbol{a}) &= n! \int_D \varphi_1^{(n)}\big(r, g(x)\big)\,\mathrm{d}x \\ &< n! \int_D \left(a_0^r + \sum_{i=1}^{n}(a_i^r - a_0^r)x_i\right)\mathrm{d}x \\ &= \frac{1}{n+1}\sum_{i=0}^{n} a_i^r \\ &= M_r^r(\boldsymbol{a}).\end{aligned}$$

Therefore, it is deduced that $S_r(\boldsymbol{a}) < M_r(\boldsymbol{a})$ for $r > 1$ and $S_r(\boldsymbol{a}) > M_r(\boldsymbol{a})$ for $r < 0$.

By the same argument as above, if $0 < r < 1$, then $S_r(\boldsymbol{a}) > M_r(\boldsymbol{a})$ and

$$S_0(\boldsymbol{a}) < S_{-(n+1)}(\boldsymbol{a}) = G(\boldsymbol{a}) = M_0(\boldsymbol{a}).$$

Thus, if $r < 1$, then $S_r(\boldsymbol{a}) > M_r(\boldsymbol{a})$. The proof is complete. □

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