A Completely Monotonic Function Involving Divided Differences of Psi and Polygamma Functions and an Application

A COMPLETELY MONOTONIC FUNCTION INVOLVING DIVIDED DIFFERENCES OF PSI AND POLYGAMMA FUNCTIONS AND AN APPLICATION

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Abstract. A function involving the divided differences of the psi function and the polygamma functions is proved to be completely monotonic. As an application of this result, the monotonicity and convexity of a function originated from establishing the best upper and lower bounds in Kershaw's inequality is deduced.

1. Introduction

Recall [5] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and $n \geq 0$. For information about the history, applications and recent developments on the completely monotonic function, please refer to the expository article [5] and the references therein.

The Kershaw's inequality [4] states that

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s} \tag{1}$$

for $0 < s < 1$ and $x \geq 1$, where Γ denotes the classical Euler's gamma function and the middle term in (1) is a special case of the Wallis' function $\frac{\Gamma(x+p)}{\Gamma(x+q)}$ for $x+p>0$ and $x+q>0$. It is clear that inequality (1) can be rearranged as

$$\frac{s}{2} < \left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)} - x < \sqrt{s+\frac{1}{4}} - \frac{1}{2}. \tag{2}$$

Let s and t be nonnegative numbers and $\alpha = \min\{s,t\}$. Define

$$z_{s,t}(x) = \begin{cases} \left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x, & s \neq t \\ e^{\psi(x+s)} - x, & s = t \end{cases} \tag{3}$$

in $x \in (-\alpha, \infty)$. Standard differentiating and simplifying yields

$$z'_{s,t}(x) = [z_{s,t}(x)+x]\frac{\psi(x+t)-\psi(x+s)}{t-s} - 1, \tag{4}$$

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$$z''_{s,t}(x) = [z_{s,t}(x) + x]\left\{\left[\frac{\psi(x+t) - \psi(x+s)}{t-s}\right]^2 + \frac{\psi'(x+t) - \psi'(x+s)}{t-s}\right\} \tag{5}$$

$$= \frac{z_{s,t}(x) + x}{(t-s)^2}\left\{[\psi(x+t) - \psi(x+s)]^2 + (t-s)[\psi'(x+t) - \psi'(x+s)]\right\}. \tag{6}$$

In order to obtain the best upper and lower bounds for double inequality (1) or (2), the monotonicity and convexity properties of the function $z_{s,t}(x)$ in $x \in (-\alpha, \infty)$ is showed in [2, 3, 7] by using Laplace transform and other complicated techniques respectively.

Let

$$\Theta_{s,t}(x) = [\psi(x+t) - \psi(x+s)]^2 + (t-s)[\psi'(x+t) - \psi'(x+s)] \tag{7}$$

and

$$\Delta_{s,t}(x) = \begin{cases} \left[\dfrac{\psi(x+t) - \psi(x+s)}{t-s}\right]^2 + \dfrac{\psi'(x+t) - \psi'(x+s)}{t-s}, & s \ne t \\ [\psi'(x+s)]^2 + \psi''(x+s), & s = t \end{cases} \tag{8}$$

in $x \in (-\alpha, \infty)$. It is clear from (5) and (6) that

$$z''_{s,t}(x) = [z_{s,t}(x) + x]\Delta_{s,t}(x) = \frac{z_{s,t}(x) + x}{(t-s)^2}\Theta_{s,t}(x) \tag{9}$$

for $t \ne s$.

The aim of this paper is to prove the completely monotonic property of the functions $\Theta_{s,t}(x)$ and $\Delta_{s,t}(x)$ in $(-\alpha, \infty)$.

Theorem 1. *The functions $\Theta_{s,t}(x)$ for $|t-s| < 1$ and $-\Theta_{s,t}(x)$ for $|t-s| > 1$ are completely monotonic in $(-\alpha, \infty)$. The functions $\Delta_{s,t}(x)$ for $|t-s| < 1$ and $-\Delta_{s,t}(x)$ for $|t-s| > 1$ are completely monotonic in $x \in (-\alpha, \infty)$.*

Remark 1. Note that, among other things, the positivity of the function $\Delta_{0,0}(x) = [\psi'(x)]^2 + \psi''(x)$ in (8) has been verified in [1].

As a straightforward application of Theorem 1, the monotonicity and convexity of the function $z_{s,t}(x)$ is obtained.

Theorem 2 ([2, 3, 7])**.** *The function $z_{s,t}(x)$ in $(-\alpha, \infty)$ is either convex and decreasing for $|t-s| < 1$ or concave and increasing for $|t-s| > 1$.*

2. Proofs of theorems

The basic tool of this paper is the following lemma.

Lemma 1. *Let $f(x)$ be defined in an infinite interval I. If $\lim_{x\to\infty} f(x) = 0$ and $f(x) - f(x+\varepsilon) > 0$ for any given $\varepsilon > 0$, then $f(x) > 0$ in I.*

Proof. By induction, for any $x \in I$, we have

$$f(x) > f(x+\varepsilon) > f(x+2\varepsilon) > \cdots > f(x+k\varepsilon) \to 0$$

as $k \to \infty$. The proof of Lemma 1 is complete. □

2.1. **Proof of Theorem 1.** It is well known that for any positive integer $n \in \mathbb{N}$ the psi function $\psi(x)$ and the polygamma or multigamma functions $\psi^{(n)}(x)$ have the following integral expressions

$$\psi(x) = \ln x + \int_0^\infty \left[\frac{1}{u} - \frac{1}{1-e^{-u}}\right] e^{-xu} \, \mathrm{d}u \tag{10}$$

and

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{u^n}{1-e^{-u}} e^{-xu} \, \mathrm{d}u. \tag{11}$$

Using

$$\psi^{(i-1)}(x+1) = \psi^{(i-1)}(x) + \frac{(-1)^{i-1}(i-1)!}{x^i} \tag{12}$$

for $i \in \mathbb{N}$ and $x > 0$ and direct computing gives

$$\begin{aligned}
\Theta_{s,t}(x) - \Theta_{s,t}(x+1) &= \big\{[\psi(x+t) + \psi(x+t+1)] - [\psi(x+s) + \psi(x+s+1)]\big\} \\
&\quad \times \big\{[\psi(x+t) - \psi(x+t+1)] - [\psi(x+s) - \psi(x+s+1)]\big\} \\
&\quad + (t-s)\big\{[\psi'(x+t) - \psi'(x+t+1)] - [\psi'(x+s) - \psi'(x+s+1)]\big\} \\
&= \left\{\frac{[\psi(x+t+1) + \psi(x+t)] - [\psi(x+s+1) + \psi(x+s)]}{t-s}\right. \\
&\quad \left. - \frac{2x+s+t}{(x+s)(x+t)}\right\} \frac{(t-s)^2}{(x+s)(x+t)} \triangleq \Lambda_{s,t}(x) \frac{(t-s)^2}{(x+s)(x+t)}
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\Lambda_{s,t}(x) - \Lambda_{s,t}(x+1) &= \frac{1}{t-s}\left(\frac{1}{x+s} + \frac{1}{x+s+1} - \frac{1}{x+t} - \frac{1}{x+t+1}\right) \\
&\quad - \frac{2x^2 + 2(s+t+1)x + s^2 + t^2 + s + t}{(x+s)(x+s+1)(x+t)(x+t+1)} \\
&= \frac{1-(s-t)^2}{(x+s)(x+s+1)(x+t)(x+t+1)}.
\end{aligned}$$

Since $\lim_{x\to\infty} \Lambda_{s,t}^{(i)}(x) = 0$ for any nonnegative integer i by (10) and (11), and the function $\frac{\Lambda_{s,t}(x) - \Lambda_{s,t}(x+1)}{1-(s-t)^2}$ is completely monotonic, that is,

$$(-1)^i \frac{[\Lambda_{s,t}(x) - \Lambda_{s,t}(x+1)]^{(i)}}{1-(s-t)^2} = \frac{(-1)^i \Lambda_{s,t}^{(i)}(x) - (-1)^i \Lambda_{s,t}^{(i)}(x+1)}{1-(s-t)^2} \ge 0,$$

in $(-\alpha, \infty)$, then $\frac{(-1)^i \Lambda_{s,t}^{(i)}(x)}{1-(s-t)^2} \ge 0$ follows from Lemma 1. This means the function $\frac{\Lambda_{s,t}(x)}{1-(s-t)^2}$ is completely monotonic in $(-\alpha, \infty)$.

Since the function $\frac{(t-s)^2}{(x+s)(x+t)}$ is completely monotonic and a product of two completely monotonic functions is also completely monotonic, then the function $\frac{\Theta_{s,t}(x) - \Theta_{s,t}(x+1)}{1-(s-t)^2}$ is completely monotonic in $(-\alpha, \infty)$ by considering (13), which is equivalent to

$$(-1)^k \left[\frac{\Theta_{s,t}(x) - \Theta_{s,t}(x+1)}{1-(s-t)^2}\right]^{(k)} = \frac{(-1)^k \Theta_{s,t}^{(k)}(x) - (-1)^k \Theta_{s,t}^{(k)}(x+1)}{1-(s-t)^2} \ge 0$$

for nonnegative integer k. Further, from $\lim_{x\to\infty} \Theta_{s,t}^{(k)}(x) = 0$ for nonnegative integer k, which can be deduced by utilizing (10) and (11), and Lemma 1, it is concluded that $\frac{(-1)^k\Theta_{s,t}^{(k)}(x)}{1-(s-t)^2}\geq 0$ for any nonnegative integer k. This implies $(-1)^k\Theta_{s,t}^{(k)}(x)\gtreqless 0$ if and only if $|t-s|\lesseqgtr 1$. Therefore, the functions $\Theta_{s,t}(x)$ for $|t-s|<1$ and $-\Theta_{s,t}(x)$ for $|t-s|>1$ are completely monotonic in $(-\alpha,\infty)$.

Since $\Theta_{s,t}(x)=(t-s)^2\Delta_{s,t}(x)$, the function $\Delta_{s,t}(x)$ has the same monotonicity property as $\Theta_{s,t}(x)$ in $(-\alpha,\infty)$. The proof of Theorem 1 is complete.

2.2. **Proof of Theorem 2.** By Theorem 1, it is easy to see that $\Theta_{s,t}(x)\gtreqless 0$ and $\Delta_{s,t}(x)\gtreqless 0$ in $(-\alpha,\infty)$ if and only if $|t-s|\lesseqgtr 1$. Then $z''_{s,t}(x)\gtreqless 0$ for $|t-s|\lesseqgtr 1$ follows from formula (9). The convexity and concavity of the function $z_{s,t}(x)$ is proved.

In [6], the inequality

$$\exp[(s-r)\psi(s)]>\frac{\Gamma(s)}{\Gamma(r)}>\exp[(s-r)\psi(r)] \tag{14}$$

for $s>r>0$ was obtained, which is equivalent to

$$\max\left\{e^{\psi(s)},e^{\psi(r)}\right\}>\left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}>\min\left\{e^{\psi(s)},e^{\psi(r)}\right\}$$

for any positive numbers $s>0$ and $t>0$. This implies

$$\begin{aligned} z'_{s,t}(x)&=\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}\frac{\psi(x+t)-\psi(x+s)}{t-s}-1\\ &<e^{\psi(x+t)}\frac{\psi(x+t)-\psi(x+s)}{t-s}-1\\ &=e^{\psi(x+t)}\psi'(x+\xi)-1<\psi'(x+t)e^{\psi(x+t)}-1 \end{aligned} \tag{15}$$

and

$$\begin{aligned} z'_{s,t}(x)&>e^{\psi(x+s)}\frac{\psi(x+t)-\psi(x+s)}{t-s}-1\\ &=e^{\psi(x+s)}\psi'(x+\xi)-1\\ &>\psi'(x+s)e^{\psi(x+s)}-1, \end{aligned} \tag{16}$$

if assuming $t>s>0$ without loss of generality, where $\xi\in(s,t)$.

By inequality

$$\ln x-\frac{1}{x}<\psi(x)<\ln x-\frac{1}{2x} \tag{17}$$

for $x>0$, we obtain

$$x\psi'(x)e^{-1/x}<\psi'(x)e^{\psi(x)}<x\psi'(x)e^{-1/2x} \tag{18}$$

for $x>0$. Using the asymptotic representation

$$\psi'(x)\sim\frac{1}{x}+\frac{1}{2x^2}+\cdots \tag{19}$$

as $x\to\infty$ yields

$$\lim_{x\to\infty}\left[x\psi'(x)e^{-1/x}\right]=1 \qquad \text{and} \qquad \lim_{x\to\infty}\left[x\psi'(x)e^{-1/2x}\right]=1. \tag{20}$$

Hence,

$$\lim_{x\to\infty}\left[\psi'(x)e^{\psi(x)}\right]=1. \tag{21}$$

Combining (21) with (15) and (16) leads to

$$\lim_{x\to\infty} z'_{s,t}(x) \le \lim_{x\to\infty} \left[\psi'(x+t)e^{\psi(x+t)}\right] - 1 = \lim_{x+t\to\infty} \left[\psi'(x+t)e^{\psi(x+t)}\right] - 1 = 0$$

and

$$\lim_{x\to\infty} z'_{s,t}(x) \ge \lim_{x\to\infty} \left[\psi'(x+s)e^{\psi(x+s)}\right] - 1 = \lim_{x+s\to\infty} \left[\psi'(x+s)e^{\psi(x+s)}\right] - 1 = 0.$$

Thus, it is concluded that $\lim_{x\to\infty} z'_{s,t}(x) = 0$.

Since $z''_{s,t}(x) \gtreqless 0$ in $x \in (-\alpha, \infty)$ for $|t-s| \lesseqgtr 1$, then the function $z'_{s,t}(x)$ is increasing/decreasing in $x \in (-\alpha, \infty)$ for $|t-s| \lesseqgtr 1$. Thus, it follows that $z'_{s,t}(x) \lesseqgtr 0$ and $z_{s,t}(x)$ is decreasing/increasing in $x \in (-\alpha, \infty)$ for $|t-s| \lesseqgtr 1$. The monotonicity of the function $z_{s,t}(x)$ is proved.

References

[1] N. Batir, *Some new inequalities for gamma and polygamma functions*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Art. 103. Available online at http://jipam.vu.edu.au/article.php?sid=577. RGMIA Res. Rep. Coll. **7** (2004), no. 3, Art. 1, 371–381. Available online at http://rgmia.vu.edu.au/v7n3.html.

[2] Ch.-P. Chen, *Monotonicity and convexity for the gamma function*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Art. 100. Available online at http://jipam.vu.edu.au/article.php?sid=574.

[3] N. Elezović, C. Giordano and J. Pečarić, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. **3** (2000), 239–252. Available online at http://www.mia-journal.com/contents.asp?number=10.

[4] D. Kershaw, *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. **41** (1983), 607–611.

[5] F. Qi, *Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 5. Available online at http://rgmia.vu.edu.au/v8n3.html.

[6] F. Qi, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl. **5** (2002), no. 1, 61–67. RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 7, 1027–1034. Available online at http://rgmia.vu.edu.au/v2n7.html.

[7] F. Qi, B.-N. Guo, and Ch.-P. Chen, *The best bounds in Gautschi-Kershaw inequalities*, Math. Inequal. Appl. **9** (2006), no. 3, 427–436. RGMIA Res. Rep. Coll. **8** (2005), no. 2, Art. 17. Available online at http://rgmia.vu.edu.au/v8n2.html.

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