Bounds for the r-Weighted Gini Mean Difference of an Empirical Distribution

BOUNDS FOR THE $r-$WEIGHTED GINI MEAN DIFFERENCE OF AN EMPIRICAL DISTRIBUTION

P. CERONE AND S.S. DRAGOMIR

Abstract. Various bounds for the $r-$weighted Gini mean difference of an empirical distribution are established.

1. Introduction

The *Gini mean difference* of the *sample* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is defined by

$$G(\mathbf{a}) = \frac{1}{2n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} |a_i - a_j| = \frac{1}{n^2} \sum_{1 \le i < j \le n} |a_i - a_j|$$

and

$$R(\mathbf{a}) = \frac{1}{\bar{a}} G(\mathbf{a})$$

is the Gini index of $\mathbf{a}$, provided the sample mean $\bar{a}$ is not zero [6, p. 257].

The Gini index of $\mathbf{a}$ equals the Gini mean difference of the "scaled down" sample $\tilde{a} = \left(\frac{a_1}{\bar{a}}, \ldots, \frac{a_n}{\bar{a}}\right) \quad (\bar{a} \neq 0)$

$$R(a_1, \ldots, a_n) = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{a_i}{\bar{a}} - \frac{a_j}{\bar{a}} \right|.$$

The following elementary properties of the Gini index for an empirical distribution of nonnegative data hold [6, p. 257]:

(i) Let $(a_1, \ldots, a_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} a_i > 0$. Then

$$0 = R(\bar{a}, \ldots, \bar{a}) \le R(a_1, \ldots, a_n) \le R\left(0, \ldots, 0, \sum_{i=1}^{n} a_i\right) = 1 - \frac{1}{n} < 1,$$
$$R(\beta a_1, \ldots, \beta a_n) = R(a_1, \ldots, a_n) \quad \text{for every} \quad \beta > 0$$

and

$$R(a_1 + \lambda, \ldots, a_n + \lambda) = \frac{\bar{a}}{\bar{a} + \lambda} R(a_1, \ldots, a_n) \quad \text{for } \lambda > 0.$$

(ii) R is a continuous function on $\mathbb{R}_+^n$.

Date: 9th November, 2006.

2000 *Mathematics Subject Classification.* 26D15, 62H20, 62G10, 62P20.

Key words and phrases. Gini mean difference, $r-$weighted gini mean differences, Empirical distributions.

These and other properties have been investigated in [6], [3] and [4].

For $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{R}^n$ and $\mathbf{p}=(p_1,\dots,p_n)$ a probability sequence, meaning that $p_i\geq 0$ $(i\in\{1,\dots,n\})$ and $\sum_{i=1}^n p_i=1$, we considered in [1] the *weighted Gini mean difference* defined by formula

$$G(\mathbf{p},\mathbf{a})=\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}p_ip_j\left|a_i-a_j\right|=\sum_{1\leq i<j\leq n}p_ip_j\left|a_i-a_j\right|, \tag{1.1}$$

and proved that

$$\frac{1}{2}K(\mathbf{p},\mathbf{a})\leq G(\mathbf{p},\mathbf{a})\leq\inf_{\gamma\in\mathbb{R}}\left[\sum_{i=1}^{n}p_i\left|a_i-\gamma\right|\right]\leq K(\mathbf{p},\mathbf{a}), \tag{1.2}$$

where $K(\mathbf{p},\mathbf{a})$ is the *mean absolute deviation*, namely

$$K(\mathbf{p},\mathbf{a}):=\sum_{i=1}^{n}p_i\left|a_i-\sum_{j=1}^{n}p_ja_j\right|. \tag{1.3}$$

We have also shown that if more information on the sampling data $\mathbf{a}=(a_1,\dots,a_n)$ is available, i.e., there exists the real numbers a and A such that $a\leq a_i\leq A$ for each $i\in\{1,\dots,n\}$, then

$$G(\mathbf{p},\mathbf{a})\leq(A-a)\max_{J\subseteq\{1,\dots,n\}}\left[P_J(1-P_J)\right]\qquad\left(\leq\frac{1}{4}(A-a)\right), \tag{1.4}$$

where $P_J:=\sum_{j\in J}p_j$. Also, we have shown that

$$G(\mathbf{p},\mathbf{a})\leq\sum_{i=1}^{n}p_i\left|a_i-\frac{A+a}{2}\right|\qquad\left(\leq\frac{1}{2}(A-a)\right). \tag{1.5}$$

Notice that in general the bounds for the weighted Gini mean difference $G(\mathbf{p},\mathbf{a})$ provided by (1.4) and (1.5) cannot be compared to conclude that one is always better than the other [1].

The main aim of this paper is to continue the study begun in [1] and provide various bounds for the more general $r-$weighted Gini mean difference that has been introduced in [1].

2. Bounds for the $r-$weighted Gini Mean Difference

For $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{R}^n$ and $\mathbf{p}=(p_1,\dots,p_n)$ a probability sequence, meaning that $p_i\geq 0$ $(i\in\{1,\dots,n\})$ and $\sum_{i=1}^n p_i=1$, define the *$r-$weighted Gini mean difference,* for $r\in[1,\infty)$, by the formula [1, 291]:

$$G_r(\mathbf{p},\mathbf{a}):=\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}p_ip_j\left|a_i-a_j\right|^r=\sum_{1\leq i<j\leq n}p_ip_j\left|a_i-a_j\right|^r. \tag{2.1}$$

For $r=1$ we have the *weighted Gini mean difference* $G(\mathbf{p},\mathbf{a})$ of (1.1) which becomes, for the uniform probability distribution $\mathbf{p}=\left(\frac{1}{n},\dots,\frac{1}{n}\right)$ the *Gini mean difference*

$$G(\mathbf{a}):=\frac{1}{2n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left|a_i-a_j\right|=\frac{1}{n^2}\sum_{1\leq i<j\leq n}\left|a_i-a_j\right|.$$

For the uniform probability distribution $\mathbf{p}=\left(\frac{1}{n},\dots,\frac{1}{n}\right)$ we denote

$$G_r(\mathbf{a}) := G_r(\mathbf{p},\mathbf{a}) = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|a_i-a_j|^r = \frac{1}{n^2}\sum_{1\le i<j\le n}|a_i-a_j|^r.$$

Now, if we define $\Delta := \{(i,j)\,|i,j\in\{1,\dots,n\}\}$, then we can simply write from (2.1)

$$G_r(\mathbf{p},\mathbf{a}) = \frac{1}{2}\sum_{(i,j)\in\Delta}p_ip_j|a_i-a_j|^r, \qquad r\ge 1. \tag{2.2}$$

The following result concerning upper and lower bounds for $G_r(\mathbf{p},\mathbf{a})$ may be stated:

Theorem 1. *For any* $p_i\in(0,1)$, $i\in\{1,\dots,n\}$ *with* $\sum_{i=1}^n p_i=1$ *and* $a_i\in\mathbb{R}$, $i\in\{1,\dots,n\}$, *we have the inequalities*

$$\frac{1}{2}\max_{(i,j)\in\Delta}\left\{\frac{p_i^rp_j^r+p_ip_j(1-p_ip_j)^{r-1}}{(1-p_ip_j)^{r-1}}|a_i-a_j|^r\right\} \le G_r(\mathbf{p},\mathbf{a}) \le \frac{1}{2}\max_{(i,j)\in\Delta}|a_i-a_j|^r, \tag{2.3}$$

where $r\in(0,\infty)$.

Proof. Observe that

$$\sum_{(i,j)\in\Delta}p_ip_j(a_i-a_j)=0.$$

Then, for any fixed $(i,j)\in\Delta$ we have

$$p_ip_j(a_i-a_j) = -\sum_{(k,l)\in\Delta\setminus\{(i,j)\}}p_kp_l(a_k-a_l). \tag{2.4}$$

Taking the modulus in (2.4) and utilising the Hölder discrete inequality for multiple indices and $r>1$, $\frac{1}{r}+\frac{1}{q}=1$ $\left(q=\frac{r}{r-1}\right)$, we have successively:

$$\begin{aligned} p_ip_j|a_i-a_j| &= \left|\sum_{(k,l)\in\Delta\setminus\{(i,j)\}}p_kp_l(a_k-a_l)\right| \\ &\le \left(\sum_{(k,l)\in\Delta\setminus\{(i,j)\}}p_kp_l\right)^{\frac{1}{q}}\left(\sum_{(k,l)\in\Delta\setminus\{(i,j)\}}p_kp_l|a_k-a_l|^r\right)^{\frac{1}{r}} \\ &= \left(\sum_{(k,l)\in\Delta}p_kp_l-p_ip_j\right)^{\frac{1}{q}} \\ &\qquad\times\left(\sum_{(k,l)\in\Delta}p_kp_l|a_k-a_l|^r-p_ip_j|a_i-a_j|^r\right)^{\frac{1}{r}} \\ &= (1-p_ip_j)^{\frac{r-1}{r}}\left(2G_r(\mathbf{p},\mathbf{a})-p_ip_j|a_i-a_j|^r\right)^{\frac{1}{r}} \end{aligned} \tag{2.5}$$

for each $(i,j)\in\Delta$.

Taking the power r in (2.5) we have

$$p_i^r p_j^r |a_i - a_j|^r \leq (1 - p_i p_j)^{r-1} (2G_r(\mathbf{p}, \mathbf{a}) - p_i p_j |a_i - a_j|^r),$$

giving

$$\left[p_i^r p_j^r + p_i p_j (1 - p_i p_j)^{r-1}\right] |a_i - a_j|^r \leq 2(1 - p_i p_j)^{r-1} G_r(\mathbf{p}, \mathbf{a}),$$

so that

$$\frac{1}{2} \cdot \frac{p_i^r p_j^r + p_i p_j (1 - p_i p_j)^{r-1}}{(1 - p_i p_j)^{r-1}} |a_i - a_j|^r \leq G_r(\mathbf{p}, \mathbf{a}) \tag{2.6}$$

for each $(i, j) \in \Delta$.

Taking the maximum over $(i, j) \in \Delta$ in (2.6), we deduce the first inequality in (2.3).

The second inequality is obvious on observing that

$$G_r(\mathbf{p}, \mathbf{a}) \leq \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j \max_{(i,j) \in \Delta} |a_i - a_j|^r = \frac{1}{2} \max_{(i,j) \in \Delta} |a_i - a_j|^r.$$

The proof is complete. ∎

Remark 1. *The case $r = 2$ is of interest, since*

$$G_2(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j |a_i - a_j|^2 = \sum_{i=1}^n p_i a_i^2 - \left(\sum_{i=1}^n p_i a_i\right)^2,$$

for which we can obtain from Theorem 1 the following bounds:

$$\frac{1}{2} \max_{(i,j) \in \Delta} \left\{ \frac{p_i p_j}{1 - p_i p_j} (a_i - a_j)^2 \right\} \leq G_2(\mathbf{p}, \mathbf{a}) \leq \frac{1}{2} \max_{(i,j) \in \Delta} (a_i - a_j)^2. \tag{2.7}$$

Remark 2. *Consider the function*

$$h_r(t) := \frac{t^r + t(1 - t)^{r-1}}{(1 - t)^{r-1}} = t + t^r (1 - t)^{1-r}$$

defined for $t \in [0, 1)$ and $r > 1$. Then

$$h_r'(t) = 1 + r t^{r-1} (1 - t)^{1-r} + (r - 1) t^r (1 - t)^{-r}$$

which shows that h_r is strictly increasing on $[0, 1)$.

Therefore

$$\begin{aligned} \min_{(i,j) \in \Delta} \left\{ \frac{p_i^r p_j^r + p_i p_j (1 - p_i p_j)^{r-1}}{(1 - p_i p_j)^{r-1}} \right\} &= \min_{(i,j) \in \Delta} h_r(p_i p_j) \\ &\geq h_r\left[\min_{(i,j) \in \Delta} (p_i p_j)\right] \\ &\geq h_r\left(\min_{i \in \{1,\ldots,n\}} p_i \cdot \min_{j \in \{1,\ldots,n\}} p_j\right) \\ &= h_r(p_m^2) \\ &= \frac{p_m^{2r} + p_m^2 (1 - p_m^2)^{r-1}}{(1 - p_m^2)^{r-1}}, \end{aligned}$$

where $p_m := \min_{i \in \{1,\ldots,n\}} p_i > 0$.

In conclusion, from Theorem 1 we can obtain a coarser but, perhaps, a more useful lower bound for the $r-$weighted Gini mean difference, namely:

$$G_r(\mathbf{p},\mathbf{a}) \geq \frac{1}{2}\cdot\frac{p_m^{2r}+p_m^2\left(1-p_m^2\right)^{r-1}}{\left(1-p_m^2\right)^{r-1}}\cdot\max_{(i,j)\in\Delta}|a_i-a_j|^r, \tag{2.8}$$

where p_m is defined above.

For $r=2$, we then have:

$$G_2(\mathbf{p},\mathbf{a}) \geq \frac{1}{2}\cdot\frac{p_m^2}{1-p_m^2}\cdot\max_{(i,j)\in\Delta}(a_i-a_j)^2. \tag{2.9}$$

The following result for the weighted Gini mean difference can be stated:

Theorem 2. *For any $p_i \in (0,1)$, $i \in \{1,\dots,n\}$ with $\sum_{i=1}^n p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1,\dots,n\}$, we have the bounds:*

$$\frac{1}{2}\max_{(i,j)\in\Delta}\left\{p_ip_j\left[1+\frac{1}{\max_{(k,l)\in\Delta\setminus\{(i,j)\}}\{p_kp_l\}}\right]\cdot|a_i-a_j|\right\} \leq G(\mathbf{p},\mathbf{a}) \leq \frac{1}{2}\max_{(i,j)\in\Delta}|a_i-a_j|. \tag{2.10}$$

Proof. As in the proof of Theorem 1 we have

$$\begin{aligned}
p_ip_j|a_i-a_j| &= \left|\sum_{(k,l)\in\Delta\setminus\{(i,j)\}}p_kp_l(a_k-a_l)\right| \\
&\leq \max_{(k,l)\in\Delta\setminus\{(i,j)\}}\{p_kp_l\}\cdot\sum_{(k,l)\in\Delta\setminus\{(i,j)\}}p_kp_l|a_k-a_l| \\
&= \max_{(k,l)\in\Delta\setminus\{(i,j)\}}\{p_kp_l\}\left[\sum_{(k,l)\in\Delta}p_kp_l|a_k-a_l|-p_ip_j|a_i-a_j|\right]
\end{aligned}$$

which gives:

$$p_ip_j\left[1+\max_{(k,l)\in\Delta\setminus\{(i,j)\}}\{p_kp_l\}\right]|a_i-a_j| \leq \max_{(k,l)\in\Delta\setminus\{(i,j)\}}\{p_kp_l\}\cdot\sum_{(k,l)\in\Delta}p_kp_l|a_k-a_l|.$$

That is

$$p_ip_j\left[\frac{1+\max_{(k,l)\in\Delta\setminus\{(i,j)\}}\{p_kp_l\}}{\max_{(k,l)\in\Delta\setminus\{(i,j)\}}\{p_kp_l\}}\right]|a_i-a_j| \leq \sum_{(k,l)\in\Delta}p_kp_l|a_k-a_l|,$$

which, by taking the maximum over $(i,j)\in\Delta$ implies the first part of (2.10).

The second part is obvious. ∎

Remark 3. *Since*

$$\max_{(k,l)\in\Delta\setminus\{(i,j)\}}\{p_kp_l\} \leq \max_{(k,l)\in\Delta}\{p_kp_l\} = p_M^2,$$

where $p_M := \max_{k\in\{1,\dots,n\}}p_k$, hence

$$1+\frac{1}{\max_{(k,l)\in\Delta\setminus\{(i,j)\}}\{p_kp_l\}} \geq 1+\frac{1}{p_M^2}$$

and we get from Theorem 2 the following lower bounds for $G(\mathbf{p},\mathbf{a})$

$$G(\mathbf{p},\mathbf{a}) \geq \frac{1}{2}\left(\frac{p_M^2+1}{p_M^2}\right)\max_{(i,j)\in\Delta}\{p_ip_j|a_i-a_j|\} \geq \frac{1}{2}p_m^2\left(\frac{p_M^2+1}{p_M^2}\right)\max_{(i,j)\in\Delta}|a_i-a_j|, \tag{2.11}$$

where $p_m := \min_{k\in\{1,\dots,n\}} p_k$ *and* $p_M := \max_{k\in\{1,\dots,n\}} p_k$.

3. Related Results

The following result is due to Izumino and Pečarić [5] (see also [2, p. 174 - 175]):

Lemma 1. *Let* f *be a convex even function defined on* $[m-M, M-m]$ $(0<m<M)$ *with* $f(0)=0$. *Then for each* $n-$*tuple* $x=(x_1,\dots,x_n)$ *satisfying the condition* $m\leq x_k\leq M$ $(k=1,\dots,n)$ *and for each positive weight* $q=(q_1,\dots,q_n)$ *we have*

$$\sum_{1\leq i<j\leq n} q_iq_jf(x_i-x_j) \leq f(M-m)\max_{J\subseteq\{1,\dots,n\}}[Q_J(1-Q_J)] \leq \frac{1}{4}f(M-m), \tag{3.1}$$

where $Q_j := \sum_{j\in J} q_j$.

The following result holds concerning upper bounds for the $r-$weighted Gini mean difference when some information on the size of the elements a_i, $i\in\{1,\dots,n\}$ are available.

Theorem 3. *For any* $p_i\in(0,1)$, $i\in\{1,\dots,n\}$ *with* $\sum_{i=1}^n p_i = 1$ *and* $a_i\in\mathbb{R}$, $i\in\{1,\dots,n\}$ *with the property that*

$$-\infty < a \leq a_i \leq A < \infty \quad \textit{for each} \quad i\in\{1,\dots,n\}, \tag{3.2}$$

we have the inequality:

$$G_r(\mathbf{p},\mathbf{a}) \leq (A-a)^r \max_{J\subseteq\{1,\dots,n\}}[P_J(1-P_J)] \quad \left(\leq \frac{1}{4}(A-a)^r\right), \tag{3.3}$$

for $r\geq 1$.

Proof. Without loss of generality, we may assume that $a\geq 0$.

Now, if we apply Lemma 1 for $f(x)=|x|^r$, $x_i=a_i$ and $q_i=p_i$, $i\in\{1,\dots,n\}$, we get

$$G_r(\mathbf{p},\mathbf{a}) = \frac{1}{2}\sum_{i,j=1} p_ip_j|a_i-a_j|^r \leq |A-a|^r \max_{J\subseteq\{1,\dots,n\}}[P_J(1-P_J)]$$

and the result is proved. ∎

Finally, the following result that provides a connection between

$$G_2(\mathbf{p},\mathbf{a}) = \sum_{i=1}^n p_ia_i^2 - \left(\sum_{i=1}^n p_ia_i\right)^2,$$

and

$$G_2(\mathbf{a}) = \frac{1}{n}\sum_{i=1}^n a_i^2 - \left(\frac{1}{n}\sum_{i=1}^n a_i\right)^2,$$

can be stated.

Theorem 4. *If $p_i \in (0,1)$ for $i \in \{1,\dots,n\}$ with $\sum_{i=1}^n p_i = 1$, then for any $a_i \in \mathbb{R}$ $i \in \{1,\dots,n\}$ we have the inequality:*

$$G_2(\mathbf{p},\mathbf{a}) \le n^2 \left[1 - \frac{\left(\sum_{i=1}^n p_i^3\right)^2}{\left(\sum_{i=1}^n p_i^2\right)^2}\right] G_2(\mathbf{a}). \tag{3.4}$$

Proof. Utilising the Cauchy-Bunyakovsky-Schwarz inequality, we have that:

$$\begin{aligned} & p_i p_j |a_i - a_j| \\ &= \left| \sum_{(k,l)\in\Delta\setminus\{(i,j)\}} p_k p_l (a_k - a_l) \right| \\ &\le \left(\sum_{(k,l)\in\Delta\setminus\{(i,j)\}} p_k^2 p_l^2 \right)^{\frac{1}{2}} \left(\sum_{(k,l)\in\Delta\setminus\{(i,j)\}} |a_k - a_l|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{(k,l)\in\Delta} p_k^2 p_l^2 - p_i^2 p_j^2 \right)^{\frac{1}{2}} \left(\sum_{(k,l)\in\Delta} |a_k - a_l|^2 - |a_i - a_j|^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{i=1}^n p_k^2 \right)^2 - p_i^2 p_j^2 \right]^{\frac{1}{2}} \left(\sum_{(k,l)\in\Delta} |a_k - a_l|^2 - |a_i - a_j|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{3.5}$$

The square of (3.5) produces

$$p_i^2 p_j^2 |a_i - a_j|^2 \le \left[\left(\sum_{k=1}^n p_k^2 \right)^2 - p_i^2 p_j^2 \right] \left[\sum_{(k,l)\in\Delta} |a_k - a_l|^2 - |a_i - a_j|^2 \right],$$

giving

$$\begin{aligned} & \left[p_i^2 p_j^2 + \left(\sum_{k=1}^n p_k^2 \right)^2 - p_i^2 p_j^2 \right] |a_i - a_j|^2 \\ & \qquad \le \left[\left(\sum_{k=1}^n p_k^2 \right)^2 - p_i^2 p_j^2 \right] \sum_{(k,l)\in\Delta} |a_k - a_l|^2 \end{aligned}$$

from which we get

$$|a_i - a_j|^2 \le \left[1 - \frac{p_i^2 p_j^2}{\left(\sum_{k=1}^n p_k^2\right)^2} \right] \sum_{(k,l)\in\Delta} |a_k - a_l|^2. \tag{3.6}$$

Now, if we multiply (3.6) with $p_i p_j \ge 0$ and sum over $(i,j) \in \Delta$ then we get

$$G_2(\mathbf{p},\mathbf{a}) \le n^2 \left[1 - \frac{\left(\sum_{i=1}^n p_i^3\right)^2}{\left(\sum_{i=1}^n p_i^2\right)^2}\right] G_2(\mathbf{a}), \tag{3.7}$$

and the result is proved. ∎

Remark 4. *It is obvious, by the definition of $G_r(\mathbf{p},\mathbf{a})$ in (2.2) that for $r=2$*

$$G_2(\mathbf{p},\mathbf{a}) = \frac{1}{2}\sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j|^2 \le \frac{1}{2}\max_{(i,j)\in\Delta}\{p_i p_j\}\sum_{(i,j)\in\Delta}|a_i - a_j|^2 = n^2 \max_{(i,j)\in\Delta}\{p_i p_j\} G_2(\mathbf{a}). \tag{3.8}$$

Then, it is natural to ask when comparing (3.7) and (3.8) the question, when is the bound

$$B_1(\mathbf{p}) := 1 - \frac{\left(\sum_{i=1}^n p_i^3\right)^2}{\left(\sum_{i=1}^n p_i^2\right)^2}$$

better than

$$B_2(\mathbf{p}) := \max_{(i,j)\in\Delta}\{p_i p_j\}.$$

If we take $n=2$ and $p_1 = p$, $p_2 = 1-p$, $p \in (0,1)$ then

$$B_1(p) = 1 - \left[\frac{p^3 + (1-p)^3}{p^2 + (1-p)^2}\right]^2$$

and

$$B_2(p) = \max\left\{p^2, p(1-p), (1-p)^2\right\}.$$

The variation of the bounds $B_1(p)$ and $B_2(p)$ are depicted in Figure 1 and Figure 2, respectively. The plot of the difference $D(p) := B_1(p) - B_2(p)$ shows that one bound is not always better than the other (see Figure 3).

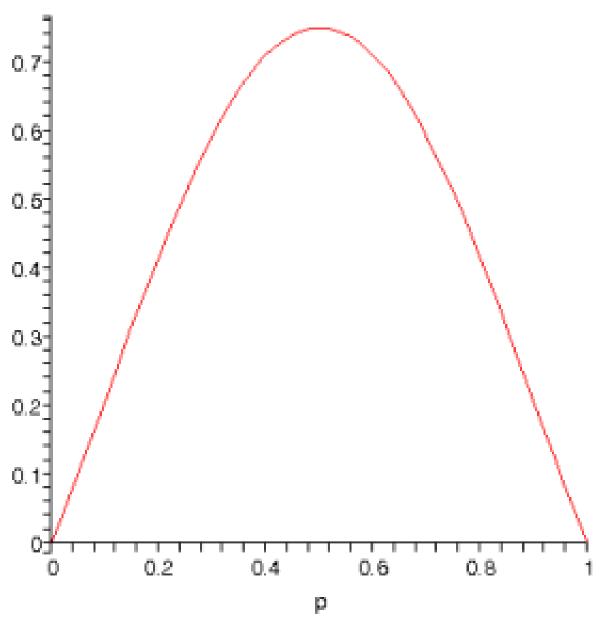

FIGURE 1. The plot of $B_1(p)$.

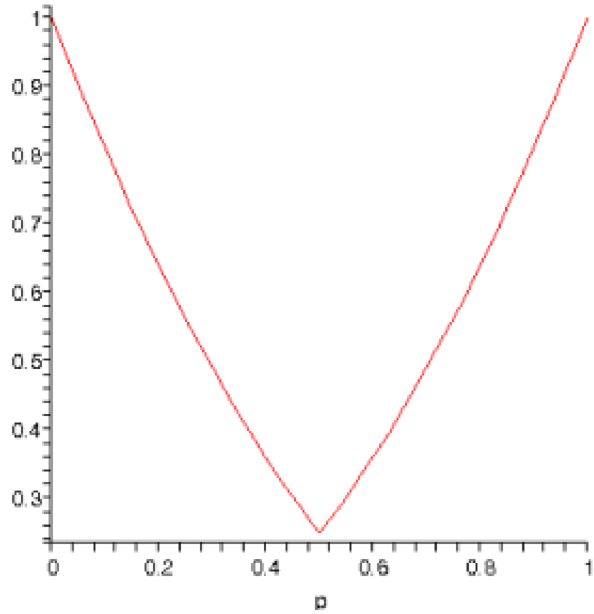

FIGURE 2. The plot of $B_2(p)$.

Finally, the following result in comparing the weighted Gini mean difference $G(\mathbf{p},\mathbf{a})$ with the unweighted means $G_r(\mathbf{a})$ may be stated:

Theorem 5. *If $p_i \in (0,1)$ for $i \in \{1,\dots,n\}$ with $\sum_{i=1}^n p_i = 1$, and $q, r > 1$ with $\frac{1}{q}+\frac{1}{r}=1$, then for any $a_i \in \mathbb{R}$ $i \in \{1,\dots,n\}$ we have the inequality:*

$$G(\mathbf{p},\mathbf{a}) \le 2^{1/r-2} n^{2/r+2} \left(\sum_{i=1}^n p_i^q\right)^{2/q} [G_r(\mathbf{a})]^{1/r}. \tag{3.9}$$

Proof. We use Hölder's inequality for double sums to get

$$\begin{aligned} p_i p_j |a_i - a_j| &= \left| \sum_{(k,l)\in\Delta\setminus\{(i,j)\}} p_k p_l (a_k - a_l) \right| \\ &\le \left(\sum_{(k,l)\in\Delta\setminus\{(i,j)\}} p_k^q p_l^q \right)^{1/q} \left(\sum_{(k,l)\in\Delta\setminus\{(i,j)\}} |a_k - a_l|^r \right)^{1/r} \\ &\le \left(\sum_{(k,l)\in\Delta} p_k^q p_l^q - p_i^q p_j^q \right)^{1/q} \left(\sum_{(k,l)\in\Delta} |a_k - a_l|^r - |a_i - a_j|^r \right)^{1/r} \\ &= \left[\left(\sum_{k=1}^n p_k^q \right)^2 - p_i^q p_j^q \right]^{1/q} \left(2n^2 G_r(\mathbf{a}) - |a_i - a_j|^r \right)^{1/r} \end{aligned} \tag{3.10}$$

for each $(i,j) \in \Delta$.

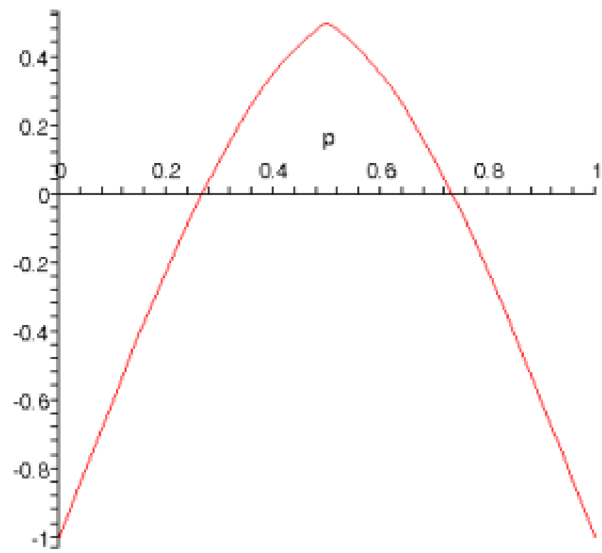

FIGURE 3. The plot of the difference $D_1(p)$.

Utilising the elementary inequality

$$\left(\alpha^r-\beta^r\right)^{1/r}\left(\gamma^q-\delta^q\right)^{1/q}\le \alpha\gamma-\beta\delta$$

provided $\alpha\ge\beta,\gamma\ge\delta$ and $q,r>1$ with $\frac{1}{q}+\frac{1}{r}=1$, we can get that

$$p_ip_j\left|a_i-a_j\right|\le\left(\sum_{i=1}^n p_i^q\right)^{2/q}\left[2n^2G_r\left(\mathbf{a}\right)\right]^{1/r}-p_ip_j\left|a_i-a_j\right|$$

which gives

$$2p_ip_j\left|a_i-a_j\right|\le\left(\sum_{i=1}^n p_i^q\right)^{2/q}\left[2n^2G_r\left(\mathbf{a}\right)\right]^{1/r}, \tag{3.11}$$

for each $(i,j)\in\Delta$.

Summing in the inequality (3.11) over $(i,j)\in\Delta$ we deduce the desired result (3.9). ∎

Remark 5. *The particular case $q=r=2$ provides the follwing simple inequality*

$$G\left(\mathbf{p},\mathbf{a}\right)\le 2^{-3/2}n^3\left(\sum_{i=1}^n p_i^2\right)\left[G_2\left(\mathbf{a}\right)\right]^{1/2}. \tag{3.12}$$

REFERENCES

[1] P. CERONE and S.S. DRAGOMIR, Bounds for the Gini mean difference of an empirical distribution, *Applied Math. Letters,* **19** (2006), 283-293.

[2] S.S. DRAGOMIR, *Discrete Inequalities of the Cauchy-Bunyakovsky-Schwarz Type*, Nova Science Publishers, N.Y., 2004.
[3] G.M. GIORGI, Bibliographic portrait of the Grüss concentration ratio, *Metron,* **48** (1990), 183-221.
[4] G.M. GIORGI, Il rapperto di concentrazionne di Gini, Liberia Editrice Ticci, Sienna, 1992.
[5] S. IZUMINO and E. PEČARIĆ, Some extensions of Grüss' inequality and its applications, *Nihonkai Math. J.,* **13** (2002), 159-166.
[6] G.A. KOSHEVOY and K. MOSLER, Multivariate Gini indices, *J. Multivariate Analysis*, **60** (1997), 252-276.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, VIC 8001, Australia.
E-mail address: `pietro.cerone@vu.edu.au`
URL: `http://rgmia.vu.edu.au/cerone`

E-mail address: `sever.dragomir@vu.edu.au`
URL: `http://rgmia.vu.edu.au/dragomir`