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# HARDY-TYPE INEQUALITIES VIA AUXILIARY SEQUENCES

PENG GAO

ABSTRACT. We prove some Hardy-type inequalities via an approach that involves constructing auxiliary sequences.

## 1. INTRODUCTION

Suppose throughout that  $p \neq 0$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $l^p$  be the Banach space of all complex sequences  $\mathbf{a} = (a_n)_{n \geq 1}$  with norm

$$\|\mathbf{a}\| := \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty.$$

The celebrated Hardy's inequality ([5, Theorem 326]) asserts that for  $p > 1$ ,

$$(1.1) \quad \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^n a_k \right|^p \leq \left( \frac{p}{p-1} \right)^p \sum_{k=1}^{\infty} |a_k|^p.$$

Hardy's inequality can be regarded as a special case of the following inequality:

$$\sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} c_{j,k} a_k \right|^p \leq U \sum_{k=1}^{\infty} |a_k|^p,$$

in which  $C = (c_{j,k})$  and the parameter  $p$  are assumed fixed ( $p > 1$ ), and the estimate is to hold for all complex sequences  $\mathbf{a}$ . The  $l^p$  operator norm of  $C$  is then defined as the  $p$ -th root of the smallest value of the constant  $U$ :

$$\|C\|_{p,p} = U^{\frac{1}{p}}.$$

Hardy's inequality thus asserts that the Cesàro matrix operator  $C$ , given by  $c_{j,k} = 1/j, k \leq j$  and 0 otherwise, is bounded on  $l^p$  and has norm  $\leq p/(p-1)$ . (The norm is in fact  $p/(p-1)$ .)

We say a matrix  $A$  is a summability matrix if its entries satisfy:  $a_{j,k} \geq 0$ ,  $a_{j,k} = 0$  for  $k > j$  and  $\sum_{k=1}^j a_{j,k} = 1$ . We say a summability matrix  $A$  is a weighted mean matrix if its entries satisfy:

$$a_{j,k} = \lambda_k / \Lambda_j, \quad 1 \leq k \leq j; \quad \Lambda_j = \sum_{i=1}^j \lambda_i, \quad \lambda_i \geq 0, \quad \lambda_1 > 0.$$

Hardy's inequality (1.1) now motivates one to determine the  $l^p$  operator norm of an arbitrary summability matrix  $A$ . For examples, the following two inequalities were claimed to hold by Bennett ([1, p. 40-41]; see also [2, p. 407]):

$$(1.2) \quad \sum_{n=1}^{\infty} \left| \frac{1}{n^\alpha} \sum_{i=1}^n (i^\alpha - (i-1)^\alpha) a_i \right|^p \leq \left( \frac{\alpha p}{\alpha p - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p,$$

$$(1.3) \quad \sum_{n=1}^{\infty} \left| \frac{1}{\sum_{i=1}^n i^{\alpha-1}} \sum_{i=1}^n i^{\alpha-1} a_i \right|^p \leq \left( \frac{\alpha p}{\alpha p - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p,$$

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whenever  $\alpha > 0, p > 1, \alpha p > 1$ .

No proofs of the above two inequalities were supplied in [1]-[2] and recently, the author [4] and Bennett himself [3] proved inequalities (1.2) for  $p > 1, \alpha \geq 1, \alpha p > 1$  and (1.3) for  $p > 1, \alpha \geq 2$  or  $0 < \alpha \leq 1, \alpha p > 1$  independently.

We point out here that Bennett in fact was able to prove (1.2) for  $p \geq 1, \alpha > 0, \alpha p > 1$  (see [3, Theorem 1] with  $\beta = 1$  there) which now leaves the case  $p > 1, 1 < \alpha < 2$  of inequality (1.3) the only case open to us. For this, Bennett expects inequality (1.3) to hold for  $1 + 1/p < \alpha < 2$  (see page 830 of [3]) and as a support, Bennett [3, Theorem 18] has shown that inequality (1.3) holds for  $\alpha = 1 + 1/p, p \geq 1$ .

In this paper, we will study inequality (1.3) using a method of Knopp [6] which involves constructing auxiliary sequences. We will partially resolve the remaining case  $p > 1, 1 < \alpha < 2$  of inequality (1.3) by proving in Section 2 the following:

**Theorem 1.1.** *Inequality (1.3) holds for  $p \geq 2, 1 \leq \alpha \leq 1 + 1/p$  or  $1 < p \leq 4/3, 1 + 1/p \leq \alpha \leq 2$ .*

We shall leave the explanation of Knopp's approach in detail in Section 2 by pointing out here that it can be applied to prove other types of inequalities similar to that of Hardy's. As an example, we note that Theorem 359 of [5] states:

**Theorem 1.2.** *For  $0 < p < 1$  and  $a_n \geq 0$ ,*

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k \right)^p \geq p^p \sum_{n=1}^{\infty} a_n^p.$$

The constant  $p^p$  in Theorem 1.2 is not best possible and this was fixed by Levin and Stečkin [7, Theorem 61] for  $0 < p \leq 1/3$  in the following

**Theorem 1.3.** *For  $0 < p \leq 1/3$  and  $a_n \geq 0$ ,*

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k \right)^p \geq \left( \frac{p}{1-p} \right)^p \sum_{n=1}^{\infty} a_n^p.$$

We shall give another proof of this result in Section 3 using Knopp's approach. We point out here for each  $1/3 < p < 1$ , Levin and Stečkin also gave a better constant than the one  $p^p$  given in Theorem 1.2. For example, when  $p = 1/2$ , they gave  $\sqrt{3}/2$  instead of  $1/\sqrt{2}$ . In Section 4, we shall further improve this constant by proving the following

**Theorem 1.4.** *For  $a_n \geq 0$ ,*

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k \right)^{1/2} \geq \sqrt{7}/3 \sum_{n=1}^{\infty} a_n^{1/2}.$$

In our proofs of Theorems 1.1-1.2, certain auxiliary sequences are constructed and there can be many ways to construct such sequences. In Section 5, we give an example regarding these possibilities by answering a question of Bennett.

## 2. PROOF OF THEOREM 1.1

We begin this section by explaining Knopp's idea [6] on proving Hardy's inequality (1.1). In fact, we will explain this more generally for the case involving weighted mean matrices. For real numbers  $\lambda_1 > 0, \lambda_i \geq 0, i \geq 2$ , we write  $\Lambda_n = \sum_{i=1}^n \lambda_i$  and we are looking for a positive constant  $U$  such that

$$(2.1) \quad \sum_{n=1}^{\infty} \left| \frac{1}{\Lambda_n} \sum_{k=1}^n \lambda_k a_k \right|^p \leq U \sum_{k=1}^{\infty} |a_k|^p$$

holds for all complex sequences  $\mathbf{a}$  with  $p > 1$  being fixed. Knopp's idea is to find an auxiliary sequence  $\mathbf{w} = \{w_i\}_{i=1}^{\infty}$  of positive terms such that by Hölder's inequality,

$$\begin{aligned} \left( \sum_{k=1}^n \lambda_k |a_k| \right)^p &= \left( \sum_{k=1}^n \lambda_k |a_k| w_k^{-\frac{1}{p^*}} \cdot w_k^{\frac{1}{p^*}} \right)^p \\ &\leq \left( \sum_{k=1}^n \lambda_k^p |a_k|^p w_k^{-(p-1)} \right) \left( \sum_{j=1}^n w_j \right)^{p-1} \end{aligned}$$

so that

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \frac{1}{\Lambda_n} \sum_{k=1}^n \lambda_k a_k \right|^p &\leq \sum_{n=1}^{\infty} \frac{1}{\Lambda_n^p} \left( \sum_{k=1}^n \lambda_k^p |a_k|^p w_k^{-(p-1)} \right) \left( \sum_{j=1}^n w_j \right)^{p-1} \\ &= \sum_{k=1}^{\infty} w_k^{-(p-1)} \lambda_k^p \left( \sum_{n=k}^{\infty} \frac{1}{\Lambda_n^p} \left( \sum_{j=1}^n w_j \right)^{p-1} \right) |a_k|^p. \end{aligned}$$

Suppose now one can find for each  $p > 1$  a positive constant  $U$ , a sequence  $\mathbf{w}$  of positive terms with  $w_n^{p-1}/\lambda_n^p$  decreasing to 0, such that for any integer  $n \geq 1$ ,

$$(2.2) \quad (w_1 + \dots + w_n)^{p-1} < U \Lambda_n^p \left( \frac{w_n^{p-1}}{\lambda_n^p} - \frac{w_{n+1}^{p-1}}{\lambda_{n+1}^p} \right),$$

then it is easy to see that inequality (2.1) follows from this. When  $\lambda_n = 1$  for all  $n$ , Knopp's choice for  $\mathbf{w}$  is given by  $w_n = \binom{n-1-1/p}{n-1}$  and one can show that (2.2) holds in this case with  $U = (p^*)^p$  and Hardy's inequality (1.1) follows from this.

We now want to apply Knopp's approach to prove Theorem 1.1. For this, we replace  $\alpha - 1$  by  $\alpha$  and rewrite (1.3) as

$$\sum_{n=1}^{\infty} \left| \frac{1}{\sum_{i=1}^n i^\alpha} \sum_{i=1}^n i^\alpha a_i \right|^p \leq \left( \frac{(\alpha+1)p}{(\alpha+1)p-1} \right)^p \sum_{n=1}^{\infty} |a_n|^p.$$

Note that we are interested in the case  $0 \leq \alpha \leq 1$  here. From our discussions above, we are looking for a sequence  $\mathbf{w}$  of positive terms with  $w_n^{p-1}/\lambda_n^p$  decreasing to 0, such that for any integer  $n \geq 1$ ,

$$(2.3) \quad (w_1 + \dots + w_n)^{p-1} < \left( \frac{(\alpha+1)p}{(\alpha+1)p-1} \right)^p \left( \sum_{i=1}^n i^\alpha \right)^p \left( \frac{w_n^{p-1}}{n^{\alpha p}} - \frac{w_{n+1}^{p-1}}{(n+1)^{\alpha p}} \right).$$

Following Knopp's choice, we define a sequence  $\mathbf{w}$  such that

$$(2.4) \quad w_{n+1} = \frac{n + \alpha - 1/p}{n} w_n, \quad n \geq 1.$$

Note that the above sequence is uniquely determined for any given positive  $w_1$  and therefore we may assume  $w_1 = 1$  here. We note further that we need  $\alpha > -1/p^*$  in order for  $w_n > 0$  for all  $n$  and we also point out that it is easy to show by induction that

$$(2.5) \quad \sum_{i=1}^n w_i = \frac{n + \alpha - 1/p}{1 + \alpha - 1/p} w_n.$$

Moreover, one can easily check that

$$\frac{w_n^{p-1}}{n^{\alpha p}} = O(n^{-\alpha-1/p^*}),$$

so that  $w_n^{p-1}/\lambda_n^p$  decreases to 0 as  $n$  approaches infinity as long as  $\alpha > -1/p^*$ .

Now we need a lemma on sums of powers, which is due to Levin and Stečkin [7, Lemma 1, 2, p.18]:

**Lemma 2.1.** *For an integer  $n \geq 1$ ,*

$$(2.6) \quad \sum_{i=1}^n i^r \geq \frac{1}{r+1} n(n+1)^r, \quad 0 \leq r \leq 1,$$

$$(2.7) \quad \sum_{i=1}^n i^r \geq \frac{r}{r+1} \frac{n^r(n+1)^r}{(n+1)^r - n^r}, \quad r \geq 1.$$

*Inequality (2.7) reverses when  $-1 < r \leq 1$ .*

We note here only the case  $r \geq 0$  for (2.7) was proved in [7] but one checks easily that the proof extends to the case  $r > -1$ .

As we are interested in  $0 \leq \alpha \leq 1$  here, we can now combine (2.4)-(2.6) to deduce that inequality (2.3) will follow from

$$\left(1 + \frac{\alpha - 1/p}{n}\right)^{p-1} < \frac{n}{1 + \alpha - 1/p} \left( \left(1 + \frac{1}{n}\right)^{\alpha p} - \left(1 + \frac{\alpha - 1/p}{n}\right)^{p-1} \right).$$

We can simplify the above inequality further by recasting it as

$$(2.8) \quad \left(1 + \frac{\alpha + 1/p^*}{n}\right)^{1/p} \left(1 + \frac{\alpha - 1/p}{n}\right)^{1/p^*} < \left(1 + \frac{1}{n}\right)^\alpha.$$

Now we define for fixed  $n \geq 1, p > 1$ ,

$$f(x) = x \ln(1 + 1/n) - \frac{1}{p} \ln\left(1 + \frac{x + 1/p^*}{n}\right) - \frac{1}{p^*} \ln\left(1 + \frac{x - 1/p}{n}\right).$$

It is easy to see here that inequality (2.8) is equivalent to  $f(\alpha) > 0$ . It is also easy to see that  $f(x)$  is a convex function of  $x$  for  $0 \leq x \leq 1$  and that  $f(1/p) = 0$ . It follows from this that if  $f'(1/p) \leq 0$  then  $f(x) > 0$  for  $0 \leq x < 1/p$  and if  $f'(1/p) \geq 0$  then  $f(x) > 0$  for  $1/p < x \leq 1$ . We have

$$f'(1/p) = \ln(1 + 1/n) - \frac{1}{n} + \frac{1}{pn(n+1)}.$$

We now use Taylor expansion to conclude for  $x > 0$ ,

$$(2.9) \quad x - x^2/2 < \ln(1 + x) < x - x^2/2 + x^3/3.$$

It follows from this that for  $p \geq 2, n \geq 2$ ,

$$f'(1/p) < -\frac{1}{2n^2} + \frac{1}{3n^3} + \frac{1}{pn(n+1)} \leq -\frac{1}{2n^2} + \frac{1}{3n^3} + \frac{1}{2n(n+1)} = \frac{1}{3n^3} - \frac{1}{2n^2(n+1)} \leq 0.$$

and for  $n = 1$ ,

$$f'(1/p) = \ln 2 - 1 + \frac{1}{2p} \leq \ln 2 - 1 + \frac{1}{4} < 0,$$

It's also easy to check that for  $1 < p \leq 4/3, n = 1$ ,

$$f'(1/p) = \ln 2 - 1 + \frac{1}{2p} > 0.$$

For  $n \geq 2, 1 < p \leq 4/3$ , by using the first inequality of (2.9) we get

$$f'(1/p) > -\frac{1}{2n^2} + \frac{1}{pn(n+1)} \geq 0.$$

This now enables us to conclude the proof of Theorem 1.1.

## 3. ANOTHER PROOF OF THEOREM 1.3

We use the idea of Levin and Stečkin in the proof of Theorem 62 in [7] to find an auxiliary sequence  $\mathbf{w} = \{w_i\}_{i=1}^{\infty}$  of positive terms so that for any finite summation from  $n = 1$  to  $N$  with  $N \geq 1$ , we have

$$\sum_{n=1}^N a_n^p = \sum_{n=1}^N \frac{a_n^p}{\sum_{i=1}^n w_i} \sum_{k=1}^n w_k = \sum_{n=1}^N w_n \sum_{k=n}^N \frac{a_k^p}{\sum_{i=1}^k w_i}.$$

On letting  $N \rightarrow \infty$ , we then have

$$\sum_{n=1}^{\infty} a_n^p = \sum_{n=1}^{\infty} w_n \sum_{k=n}^{\infty} \frac{a_k^p}{\sum_{i=1}^k w_i}.$$

By Hölder's inequality, we have

$$\sum_{k=n}^{\infty} \frac{a_k^p}{\sum_{i=1}^k w_i} \leq \left( \sum_{k=n}^{\infty} \left( \sum_{i=1}^k w_i \right)^{-1/(1-p)} \right)^{1-p} \left( \sum_{k=n}^{\infty} a_k \right)^p.$$

Suppose now one can find a sequence  $\mathbf{w}$  of positive terms with  $w_n^{-1/(1-p)} n^{-p/(1-p)}$  decreasing to 0 for each  $0 < p \leq 1/3$ , such that for any integer  $n \geq 1$ ,

$$(3.1) \quad (w_1 + \cdots + w_n)^{-1/(1-p)} \leq \left( \frac{1-p}{p} \right)^{p/(1-p)} \left( \frac{w_n^{-1/(1-p)}}{n^{p/(1-p)}} - \frac{w_{n+1}^{-1/(1-p)}}{(n+1)^{p/(1-p)}} \right),$$

then it is easy to see that Theorem 1.3 follows from this.

We now define our sequence  $\mathbf{w}$  to be

$$(3.2) \quad w_{n+1} = \frac{n+1/p-2}{n} w_n, \quad n \geq 1.$$

Note that the above sequence is uniquely determined for any given positive  $w_1$  and therefore we may assume  $w_1 = 1$  here. We note further that  $w_n > 0$  for all  $n$  as  $0 < p \leq 1/3$  and it is easy to show by induction that

$$(3.3) \quad \sum_{i=1}^n w_i = \frac{n+1/p-2}{1/p-1} w_n.$$

Moreover, one can easily check that

$$\frac{w_n^{-1/(1-p)}}{n^{p/(1-p)}} = O(n^{-(1-p)/p}),$$

so that  $w_n^{-1/(1-p)} n^{-p/(1-p)}$  decreases to 0 as  $n$  approaches infinity.

We now combine (3.2)-(3.3) to recast inequality (3.1) as

$$(n+1/p-2)^{-1/(1-p)} \leq \frac{p}{1-p} \left( n^{-p/(1-p)} - (n+1)^{-p/(1-p)} n^{1/(1-p)} (n+1/p-2)^{-1/(1-p)} \right).$$

We further rewrite the above inequality as

$$\begin{aligned} \frac{1-p}{p} &\leq n^{-p/(1-p)} (n+1/p-2)^{1/(1-p)} - (n+1)^{-p/(1-p)} n^{1/(1-p)} \\ &= n \left( \left( 1 + \frac{1/p-2}{n} \right)^{1/(1-p)} - \left( 1 + \frac{1}{n} \right)^{-p/(1-p)} \right). \end{aligned}$$

It is easy to see that the above inequality follows from  $f(1/n) \geq 0$  where we define for  $x \geq 0$ ,

$$f(x) = \left( 1 + (1/p-2)x \right)^{1/(1-p)} - \left( 1 + x \right)^{-p/(1-p)} - \frac{1-p}{p} x.$$

We now prove that  $f(x) \geq 0$  for  $x \geq 0$  for  $0 < p \leq 1/3$  and this will conclude the proof of Theorem 1.3. We note that

$$\begin{aligned} f'(x) &= \frac{1/p-2}{1-p} \left(1 + (1/p-2)x\right)^{p/(1-p)} + \frac{p}{1-p} \left(1+x\right)^{-p/(1-p)-1} - \frac{1-p}{p}, \\ f''(x) &= \frac{p(1/p-2)^2}{(1-p)^2} \left(1 + (1/p-2)x\right)^{p/(1-p)-1} - \frac{p}{(1-p)^2} \left(1+x\right)^{-p/(1-p)-2}. \end{aligned}$$

We now define for  $x \geq 0$ ,

$$g(x) = (1/p-2)^{2(1-p)/(1-2p)} (1+x)^{(2-p)/(1-2p)} - (1 + (1/p-2)x).$$

It is easy to see that  $g(x) \geq 0$  implies  $f''(x) \geq 0$ . Note that  $(2-p)/(1-2p) \geq 1$  so that

$$\begin{aligned} g'(x) &= (1/p-2)^{2(1-p)/(1-2p)} (2-p)/(1-2p) (1+x)^{(2-p)/(1-2p)-1} - (1/p-2) \\ &\geq (1/p-2)^{2(1-p)/(1-2p)} - (1/p-2) \geq 0, \end{aligned}$$

where the last inequality above follows from  $2(1-p)/(1-2p) \geq 1$  and  $0 < p \leq 1/3$  so that  $1/p-2 \geq 1$ . It follows from this that  $f''(x) \geq 0$  and as one checks easily that  $f'(0) = 0$ , which implies  $f'(x) \geq 0$  so that  $f(x) \geq f(0) = 0$  which is just what we want to prove.

#### 4. PROOF OF THEOREM 1.4

We follow our strategy in the previous section to look for a sequence  $\mathbf{w}$  of positive terms with  $w_n^{-2}n^{-1}$  decreasing to 0, such that for any integer  $n \geq 1$ ,

$$(4.1) \quad (w_1 + \cdots + w_n)^{-2} \leq C \left( \frac{w_n^{-2}}{n} - \frac{w_{n+1}^{-2}}{(n+1)} \right),$$

then it is easy to see that Theorem 1.4 with  $\sqrt{7}/3$  replaced by  $C^{-1/2}$  follows from this.

We now define our sequence  $\mathbf{w}$  to be

$$(4.2) \quad w_{n+1} = \frac{n+\alpha}{n} w_n, \quad n \geq 1.$$

Here  $\alpha > -1/2$  is our parameter so that we hope to optimize the constant  $C$  in (4.1) later by choosing a suitable  $\alpha$ . Similar to our treatment in the previous section, we let  $w_1 = 1$  here and note that  $w_n > 0$  for all  $n$  and that

$$(4.3) \quad \sum_{i=1}^n w_i = \frac{n+\alpha}{1+\alpha} w_n.$$

Moreover, one can easily check that

$$\frac{w_n^{-2}}{n} = O(n^{-1-2\alpha}),$$

so that  $w_n^{-2}n^{-1}$  decreases to 0 as  $n$  approaches infinity.

We now combine (4.2)-(4.3) to recast inequality (4.1) as

$$\frac{(1+\alpha)^2}{C} \leq \frac{(n+\alpha)^2}{n} - \frac{n^2}{n+1}.$$

Or equivalently,

$$(4.4) \quad (1+2\alpha)n^2 + (2\alpha+\alpha^2)n + \alpha^2 \geq \frac{(1+\alpha)^2}{C} n(n+1).$$

We point out here that the choice  $\alpha = 0$  will lead to  $C \geq 2$  in order for inequality (4.4) to hold for all  $n \geq 1$ , which corresponds Theorem 1.2 with  $\sqrt{7}/3$  replaced by  $2^{-1/2}$ . The choice  $\alpha = 1$  will lead to  $C \geq 4/3$  in order for inequality (4.4) to hold for all  $n \geq 1$ , which corresponds Theorem 1.2 with

$\sqrt{7}/3$  replaced by  $\sqrt{3}/2$ . We now choose  $\alpha = 1/2$  here with  $C = 9/7$  and one checks readily that inequality (4.4) holds for such choices and this leads to the desired constant  $\sqrt{7}/3$  in Theorem 1.2.

### 5. ANOTHER LOOK AT INEQUALITY (1.3)

In this section we return to the consideration of inequality (1.3) via our approach in Section 2, which boils down to a construction of a sequence  $\mathbf{w}$  of positive terms with  $w_n^{p-1}/\lambda_n^p$  decreasing to 0, such that for any integer  $n \geq 1$ , inequality (2.3) is satisfied. Certainly here the choice for  $\mathbf{w}$  may not be unique and in fact in the case  $\alpha = 0$ , Bennett asked in [1] (see the paragraph below Lemma 4.11) for other sequences, not multiples of Knopp's, that satisfy (2.3). He also mentioned that the obvious choice,  $w_n = n^{-1/p}$ , does not work.

We point out here even though the choice  $w_n = n^{-1/p}$  does not satisfy (2.3) when  $\alpha = 0$  for all  $p > 1$ , as one can see by considering inequality (2.3) for the case  $n = 1$  with  $p \rightarrow 1^+$ , it nevertheless works for  $p \geq 3$ , which we now show by first rewriting (2.3) in our case as

$$(5.1) \quad \left( \sum_{i=1}^n i^{-1/p} \right)^{p-1} < \left( \frac{p}{p-1} \right)^p n^p \left( n^{-(p-1)/p} - (n+1)^{-(p-1)/p} \right).$$

We note that the case  $n = 1$  of (5.1) follows from the case  $\alpha = 0$  of the following inequality,

$$(5.2) \quad 1 - 2^{-(p-1)/p-\alpha} > \left( 1 - \frac{1}{(\alpha+1)p} \right)^p, \quad 0 \leq \alpha \leq 1/p.$$

To show (5.2), we see by Taylor expansion, that for  $p \geq 2, x < 0$ ,

$$(1+x)^p < 1 + px + \frac{p(p-1)x^2}{2}.$$

Apply the above inequality with  $x = -1/(\alpha p + p)$ , we obtain for  $p \geq 3$ ,

$$\left( 1 - \frac{1}{(\alpha+1)p} \right)^p < 1 - \frac{1}{(\alpha+1)} + \frac{(p-1)}{2(\alpha+1)^2 p}.$$

Hence inequality (5.2) will follow from

$$1 - \frac{p-1}{2(\alpha+1)p} - 2^{-(p-1)/p} \frac{(\alpha+1)}{2^\alpha} > 0.$$

It is easy to see that when  $p \geq 3$ , the function  $\alpha \mapsto (1+\alpha)2^{-\alpha}$  is an increasing function of  $\alpha$  for  $0 \leq \alpha \leq 1/p$ . It follows from this that for  $0 \leq \alpha \leq 1/p$ ,

$$1 - \frac{p-1}{2(\alpha+1)p} - 2^{-(p-1)/p} \frac{(\alpha+1)}{2^\alpha} > 1 - \frac{p-1}{2p} - 2^{-(p-1)/p} \frac{(1/p+1)}{2^{1/p}} = 0,$$

and from which inequality (5.2) follows.

Now, to show (5.1) holds for all  $n \geq 2, p \geq 3$ , we first note that for  $p > 1$ ,

$$\sum_{i=1}^n i^{-1/p} < 1 + \int_1^n x^{-1/p} dx = \frac{p}{p-1} n^{1-1/p} - \frac{1}{p-1}.$$

On the other hand, by Hadamard's inequality, which asserts that for a continuous convex function  $f(x)$  on  $[a, b]$ ,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2},$$

we have for  $p > 1$ ,

$$n^{-(p-1)/p} - (n+1)^{-(p-1)/p} = \frac{p-1}{p} \int_n^{n+1} x^{-1-1/p^*} dx \geq \frac{p-1}{p} (n+1/2)^{-1-1/p^*}.$$



Hence inequality (5.1) will follow from the following inequality for  $n \geq 2$ ,

$$\frac{p}{p-1}n^{1-1/p} - \frac{1}{p-1} \leq p^*n^{1/p^*} \left(1 + \frac{1}{2n}\right)^{-(1+p^*)/p}.$$

It is easy to see that for  $p > 1$ ,

$$\left(1 + \frac{1}{2n}\right)^{-(1+p^*)/p} \geq 1 - \frac{1+p^*}{p} \frac{1}{2n}.$$

Hence it suffices to show

$$\frac{p}{p-1}n^{1-1/p} - \frac{1}{p-1} \leq p^*n^{1/p^*} \left(1 - \frac{1+p^*}{p} \frac{1}{2n}\right),$$

or equivalently,

$$\left(1 + \frac{1}{2p-2}\right)^p \leq n.$$

It's easy to check that the right-hand expression above is a decreasing function of  $p \geq 3$  and is equal to  $5^3/4^3 < 2$  when  $p = 3$ . Hence it follows that (5.1) holds for all  $n \geq 2, p \geq 3$ .

We consider lastly inequality (2.3) for other values of  $\alpha$  and we take  $w_n = n^{\alpha-1/p}$  for  $n \geq 1$  so that we can rewrite (2.3) as

$$(5.3) \quad \left(\sum_{i=1}^n i^{\alpha-1/p}\right)^{p-1} < \left(\frac{(\alpha+1)p}{(\alpha+1)p-1}\right)^p \left(\sum_{i=1}^n i^\alpha\right)^p \left(n^{-(p-1)/p-\alpha} - (n+1)^{-(p-1)/p-\alpha}\right).$$

We end our discussion here by considering the case  $1 \leq \alpha \leq 1 + 1/p$  and we apply Lemma 2.1 to obtain

$$\begin{aligned} \sum_{i=1}^n i^{\alpha-1/p} &\leq \frac{\alpha-1/p}{\alpha-1/p+1} \frac{n^{\alpha-1/p}(n+1)^{\alpha-1/p}}{(n+1)^{\alpha-1/p} - n^{\alpha-1/p}} = \frac{1}{\alpha-1/p+1} \left(\int_n^{n+1} x^{-\alpha+1/p-1} dx\right)^{-1}, \\ \sum_{i=1}^n i^\alpha &\geq \frac{\alpha}{\alpha+1} \frac{n^\alpha(n+1)^\alpha}{(n+1)^\alpha - n^\alpha} = \frac{1}{\alpha+1} \left(\int_n^{n+1} x^{-\alpha-1} dx\right)^{-1} \end{aligned}$$

We further write

$$n^{-(p-1)/p-\alpha} - (n+1)^{-(p-1)/p-\alpha} = (\alpha-1/p+1) \int_n^{n+1} x^{-\alpha+1/p-2} dx,$$

so that inequality (5.3) will follow from

$$\int_n^{n+1} x^{-\alpha-1} dx < \left(\int_n^{n+1} x^{-\alpha+1/p-1} dx\right)^{1-1/p} \left(\int_n^{n+1} x^{-\alpha+1/p-2} dx\right)^{1/p}.$$

One can easily see that the above inequality holds by Hölder's inequality and it follows that inequality (5.3) holds for  $p > 1, 1 \leq \alpha \leq 1 + 1/p$ . This provides another proof of inequality (1.3) for  $p > 1, 1 \leq \alpha \leq 1 + 1/p$ .

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DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES, UNIVERSITY OF TORONTO AT SCARBOROUGH, 1265 MILITARY TRAIL, TORONTO ONTARIO, CANADA M1C 1A4

E-mail address: penggao@utsc.utoronto.ca