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Abstract. In the article, the logarithmic mean is proved to be completely monotonic and an open problem about the logarithmically complete monotonicity of the extended mean values is posed.

1. Introduction

Recall [11, 28] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \ge 0$ for $x \in I$ and $n \ge 0$. Recall [2] that if $f^{(k)}(x)$ for some nonnegative integer k is completely monotonic on an interval I, but $f^{(k-1)}(x)$ is not completely monotonic on I, then $f(x)$ is called a completely monotonic function of k-th order on an interval I. Recall also [17, 18, 20] that a function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies $(-1)^k [\ln f(x)]^{(k)} \ge 0$ for $k \in \mathbb{N}$ on I. It has been proved in [3, 10, 17, 18] and other references that a logarithmically completely monotonic function on an interval I is also completely monotonic on I. The logarithmically completely monotonic functions have close relationships with both the completely monotonic functions and Stieltjes transforms. For detailed information, please refer to [3, 10, 11, 21, 28] and the references therein.

For two positive numbers a and b, the logarithmic mean $L(a,b)$ is defined by

$$L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a}, & a \ne b; \\ a, & a = b. \end{cases} \tag{1}$$

This is one of the most important means of two positive variables. See [4, 6, 12, 16] and the list of references therein. It is cited on 13 pages at least in [4], see [4, p. 532]. However, any complete monotonicity on mean values is not founded in the authoritative book [4].

The main aim of this paper is to prove the complete monotonicity of the logarithmic mean L.

Our main result is as follows.

Theorem 1. *The logarithmic mean $L_{s,t}(x) = L(x+s, x+t)$ is a completely monotonic function of first order in $x > -\min\{s,t\}$ for $s,t \in \mathbb{R}$ with $s \ne t$.*

As by-product of the proof of Theorem 1, the following logarithmically completely monotonic property of the function $(x+s)^{1-u}(x+t)^u$ for $s,t \in \mathbb{R}$ with $s \ne t$ and $u \in (0,1)$ is deduced.

2000 *Mathematics Subject Classification.* 26A48, 26A51.

Key words and phrases. completely monotonic function, logarithmically completely monotonic function, logarithmic mean, extended mean values.

This paper was typeset using $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX.

Corollary 1. *Let $s,t\in\mathbb{R}$ with $s\ne t$ and $u\in(0,1)$. Then $(x+s)^{1-u}(x+t)^u$ is a completely monotonic function of first order in $x>-\min\{s,t\}$. More strongly, the function $\frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x}=\bigl(\frac{x+t}{x+s}\bigr)^u\bigl[1+\frac{u(s-t)}{x+t}\bigr]$ is logarithmically completely monotonic in $x>-\min\{s,t\}$.*

The extended mean values $E(r,s;x,y)$ can be defined by

$$\begin{aligned}
E(r,s;x,y)&=\left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)}, && rs(r-s)(x-y)\ne0;\\
E(r,0;x,y)&=\left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{1/r}, && r(x-y)\ne0;\\
E(r,r;x,y)&=\frac{1}{e^{1/r}}\left[\frac{x^{x^r}}{y^{y^r}}\right]^{1/(x^r-y^r)}, && r(x-y)\ne0;\\
E(0,0;x,y)&=\sqrt{xy}, && x\ne y;\\
E(r,s;x,x)&=x, && x=y;
\end{aligned}$$

where x and y are positive numbers and $r,s\in\mathbb{R}$. Its monotonicity, Schur-convexity, logarithmic convexity, comparison, generalizations, applications and history have been investigated in many articles such as [4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 19, 22, 23, 24, 25, 26, 27, 29, 30, 31] and the references therein, especially the book [4] and the expository paper [16].

For $x,y>0$ and $r,s\in\mathbb{R}$, let $E^{[1]}_{r,s;x,y}(w)=E(r+w,s+w;x,y)$ with $w\in\mathbb{R}$, $E^{[2]}_{r,s;x,y}(w)=E(r,s;x+w,y+w)$ and $E^{[3]}_{r,s;x,y}(w)=E(r+w,s+w;x+w,y+w)$ with $w>-\min\{x,y\}$. Motivated by Theorem 1, it is natural to pose an open problem: What about the (logarithmically) complete monotonicity of the functions $E^{[i]}_{r,s;x,y}(w)$ in w for $1\le i\le3$?

2. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. In [4, p. 386], an integral representation of the logarithmic mean $L(a,b)$ for positive numbers a and b is given:

$$L(a,b)=\int_0^1 a^{1-u}b^u\,\mathrm{d}u. \tag{2}$$

From this, it follows easily that

$$L_{s,t}(x)=\int_0^1(x+s)^{1-u}(x+t)^u\,\mathrm{d}u \tag{3}$$

and

$$\frac{\mathrm{d}L_{s,t}(x)}{\mathrm{d}x}=\int_0^1\left(\frac{x+t}{x+s}\right)^u\frac{x+(1-u)t+us}{x+t}\,\mathrm{d}u>0. \tag{4}$$

This means that the function $L_{s,t}(x)$ is increasing, and then it is not completely monotonic in $x>-\min\{s,t\}$.

In [1, p. 230, 5.1.32], it is listed that

$$\ln\frac{b}{a}=\int_0^\infty\frac{e^{-au}-e^{-bu}}{u}\,\mathrm{d}u. \tag{5}$$

Taking logarithm on both sides of equation (4) and utilizing (5) yields

$$\begin{aligned}\ln\frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x}&=u\ln\frac{x+t}{x+s}+\ln\frac{x+(1-u)t+us}{x+t}\\&=u\int_0^\infty\frac{e^{-(x+s)v}-e^{-(x+t)v}}{v}\,\mathrm{d}v+\int_0^\infty\frac{e^{-(x+t)v}-e^{-[x+(1-u)t+us]v}}{v}\,\mathrm{d}v\\&=\int_0^\infty\frac{ue^{-(x+s)v}+(1-u)e^{-(x+t)v}-e^{-[x+(1-u)t+us]v}}{v}\,\mathrm{d}v.\end{aligned}$$

Employing the well known Jensen's inequality [4, p. 31, Theorem 12] for convex functions and considering that the function e^{-x} is convex gives

$$q_{s,t;u;v}(x)\triangleq ue^{-(x+s)v}+(1-u)e^{-(x+t)v}-e^{-[x+(1-u)t+us]v}>0. \tag{6}$$

Hence, for positive integer $m\in\mathbb{N}$,

$$(-1)^m\frac{\partial^m}{\partial x^m}\left\{\ln\frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x}\right\}=\int_0^\infty v^{m-1}q_{s,t;u;v}(x)\,\mathrm{d}v>0. \tag{7}$$

This implies that the function $\frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x}$ is logarithmically completely monotonic in $x>-\min\{s,t\}$. Further, since a logarithmically completely monotonic function is also completely monotonic (see [3, 10, 11, 17, 18, 20, 21] and the references therein), the function $\frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x}$ is completely monotonic in $x>-\min\{s,t\}$. Therefore, the function

$$\frac{\mathrm{d}L_{s,t}(x)}{\mathrm{d}x}=\int_0^1\frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x}\,\mathrm{d}u \tag{8}$$

is completely monotonic in $x>-\min\{s,t\}$. Theorem 1 is proved. □

Proof of Corollary 1. This follows from the proof of Theorem 1 directly. □

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