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COMPLETE MONOTONICITY OF LOGARITHMIC MEAN

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Abstract. In the article, the logarithmic mean is proved to be completely monotonic and an open problem about the logarithmically complete monotonicity of the extended mean values is posed.

1. Introduction

Recall [11, 28] that a function \( f \) is said to be completely monotonic on an interval \( I \) if \( f \) has derivatives of all orders on \( I \) and \((-1)^n f^{(n)}(x) \geq 0 \) for \( x \in I \) and \( n \geq 0 \).

Recall [2] that if \( f^{(k)}(x) \) for some nonnegative integer \( k \) is completely monotonic on an interval \( I \), but \( f^{(k-1)}(x) \) is not completely monotonic on \( I \), then \( f(x) \) is called a completely monotonic function of \( k \)-th order on an interval \( I \). Recall also [17, 18, 20] that a function \( f \) is said to be logarithmically completely monotonic on an interval \( I \) if its logarithm \( \ln f \) satisfies \((-1)^k[\ln f(x)]^{(k)} \geq 0 \) for \( k \in \mathbb{N} \) on \( I \). It has been proved in [3, 10, 17, 18] and other references that a logarithmically completely monotonic function on an interval \( I \) is also completely monotonic on \( I \). The logarithmically completely monotonic functions have close relationships with both the completely monotonic functions and Stieltjes transforms. For detailed information, please refer to [3, 10, 11, 21, 28] and the references therein.

For two positive numbers \( a \) and \( b \), the logarithmic mean \( L(a, b) \) is defined by

\[
L(a, b) = \begin{cases} 
\frac{b - a}{\ln b - \ln a}, & a \neq b; \\
\frac{a}{a}, & a = b.
\end{cases}
\]  

This is one of the most important means of two positive variables. See [4, 6, 12, 16] and the list of references therein. It is cited on 13 pages at least in [4], see [4, p. 532]. However, any complete monotonicity on mean values is not founded in the authoritative book [4].

The main aim of this paper is to prove the complete monotonicity of the logarithmic mean \( L \).

Our main result is as follows.

Theorem 1. The logarithmic mean \( L_{s,t}(x) = L(x+s, x+t) \) is a completely monotonic function of first order in \( x > \min\{s, t\} \) for \( s, t \in \mathbb{R} \) with \( s \neq t \).

As by-product of the proof of Theorem 1 the following logarithmically completely monotonic property of the function \( (x+s)^{1-u}(x+t)^u \) for \( s, t \in \mathbb{R} \) with \( s \neq t \) and \( u \in (0, 1) \) is deduced.

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Corollary 1. Let \(s, t \in \mathbb{R}\) with \(s \neq t\) and \(u \in (0, 1)\). Then \((x + s)^{1-u}(x + t)^u\) is a completely monotonic function of first order in \(x > -\min\{s, t\}\). More strongly, the function \(\frac{\partial ((x + s)^{1-u}(x + t)^u)}{\partial x} = (\frac{x+t}{x+s})^u \left[1 + \frac{u(s-t)}{x+t}\right]\) is logarithmically completely monotonic in \(x > -\min\{s, t\}\).

The extended mean values \(E(r, s; x, y)\) can be defined by:

\[
E(r, s; x, y) = \left[ \frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r} \right]^{1/(s-r)}, \quad rs(r-s)(x-y) \neq 0;
\]

\[
E(r, 0; x, y) = \left[ 1 + \frac{1}{r} \cdot \frac{y^r - x^r}{\ln y - \ln x} \right]^{1/r}, \quad r(x-y) \neq 0;
\]

\[
E(r, r; x, y) = \frac{1}{e^{1/r}} \left[ \frac{x^{(1/r)-y^{(1/r)}}}{y^{(1/r)}} \right], \quad r(x-y) \neq 0;
\]

\[
E(0, 0; x, y) = \sqrt{xy}, \quad x \neq y;
\]

\[
E(r, s; x, x) = x, \quad x = y;
\]

where \(x\) and \(y\) are positive numbers and \(r, s \in \mathbb{R}\). Its monotonicity, Schur-convexity, logarithmic convexity, comparison, generalizations, applications and history have been investigated in many articles such as [4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 19, 22, 23, 24, 25, 26, 27, 29, 30, 31] and the references therein, especially the book [4] and the expository paper [16].

For \(x, y > 0\) and \(r, s \in \mathbb{R}\), let \(E_{r, s; x, y}^{[1]}(w) = E(r + w, s + w; x, y)\) with \(w \in \mathbb{R}\), \(E_{r, x, y}^{[2]}(w) = E(r, s; x + w, y + w)\) and \(E_{r, x, y}^{[3]}(w) = E(r + w, s + w; x + w, y + w)\) with \(w > -\min\{x, y\}\). Motivated by Theorem 1, it is natural to pose an open problem: What about the (logarithmically) complete monotonicity of the functions \(E_{r, s; x, y}^{[i]}(w)\) in \(w\) for \(1 \leq i \leq 3\)?

2. Proofs of Theorem 1 and Corollary 1

Proof of Theorem [7]. In [4] p. 386, an integral representation of the logarithmic mean \(L(a, b)\) for positive numbers \(a\) and \(b\) is given:

\[
L(a, b) = \int_0^1 a^{1-u}b^u \, du. \tag{2}
\]

From this, it follows easily that

\[
L_{s,t}(x) = \int_0^1 (x + s)^{1-u}(x + t)^u \, du \tag{3}
\]

and

\[
\frac{dL_{s,t}(x)}{dx} = \int_0^1 \left( \frac{x + t}{x + s} \right)^u \frac{x + us}{x + t} \, du > 0. \tag{4}
\]

This means that the function \(L_{s,t}(x)\) is increasing, and then it is not completely monotonic in \(x > -\min\{s,t\}\).

In [11] p. 230, 5.1.32, it is listed that

\[
\ln \frac{b}{a} = \int_0^\infty e^{-au} - e^{-bu} \, du. \tag{5}
\]
Taking logarithm on both sides of equation (4) and utilizing (5) yields
\[
\ln \frac{\partial \left((x + s)^{1-u} (x + t)^u\right)}{\partial x} = u \ln \frac{x + t}{x + s} + \ln \frac{x + (1 - u)t + us}{x + t} = \int_0^\infty \frac{e^{-(x+s)v} - e^{-(x+t)v}}{v} dv + \int_0^\infty \frac{e^{-(x+t)v} - e^{-(x+(1-u)t+us)v}}{v} dv.
\]
Employing the well known Jensen's inequality [4, p. 31, Theorem 12] for convex functions and considering that the function \(e^{-x}\) is convex gives
\[q_{s,t;u,v}(x) = ue^{-(x+s)v} + (1-u)e^{-(x+t)v} - e^{-(x+(1-u)t+us)v} > 0.\]
Hence, for positive integer \(m \in \mathbb{N}\),
\[(-1)^m \frac{\partial^m}{\partial x^m} \ln \frac{\partial \left((x + s)^{1-u} (x + t)^u\right)}{\partial x} = \int_0^\infty v^{m-1}q_{s,t;u,v}(x) \, dv > 0.\]
This implies that the function \(\frac{\partial \left((x + s)^{1-u} (x + t)^u\right)}{\partial x}\) is logarithmically completely monotonic in \(x > -\min\{s,t\}\). Further, since a logarithmically completely monotonic function is also completely monotonic (see [3, 10, 11, 17, 18, 20, 21] and the references therein), the function \(\frac{\partial \left((x + s)^{1-u} (x + t)^u\right)}{\partial x}\) is completely monotonic in \(x > -\min\{s,t\}\). Therefore, the function
\[\frac{dL_{s,t}(x)}{dx} = \int_0^1 \frac{\partial \left((x + s)^{1-u} (x + t)^u\right)}{\partial x} \, du\]
is completely monotonic in \(x > -\min\{s,t\}\). Theorem 1 is proved. \(\square\)

**Proof of Corollary 1.** This follows from the proof of Theorem 1 directly. \(\square\)

**References**


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