Wendel-Gautschi-Kershaw's Inequalities and Sufficient and Necessary Conditions that a Class of Functions Involving Ratio of Gamma Functions are Logarithmically Completely Monotonic

WENDEL-GAUTSCHI-KERSHAW'S INEQUALITIES AND SUFFICIENT AND NECESSARY CONDITIONS THAT A CLASS OF FUNCTIONS INVOLVING RATIO OF GAMMA FUNCTIONS ARE LOGARITHMICALLY COMPLETELY MONOTONIC

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ABSTRACT. In the article, sufficient and necessary conditions that a class of functions involving ratio of Euler's gamma functions and originating from Wendel-Gautschi-Kershaw's double inequalities are logarithmically completely monotonic are presented. From this, Wendel-Gautschi-Kershaw's double inequalities are refined, extended and sharpened.

1. INTRODUCTION

In order to establish the classical asymptotic relation $\lim_{x\to\infty}\frac{\Gamma(x+a)}{x^a\Gamma(x)}=1$ for real a and x, using Hölder's integral inequality, the following double inequality was proved in [41]:

$$\left(\frac{x}{x+a}\right)^{1-a}\le\frac{\Gamma(x+a)}{x^a\Gamma(x)}\le 1 \tag{1}$$

for $0<a<1$ and $x>0$, where $\Gamma(x)$ denotes the well known classical Euler's gamma function Γ defined for $x>0$ as $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t$. This inequality can be rewritten for $0<a<1$ and $x>0$ as

$$(x+a)^{1-a}\ge\frac{\Gamma(x+1)}{\Gamma(x+a)}\ge x^{1-a}. \tag{2}$$

In [11], along with another line, the following two double inequalities were established for $n\in\mathbb{N}$ and $0\le s\le 1$:

$$\exp[(1-s)\psi(n+1)]\ge\frac{\Gamma(n+1)}{\Gamma(n+s)}\ge n^{1-s} \tag{3}$$

and

$$(n+1)^{1-s}\ge\frac{\Gamma(n+1)}{\Gamma(n+s)}\ge n^{1-s}. \tag{4}$$

It is clear that the upper bound in inequality (4) is not better and the range in inequality (4) is not larger than the corresponding ones in (1) or (2).

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Motivated by the paper [11], among other things, the following double inequality was showed for $0 < s < 1$ and $x \ge 1$ in [14]:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{1-s}. \tag{5}$$

It is easy to see that inequality (5) refines inequalities (1), (2), (4) and the left hand side inequality in (3).

Recall [4, 10, 23, 33, 40] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \ge 0$ for $x \in I$ and $n \ge 0$. Recall also [2, 31, 33, 34, 35] that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0 \le (-1)^k[\ln f(x)]^{(k)} < \infty$ for all $k \in \mathbb{N}$ on I. It has been presented explicitly in [4, 22, 31, 33, 37] that a logarithmically completely monotonic function must be completely monotonic, but not conversely. In [4, Theorem 1.1] and [12] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [13, Theorem 4.4]. In recent years, the notion "logarithmically completely monotonic function" has been adopted in many articles such as [4, 7, 8, 9, 32, 12, 16, 17, 18, 19, 23, 28, 30, 34, 35, 38, 39, 42] and the references therein.

Inequality (5) has been investigated along with two directions.

A standard argument shows that inequality (5) can be rearranged as

$$\frac{s}{2} < \left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)} - x < \sqrt{s+\frac{1}{4}} - \frac{1}{2}. \tag{6}$$

Therefore, the first direction is to consider the monotonicity of the general function

$$z_{s,t}(x) = \begin{cases} \left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x, & s \ne t \\ e^{\psi(x+s)} - x, & s = t \end{cases} \tag{7}$$

in $x \in (-\alpha,\infty)$, where s and t are two real numbers and $\alpha = \min\{s,t\}$. In [6, 10, 20, 21, 27, 36], it was obtained that the function $z_{s,t}(x)$ is either convex and decreasing for $|t-s| < 1$ or concave and increasing for $|t-s| > 1$.

The second direction is to consider the monotonicity, complete monotonicity or logarithmically complete monotonicity of the function

$$H_{a,b,c}(x) = (x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{8}$$

for $x \in (-\rho,\infty)$, where a, b and c are real numbers and $\rho = \min\{a,b,c\}$. It is clear that $\frac{1}{H_{a,b,c}(x)} = H_{b,a,c}(x)$. In [5, Theorem 1 and Theorem 3] it was revealed for $a+1 \ge b > a$ that $H_{b,a,c}(x)$ is completely monotonic in $(\max\{-a,-c\},\infty)$ if $c \le \frac{a+b-1}{2}$ and that $H_{a,b,c}(x)$ is completely monotonic in $(\max\{-b,-c\},\infty)$ if $c \ge a$. In [5, Theorem 7] it was demonstrated that $H_{1,s,s/2}(x)$ for $0 \le s \le 1$ is completely monotonic in $(0,\infty)$. In [5, Theorem 8], it was concluded that $H_{s,1,\sqrt{s+1/4}-1/2}(x)$ for $0 < s < 1$ is strictly decreasing in $(0,\infty)$. With the help of [24, Corollary 1] which iterated [24, Theorem 1] on monotonicity results of the function

$$q_{\alpha,\beta}(t)=\begin{cases}\dfrac{e^{-\alpha t}-e^{-\beta t}}{1-e^{-t}}, & t\ne 0\\ \beta-\alpha, & t=0\end{cases}\tag{9}$$

for real numbers α and β with $(\alpha,\beta)\notin\{(0,1),(1,0)\}$ and $\alpha\ne\beta$, the logarithmically complete monotonicity of the function (8) were established in [18, Theorem 1] and the references therein:

(1) $H_{a,b,c}(x)$ is logarithmically completely monotonic in $(-\rho,\infty)$ if

$$\begin{aligned}(a,b,c)\in&\left\{a+b\ge 1, c\le b<c+\frac12\right\}\cup\left\{a>b\ge c+\frac12\right\}\\ &\cup\{2a+1\le a+b\le 1, a<c\}\cup\{b-1\le a<b\le c\}\setminus\{a=b+1=c+1\},\end{aligned}\tag{10}$$

(2) $H_{b,a,c}(x)$ is logarithmically completely monotonic in $(-\rho,\infty)$ if

$$\begin{aligned}(a,b,c)\in&\left\{a+b\ge 1, c\le a<c+\frac12\right\}\cup\left\{b>a\ge c+\frac12\right\}\\ &\cup\{b<a\le c\}\cup\{b+1\le a, c\le a\le c+1\}\\ &\cup\{b+c+1\le a+b\le 1\}\setminus\{a=c+1, b=c\}\setminus\{b=c+1, a=c\}.\end{aligned}\tag{11}$$

The monotonicity and logarithmic convexity of $q_{\alpha,\beta}(t)$ have been researched entirely in the papers [24, 25, 29, 36], since it was encountered occasionally when studying the logarithmically complete monotonicity of some functions involving gamma function Γ, the psi function ψ and the polygamma functions $\psi^{(i)}$ for $i\in\mathbb{N}$.

The monotonicity of $q_{\alpha,\beta}(t)$ in $(0,\infty)$ obtained in [24, Theorem 1] and referenced in [29] can be restated accurately and simply in [25, Corollary 2] as follows.

Proposition 1 ([25, Corollary 2])**.** *Let α and β be two real numbers satisfying $\alpha\ne\beta$ and $(\alpha,\beta)\notin\{(0,1),(1,0)\}$ and $t\in\mathbb{R}$. Then*

(1) *the function $q_{\alpha,\beta}(t)$ defined by* (9) *is increasing in $(0,\infty)$ if and only if*

$$\begin{aligned}(\alpha,\beta)\in D_1(\alpha,\beta)\triangleq\{(\alpha,\beta):(\beta-\alpha)(1-\alpha-\beta)\ge 0,&\\ (\beta-\alpha)(|\alpha-\beta|-\alpha-\beta)\ge 0\},&\end{aligned}\tag{12}$$

(2) *the function $q_{\alpha,\beta}(t)$ defined by* (9) *is decreasing in $(0,\infty)$ if and only if*

$$\begin{aligned}(\alpha,\beta)\in D_2(\alpha,\beta)\triangleq\{(\alpha,\beta):(\beta-\alpha)(1-\alpha-\beta)\le 0,&\\ (\beta-\alpha)(|\alpha-\beta|-\alpha-\beta)\le 0\}.&\end{aligned}\tag{13}$$

The (α,β)-domains $D_1(\alpha,\beta)$ and $D_2(\alpha,\beta)$ defined by (12) and (13) can be described respectively by Figure 1 and Figure 2 below. These two figures show clearly that the (α,β)-domains $D_1(\alpha,\beta)$ and $D_2(\alpha,\beta)$ are symmetric with respect to the line $\beta=\alpha$.

In this paper, with the aid of Proposition 1, the following sufficient and necessary conditions such that $H_{a,b,c}(x)$ is logarithmically completely monotonic in $(-\rho,\infty)$ are established, which extend, generalize and sharpen [18, Theorem 1] and other known results mentioned above.

Theorem 1. *Let a, b and c be real numbers and $\rho=\min\{a,b,c\}$. Then*

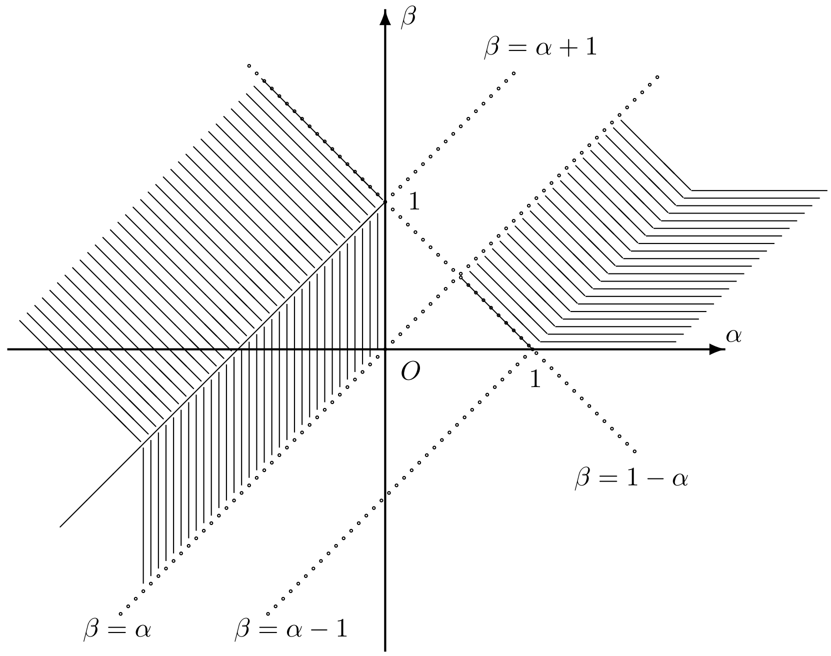

FIGURE 1. The (α, β)-domain $D_1(\alpha, \beta)$

(1) *$H_{a,b,c}(x)$ is logarithmically completely monotonic in $(-\rho, \infty)$ if and only if*

$$
\begin{aligned}
(a,b,c) \in D_1(a,b,c) &\triangleq \{(a,b,c) : (b-a)(1-a-b+2c) \ge 0\} \\
&\cap \{(a,b,c) : (b-a)(|a-b| - a - b + 2c) \ge 0\} \\
&\setminus \{(a,b,c) : a = c+1 = b+1\} \setminus \{(a,b,c) : b = c+1 = a+1\},
\end{aligned}
\tag{14}
$$

(2) *$H_{b,a,c}(x)$ is logarithmically completely monotonic in $(-\rho, \infty)$ if and only if*

$$
\begin{aligned}
(a,b,c) \in D_2(a,b,c) &\triangleq \{(a,b,c) : (b-a)(1-a-b+2c) \le 0\} \\
&\cap \{(a,b,c) : (b-a)(|a-b| - a - b + 2c) \le 0\} \\
&\setminus \{(a,b,c) : b = c+1 = a+1\} \setminus \{(a,b,c) : a = c+1 = b+1\}.
\end{aligned}
\tag{15}
$$

As applications of monotonicity results of $H_{a,b,c}(x)$ established by Theorem 1, the following refinements and sharpenings of Wendel-Gautschi-Kershaw's double inequalities from (1) to (5) are deduced straightforwardly.

Theorem 2. *Let a, b and c be real numbers, $\rho = \min\{a,b,c\}$, and δ be a given constant greater than $-\rho$. Then inequalities*

$$(x+c)^{a-b} < \frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{16}$$

in $x \in (-\rho, \infty)$ and

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \le \frac{\Gamma(\delta+a)}{\Gamma(\delta+b)} \left(\frac{x+c}{\delta+c}\right)^{a-b} \tag{17}$$

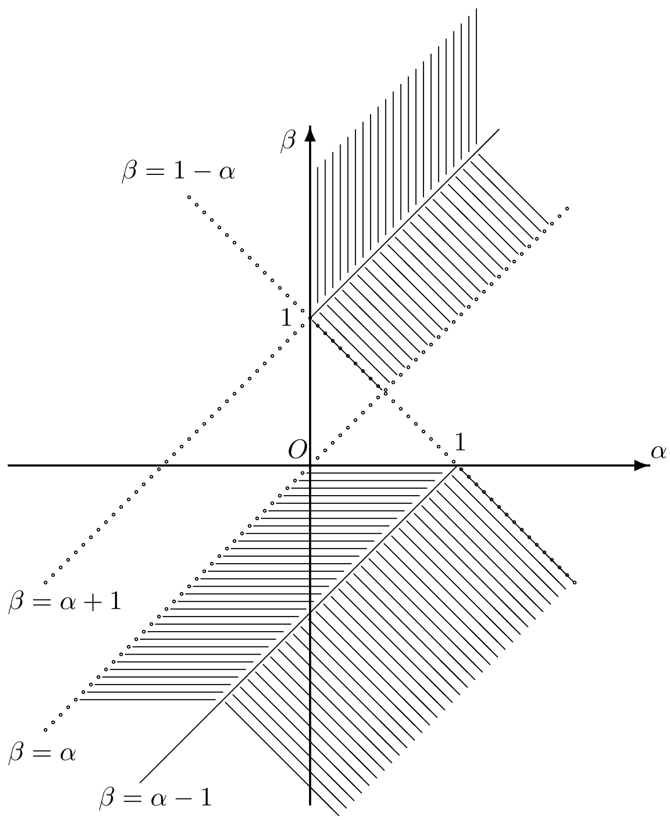

FIGURE 2. The (α, β)-domain $D_2(\alpha, \beta)$

in $x \in [\delta, \infty)$ *are valid if and only if* $(a, b, c) \in D_1(a, b, c)$. *The reversed inequalities of* (16) *and* (17) *hold in* $(-\rho, \infty)$ *and* $[\delta, \infty)$ *respectively if and only if* $(a, b, c) \in D_2(a, b, c)$.

2. Remarks

Before verifying Theorem 1 and Theorem 2, we would like to give some remarks on them and to compare them with Wendel-Gautschi-Kershaw's double inequalities from (1) to (5) and other known results.

Remark 1. The (a, b, c)-domains defined by (10) and (11) are respectively subsets of $D_1(a, b, c)$ and $D_2(a, b, c)$ defined by (14) and (15). Therefore, Theorem 1 in this paper extends [18, Theorem 1].

Remark 2. Taking $a = 1$, $0 < b < 1$ and $\delta = 1$ in (17) gives that inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+b)} \leq \frac{1}{\Gamma(1+b)}\left(\frac{x+c}{1+c}\right)^{1-b} \tag{18}$$

validates in $[1,\infty)$ if and only if

$$(b,c)\in D_1(1,b,c)\cap\{0<b<1\}\cap\{-\rho<1\}\\ =\{-1<c\le 0<b<1\}\cup\{0<2c\le b<1\}.$$

In particular, for $0<b<1$, inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+b)}\le\frac{1}{\Gamma(1+b)}\biggl(\frac{2x+b}{2+b}\biggr)^{1-b} \tag{19}$$

is sharp in $x\in[1,\infty)$.

Standard argument reveals that if

$$\biggl[\biggl(1+\frac{b}{2}\biggr)\sqrt[1-b]{\Gamma(1+b)}-1\biggr]x\triangleq x\Lambda(b)\\ \ge\biggl(\frac12-\sqrt{b+\frac14}\biggr)\biggl(1+\frac{b}{2}\biggr)\sqrt[1-b]{\Gamma(1+b)}+\frac{b}{2}\triangleq\lambda(b) \tag{20}$$

then inequality (19) would be better than the right hand side inequality in (5). It is easy to see that $\lim_{b\to1^-}\Lambda(b)=\frac12$ and $\lim_{b\to1^-}\lambda(b)=\frac14\bigl(5-3\sqrt5\bigr)<0$. This means that inequality (19) refines the right hand side inequality in (5) at least when b is closer enough to 1.

Remark 3. Let us take $a=1$ and $0<b<1$ in inequality (16). Then inequality

$$(x+c)^{1-b}<\frac{\Gamma(x+1)}{\Gamma(x+b)} \tag{21}$$

is valid in $(-\rho,\infty)$ if and only if

$$(b,c)\in D_1(1,b,c)\cap\{0<b<1\}=\{c\le0<b<1\}\cup\{0<2c\le b<1\}.$$

This implies that, in particular, inequality

$$\biggl(x+\frac{b}{2}\biggr)^{1-b}<\frac{\Gamma(x+1)}{\Gamma(x+b)} \tag{22}$$

is sharp in $\bigl(-\frac{b}{2},\infty\bigr)$ for $0<b<1$. This means also that the left hand side inequality in (5) is sharp. Moreover, inequality (22) extends the range of the argument x of the left hand side inequality in (5).

Remark 4. Since

$$[H_{a,b,c}(x)]^{1/(a-b)}=\frac{1}{x+c}\biggl[\frac{\Gamma(x+a)}{\Gamma(x+b)}\biggr]^{1/(a-b)}=\frac{z_{b,a}(x)+x}{x+c} \tag{23}$$

or

$$z_{b,a}(x)=[H_{a,b,c}(x)]^{1/(a-b)}(x+c)-x, \tag{24}$$

the monotonicity and convexity of $z_{b,a}(x)$ and the logarithmically complete monotonicity of $H_{a,b,c}(x)$ are connected.

Remark 5. It is clear that Theorem 1 of this paper and [18, Theorem 1] extend and generalize [5, Theorem 1 and Theorem 3], the complete monotonicity of the function $H_{1,s,s/2}(x)$ defined by [5, Theorem 7, 1.18], the decreasingly monotonicity of the function $H_{s,1,\sqrt{s+1/4}-1/2}(x)$ defined by [5, Theorem 8, 1.20], some results in [26] and [30, Theorem 1].

Remark 6. Stimulated by the paper [11], the following double inequality was also obtained in [14]:

$$\exp\left[(1-s)\psi\big(x+\sqrt{s}\,\big)\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \tag{25}$$

for $s\in(0,1)$ and $x\ge1$. As a generalization of inequality (25), the function

$$\frac{\Gamma(x+s)}{\Gamma(x+1)}\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \tag{26}$$

was proved in [5, Theorem 7] to be completely monotonic in $(0,\infty)$ for $0\le s\le1$, and the function

$$\frac{\Gamma(x+1)}{\Gamma(x+s)}\exp\left[(s-1)\psi\big(x+\sqrt{s}\,\big)\right] \tag{27}$$

for $x>0$ and $0<s<1$ was proved in [5, Theorem 8] to be strictly decreasing. In [36], the function

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(s-t)}\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right] \tag{28}$$

for s and t being nonnegative numbers and $\alpha=\min\{s,t\}$ was verified to be logarithmically completely monotonic in $(-\alpha,\infty)$.

More generally, for a, b and c being real numbers and $\rho=\min\{a,b,c\}$, let

$$F_{a,b,c}(x)=\begin{cases}\left[\dfrac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(a-b)}\exp[\psi(x+c)], & a\ne b\\ \exp[\psi(x+c)-\psi(x+a)], & a=b\ne c\end{cases} \tag{29}$$

in $x\in(-\rho,\infty)$. In order to refine, extend and sharpen Gautschi-Kershaw's double inequality (25), the logarithmically complete monotonicity of $F_{a,b,c}(x)$ has been researched in [10, 17, 19, 28, 30] and the references therein.

Remark 7. Finally, it is remarked that there exist more other literatures about refinements, sharpenings, extensions of Wendel-Gautschi-Kershaw's double inequalities from (1) to (5) and Gautschi-Kershaw's double inequality (25), for examples, [3, 5, 10, 15, 12, 26, 36, 41] and the references therein.

3. Proofs of theorems

Now we are in a position to prove Theorem 1 and Theorem 2.

Proof of Theorem 1. In [1], the following two formulas are given: For $x>0$ and $\omega>0$,

$$\frac{1}{x^\omega}=\frac{1}{\Gamma(\omega)}\int_0^\infty t^{\omega-1}e^{-xt}\,\mathrm{d}t. \tag{30}$$

For $k\in\mathbb{N}$ and $x>0$,

$$\psi(x)=-\gamma+\int_0^\infty\frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}t. \tag{31}$$

By formulas (30) and (31), straightforward calculation gives

$$\ln H_{a,b,c}(x)=(b-a)\ln(x+c)+\ln\Gamma(x+a)-\ln\Gamma(x+b),$$

$$[\ln H_{a,b,c}(x)]'=\frac{b-a}{x+c}+\psi(x+a)-\psi(x+b)$$

$$
\begin{aligned}
&= \frac{b-a}{x+c} + \int_0^\infty \frac{e^{-bt}-e^{-at}}{1-e^{-t}} e^{-xt}\,\mathrm{d}t \\
&= -\int_0^\infty \biggl[\frac{e^{(c-a)t}-e^{(c-b)t}}{1-e^{-t}} + (a-b)\biggr] e^{-(x+c)t}\,\mathrm{d}t \\
&= -\int_0^\infty [q_{a-c,b-c}(t) + (a-b)] e^{-(x+c)t}\,\mathrm{d}t
\end{aligned}
$$

and, for $k \in \mathbb{N}$,

$$
(-1)^k[\ln H_{a,b,c}(x)]^{(k)} = \int_0^\infty [q_{a-c,b-c}(t) + (a-b)] t^{k-1} e^{-(x+c)t}\,\mathrm{d}t,
$$

where $q_{\alpha,\beta}(t)$ is the function defined by (9).

From $q_{\alpha,\beta}(0) = \beta - \alpha$ and $q_{a-c,b-c}(0) = b-a$, it is revealed that if $q_{a-c,b-c}(t)$ is increasing (or decreasing respectively) in $(0,\infty)$ then $q_{a-c,b-c}(t) + (a-b) \gtreqless 0$ in $t \in (0,\infty)$ and $(-1)^k[\ln H_{a,b,c}(x)]^{(k)} \gtreqless 0$ in $x \in (-\rho,\infty)$ for $k \in \mathbb{N}$. Combining this with Proposition 1 demonstrates that $H_{a,b,c}(x) \in \mathcal{L}[(-\rho,\infty)]$ if $(a-c,b-c) \in D_1(a-c,b-c)$ and $[H_{a,b,c}(x)]^{-1} \in \mathcal{L}[(-\rho,\infty)]$ if $(a-c,b-c) \in D_2(a-c,b-c)$. The sufficiency of Theorem 1 is proved.

If the function $H_{a,b,c}(x) \in \mathcal{L}[(-\rho,\infty)]$, then $[\ln H_{a,b,c}(x)]' \le 0$ which is equivalent to

$$
\frac{b-a}{x+c} + \psi(x+a) - \psi(x+b) \le 0 \tag{32}
$$

in $(-\rho,\infty)$. This inequality can be rearranged as

$$
c \ge \frac{b-a}{\psi(x+b) - \psi(x+a)} - x \triangleq \chi_{a,b}(x) \tag{33}
$$

for $b > a$ in $(-\rho,\infty)$.

Since $\lim_{x\to 0^+} \psi(x) = -\infty$, then $\lim_{x\to(-a)^+} \chi_{a,b}(x) = a \le c$ for $b > a$.

In [27, Theorem 2], it was established that the functions

$$
\delta_{s,t}(x) = \begin{cases} \dfrac{\psi(x+t) - \psi(x+s)}{t-s} - \dfrac{2x+s+t+1}{2(x+s)(x+t)}, & s \ne t \\[2ex] \psi'(x+s) - \dfrac{1}{x+s} - \dfrac{1}{2(x+s)^2}, & s = t \end{cases} \tag{34}
$$

for $|t-s| < 1$ and $-\delta_{s,t}(x)$ for $|t-s| > 1$ are completely monotonic in $x \in (-\alpha,\infty)$, where s and t are two real numbers and $\alpha = \min\{s,t\}$. Consequently, from $\lim_{x\to\infty} \delta_{s,t}(x) = 0$, it is deduced that

$$
c \ge \chi_{a,b}(x) \ge \frac{2(x+a)(x+b)}{2x+a+b+1} - x \to \frac{a+b-1}{2} > a \tag{35}
$$

for $b - a > 1$ and

$$
\chi_{a,b}(x) \le \frac{2(x+a)(x+b)}{2x+a+b+1} - x \to \frac{a+b-1}{2} < a \tag{36}
$$

for $b - a < 1$ as x tends to ∞. The necessity of $H_{a,b,c}(x)$ being logarithmically completely monotonic in $(-\rho,\infty)$ follows.

The proof of necessity of $H_{b,a,c}(x)$ being logarithmically completely monotonic in $(-\rho,\infty)$ is same as above. The necessity of Theorem 1 is proved. □

Proof of Theorem 2. For a and b being two constants, as x tends to ∞, the following asymptotic formula is given in [1, p. 257 and p. 259]:

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}=1+\frac{(a-b)(a+b-1)}{2x}+O\biggl(\frac{1}{x^2}\biggr). \tag{37}$$

By formula (37), it follows that

$$\begin{aligned}H_{a,b,c}(x)&=\biggl(1+\frac{c}{x}\biggr)^{b-a}\biggl[x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}\biggr]\\&=\biggl(1+\frac{c}{x}\biggr)^{b-a}\biggl[1+\frac{(a-b)(a+b-1)}{2x}+O\biggl(\frac{1}{x^2}\biggr)\biggr]\\&\to 1\end{aligned}$$

as $x\to\infty$ for all real numbers a, b and c.

If $(a,b,c)\in D_1(a,b,c)$, then the function $H_{a,b,c}(x)$ is decreasing in $(-\rho,\infty)$ and $H_{a,b,c}(x)>\lim_{x\to\infty}=1$ which can be rearranged as inequality (16). Further, if δ is a constant greater than $-\rho$, then

$$H_{a,b,c}(x)\le H_{a,b,c}(\delta)=(\delta+c)^{b-a}\frac{\Gamma(\delta+a)}{\Gamma(\delta+b)}$$

in $[\delta,\infty)$, which can be rewritten as (17) for $x\in[\delta,\infty)$.

If $(a,b,c)\in D_2(a,b,c)$ and δ is a constant greater than $-\rho$, then the function $H_{a,b,c}(x)$ is increasing in $(-\rho,\infty)$, inequalities (16) and (17) are reversed respectively. The proof of Theorem 2 is complete. □

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