Sharp Inequalities of Ostrowski Type for Convex Functions Defined on Linear Spaces and Applications

SHARP INEQUALITIES OF OSTROWSKI TYPE FOR CONVEX FUNCTIONS DEFINED ON LINEAR SPACES AND APPLICATIONS

EDER KIKIANTY, S.S. DRAGOMIR, AND P. CERONE

Abstract. An Ostrowski type inequality for general convex functions defined on linear spaces is generalised. Some inequalities which improve the Hermite-Hadamard type inequality for convex functions defined on linear spaces are derived using the obtained result. The results in normed linear spaces are used to obtain some inequalities which are related to the given norm and associated semi-inner products, and prove the sharpness of the constants in those inequalities.

1. Introduction

In 1938, A. Ostrowski (see [23, p. 226]) considered the problem of estimating the deviation of a function from its integral mean. If a function f defined on the interval $[a, b] \subset \mathbb{R}$ is continuous, then the deviation of f at a point $x \in [a, b]$ from its integral mean $\frac{1}{b-a}\int_a^b f(x)dx$ can be approximated by the difference between its maximum and minimum value. Furthermore, if f is differentiable on (a, b), and its derivative is bounded on (a, b), that is, $|f'(x)| \le M$ for all $x \in (a, b)$, then the difference between the maximum and minimum value does not exceed $(b - a)M$ (however, it may reach this value) and the absolute deviation of $f(x)$ from its integral mean does not exceed $\frac{1}{2}(b-a)M$. If x is the midpoint of the interval (that is $x = \frac{a+b}{2}$), then the absolute deviation is bounded by the value $\frac{1}{4}(b-a)M$. To be precise, for any continuous function f on $[a, b] \subset \mathbb{R}$ which is differentiable on (a, b) and $|f'(x)| \le M$ for all $x \in (a, b)$, the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)M, \tag{1}$$

holds for every $x \in [a, b]$ (see [23, p. 226–227] for the complete proof). This is then known as the Ostrowski inequality (see [21, p. 468]). The first factor on the right hand side of (1) reaches the value of $\frac{1}{4}$ at the midpoint and monotonically increases to $\frac{1}{2}$ which is attained at both endpoints [23, p. 226]. It implies that the constant $\frac{1}{4}$ is best possible, that is, it cannot be replaced by a smaller quantity (see also [1, p. 3775–3776], for an alternative proof).

Numerous developments, extensions and generalisations of Ostrowski inequality have been carried out in various directions. One way to extend this result is to consider other classes of integrable functions. The case for absolutely continuous functions, where the derivative exists almost everywhere, has been considered in

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[10,11,16] and [18, p. 2], while the case where the functions are of bounded variation can be found in [9], [14, p. 374] and [18, p. 3–4]. The case of Hölder continuous functions and Lipschitzian functions have also been pointed out [18, p. 3] (see [3–8] for other possible directions).

Another possibility of generalising Ostrowski inequality is to consider the case of real convex functions. Since any convex function is locally Lipschitzian (hence it is locally absolutely continuous), thus it can be connected to the previous mentioned cases (see [14, 16]).

For any convex function, we can also consider another well-known inequality: the Hermite-Hadamard inequality. It was first introduced by Ch. Hermite in 1881 in the journal *Mathesis* (see [20]). Hermite mentioned that the following inequality holds for any convex function f defined on $\mathbb{R}$

$$(2) \qquad (b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}.$$

But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [25]. E.F. Beckenbach, a leading expert on the history and the theory of complex functions, wrote that this inequality was proven by J. Hadamard in 1893 [2]. In 1974, D.S. Mitrinović found Hermite's note in *Mathesis* [20]. Since (2) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [25].

Various developments and generalisations have been pointed out in many directions (see [17]). Dragomir in [14, p. 378–379] obtained some inequalities which improve the Hermite-Hadamard inequalities (see also [12,13]). These results can be derived from an Ostrowski type inequality for real convex functions (see also [16, p. 15–17]). In [12,13], Dragomir examined a generalisation of the Hermite-Hadamard inequality by considering convex functions defined on linear spaces. As an application in normed linear spaces, some inequalities which are related to semi-inner product were obtained. However, the sharpness of the constants in these inequalities was not considered.

In this paper, we generalise the Ostrowski type inequality which has been pointed out in [14] to general convex functions defined on linear spaces. Using this result, we derive some inequalities which improve the Hermite-Hadamard type inequalities for convex functions on linear spaces, as mentioned in [12, 13]. In a normed linear spaces, we obtain some inequalities related to the given norm and associated semi-inner products which are more general than those in [12,13] and provide the proof of the sharpness for the constants in those inequalities. We also revisit the inequalities which were previously suggested in [12, 13], by considering some particular cases from the general one, and prove the sharpness of the constants.

2. Definitions and Preliminary Results

Let X be a vector space, $x, y \in X$, $x \neq y$. Define the segment $[x, y] := \{(1-t)x + ty,\ t \in [0, 1]\}$. We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function $g(x, y) : [0, 1] \to \mathbb{R}$, $g(x, y)(t) := f[(1-t)x + ty]$, $t \in [0, 1]$. Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

For any convex function defined on a segment $[x.y] \subset X$, we have the Hermite-Hadamard integral inequality (see [12, p. 2], [13, p. 2])

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x+ty]dt \le \frac{f(x)+f(y)}{2}, \tag{3}$$

which can be derived from the classical Hermite-Hadamard inequality (2) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

We consider the Gâteaux lateral derivatives for any $x, y \in X$ and any function f defined on X, as

$$\begin{aligned}(\nabla_+ f(x))(y) &:= \lim_{t\to 0^+}\frac{f(x+ty)-f(x)}{t},\\ (\nabla_- f(x))(y) &:= \lim_{t\to 0^-}\frac{f(x+ty)-f(x)}{t},\end{aligned}$$

if the above limits exist.

Assume that $(X, \|\cdot\|)$ is a normed space. The function $f_0(x) = \frac{1}{2}\|x\|^2$ $(x \in X)$ is convex and the following limits

$$\begin{aligned}\langle x,y\rangle_s &:= (\nabla_+ f_0(y))(x) = \lim_{t\to 0^+}\frac{\|y+tx\|^2-\|y\|^2}{2t},\\ \langle x,y\rangle_i &:= (\nabla_- f_0(y))(x) = \lim_{t\to 0^-}\frac{\|y+tx\|^2-\|y\|^2}{2t},\end{aligned}$$

exist for any $x, y \in X$. They are called the superior and inferior semi-inner products associated to the norm $\|\cdot\|$ (see [15, p. 27–39] for further properties).

The function $f_p(x) = \|x\|^p$ $(x \in X$ and $1 \le p < \infty)$ is also convex. Therefore, the following limits, which are related to superior (inferior) semi-inner products,

$$(\nabla_+ f_p(y))(x) = \lim_{t\to 0^+}\frac{\|y+tx\|^p-\|y\|^p}{t} = p\|y\|^{p-2}\langle x,y\rangle_s, \tag{4}$$

$$(\nabla_- f_p(y))(x) = \lim_{t\to 0^-}\frac{\|y+tx\|^p-\|y\|^p}{t} = p\|y\|^{p-2}\langle x,y\rangle_i, \tag{5}$$

exist for all $x, y \in X$ whenever $p \ge 2$; otherwise, they exist for any $x \in X$ and nonzero $y \in X$. In particular, if $p = 1$, then the following limits

$$\begin{aligned}(\nabla_+ f_1(y))(x) &= \lim_{t\to 0^+}\frac{\|y+tx\|-\|y\|}{t} = \left\langle x, \frac{y}{\|y\|}\right\rangle_s,\\ (\nabla_- f_1(y))(x) &= \lim_{t\to 0^-}\frac{\|y+tx\|-\|y\|}{t} = \left\langle x, \frac{y}{\|y\|}\right\rangle_i,\end{aligned}$$

exist for $x, y \in X$ and $y \ne 0$.

Since $f(x) = \|x\|^p$ $(x \in X$ and $1 \le p < \infty)$ is a convex function, we have the following norm inequality from (3) (see [24, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \le \int_0^1 \|(1-t)x+ty\|^p dt \le \frac{\|x\|^p+\|y\|^p}{2}, \tag{6}$$

for any $x, y \in X$. Particularly, if $p = 2$, then

$$\left\|\frac{x+y}{2}\right\|^2 \le \int_0^1 \|(1-t)x+ty\|^2 dt \le \frac{\|x\|^2+\|y\|^2}{2}, \tag{7}$$

holds for any $x, y \in X$. We also get the following refinement of the triangle inequality when $p = 1$ (see [22, p. 485])

$$\left\| \frac{x+y}{2} \right\| \leq \int_0^1 \|(1-t)x + ty\| dt \leq \frac{\|x\| + \|y\|}{2}. \tag{8}$$

3. The Results

A generalisation of the classical Ostrowski inequality by considering the class of real convex functions has been obtained in [14, 16]. The following result is a generalisation of an Ostrowski type inequality in [14] for convex functions defined on linear spaces.

Theorem 1. *Let X be a vector space, $I_k : 0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ be a division of the interval $[0,1]$, α_i $(i = 0, \ldots, k+1)$ be $k+2$ points such that $\alpha_0 = 0$, $\alpha_i \in [s_{i-1}, s_i]$, $(i = 1, \ldots, k)$ and $\alpha_{k+1} = 1$. If $f : [x, y] \subset X \to \mathbb{R}$ is a convex function on the segment $[x, y]$, then we have*

$$\begin{aligned}
&\frac{1}{2} \sum_{i=0}^{k-1} \{(s_{i+1} - \alpha_{i+1})^2 \bigtriangledown_+ f[(1-\alpha_{i+1})x + \alpha_{i+1} y](y-x) \\
&-(\alpha_{i+1} - s_i)^2 \bigtriangledown_- f[(1-\alpha_{i+1})x + \alpha_{i+1} y](y-x)\} \\
\leq\ & \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f[(1-s_i)x + s_i y] - \int_0^1 f[(1-t)x + ty] dt \\
\leq\ & \frac{1}{2} \sum_{i=0}^{k-1} \{(s_{i+1} - \alpha_{i+1})^2 \bigtriangledown_- f[(1-s_{i+1})x + s_{i+1} y](y-x) \\
&-(\alpha_{i+1} - s_i)^2 \bigtriangledown_+ f[(1-s_i)x + s_i y](y-x)\}.
\end{aligned} \tag{9}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. Under the above assumption, we may apply the Ostrowski type inequality which has been obtained in [14] (see Theorem 3) for any convex function h defined on $[0,1]$:

$$\begin{aligned}
&\frac{1}{2} \sum_{i=0}^{k-1} [(s_{i+1} - \alpha_{i+1})^2 h'_+(\alpha_{i+1}) - (\alpha_{i+1} - s_i)^2 h'_-(\alpha_{i+1})] \\
\leq\ & \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) h(s_i) - \int_0^1 h(t) dt \\
\leq\ & \frac{1}{2} \sum_{i=0}^{k-1} [(s_{i+1} - \alpha_{i+1})^2 h'_-(s_{i+1}) - (\alpha_{i+1} - s_i)^2 h'_+(s_i)],
\end{aligned} \tag{10}$$

where $h'_{+(-)}$ denotes the right-(left-)sided derivative.

Consider the function $h(t) = g(x, y)(t) = f[(1-t)x + ty]$ defined on $[0,1]$. Since f is a convex function on $[x, y]$, then h is also convex on $[0,1]$, therefore, we may apply the above inequality to h. Now, the right-(left-)sided derivative can be computed as follows:

$$h'_\pm(t) = g'_\pm(x, y)(t) = (\bigtriangledown_\pm f[(1-t)x + ty])(y-x), \quad t \in [0,1].$$

We obtained the desired result by writing the inequality (10) for $h(t) = g(x,y)(t)$. The sharpness of the constants follows by some particular cases which will be considered later. □

Corollary 2. *Let X be a vector space, $x, y \in X$, $x \neq y$ and $f : [x,y] \subset X \to \mathbb{R}$ be a convex function on the segment $[x,y]$. Then for any $s \in (0,1)$ one has the inequality*

$$\begin{aligned} &\frac{1}{2}[(1-s)^2(\nabla_+ f[(1-s)x+sy])(y-x) \\ &-s^2(\nabla_- f[(1-s)x+sy])(y-x)] \\ \leq\ &(1-s)f(x)+sf(y)-\int_0^1 f[(1-t)x+ty]dt \\ \leq\ &\frac{1}{2}[(1-s)^2(\nabla_- f(y))(y-x)-s^2(\nabla_+ f(x))(y-x)]. \end{aligned} \tag{11}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. The result can be obtained by choosing $k = 1$ and $s_0 = \alpha_0 = 0$, $\alpha_1 = s \in (0,1)$, and $s_1 = \alpha_2 = 1$ in Theorem 1. The sharpness of the constants will be proven later by considering some particular cases. An alternative proof can be found in Theorem 2.4 of [13]. However the sharpness of the constants was not considered in that paper. □

The following result provides an improvement for the second Hermite-Hadamard inequality (see also [13]).

Remark 3. A particular case that can be considered is by letting $s = \frac{1}{2}$ in (11). We obtain

$$\begin{aligned} &\frac{1}{8}\left[\left(\nabla_+ f\left(\frac{x+y}{2}\right)\right)(y-x)-\left(\nabla_- f\left(\frac{x+y}{2}\right)\right)(y-x)\right] \\ \leq\ &\frac{f(x)+f(y)}{2}-\int_0^1 f[(1-t)x+ty]dt \\ \leq\ &\frac{1}{8}[(\nabla_- f(y))(y-x)-(\nabla_+ f(x))(y-x)], \end{aligned} \tag{12}$$

which provides bounds for the distance between the last two terms in the Hermite-Hadamard integral inequality (3). The constant $\frac{1}{8}$ is sharp (the proof follows by a particular case which will be proven later).

Corollary 4. *Let X be a vector space, $x, y \in X$, $x \neq y$ and $f : [x,y] \subset X \to \mathbb{R}$ be a convex function on the segment $[x,y]$. Then for any $s \in (0,1)$ one has the inequality*

$$\begin{aligned} &\frac{1}{2}[(1-s)^2(\nabla_+ f[(1-s)x+sy])(y-x) \\ &-s^2(\nabla_- f[(1-s)x+sy])(y-x)] \\ \leq\ &\int_0^1 f[(1-t)x+ty]dt-f[(1-s)x+sy] \\ \leq\ &\frac{1}{2}[(1-s)^2(\nabla_- f(y))(y-x)-s^2(\nabla_+ f(x))(y-x)]. \end{aligned} \tag{13}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. Apply Theorem 1 and choose $k=2$, $0=s_0<s_1=s<s_2=1$. Let $\alpha_0=0$, $\alpha_1=a\in(0,s]$, $\alpha_2=b\in[s,1)$ and $\alpha_3=1$. By the fact that f is a convex function, which implies that $\triangledown_{\pm}f$ is a nondecreasing function, we have the following

$$\begin{aligned}(s-a)^2(\triangledown_+f(x))(y-x)&\leq(s-a)^2(\triangledown_+f((1-a)x+ay))(y-x)\\ \text{and }-(b-s)^2(\triangledown_-f(y))(y-x)&\leq-(b-s)^2(\triangledown_-f((1-b)x+by))(y-x),\end{aligned}$$

for any $a\in(0,s],\ b\in[s,1)$.
Thus, we have

$$\begin{aligned}&\frac{1}{2}\left[(s-a)^2(\triangledown_+f(x))(y-x)-a^2(\triangledown_-f((1-a)x+ay))(y-x)\right.\\ &\left.+(1-b)^2(\triangledown_+f((1-b)x+by))(y-x)-(b-s)^2(\triangledown_-f(y))(y-x)\right]\\ \leq\ &\frac{1}{2}\left[(s-a)^2(\triangledown_+f((1-a)x+ay))(y-x)-a^2(\triangledown_-f((1-a)x+ay))(y-x)\right.\\ &\left.+(1-b)^2(\triangledown_+f((1-b)x+by))(y-x)-(b-s)^2(\triangledown_-f((1-b)x+by))(y-x)\right]\\ \leq\ &af(x)+(b-a)f[(1-s)x+sy]+(1-b)f(y)-\int_0^1 f[(1-t)x+ty]dt\\ \leq\ &\frac{1}{2}\left[(s-a)^2(\triangledown_-f((1-s)x+sy))(y-x)-a^2(\triangledown_+f(x))(y-x)\right.\\ &\left.+(1-b)^2(\triangledown_-f(y))(y-x)-(b-s)^2(\triangledown_+f((1-s)x+sy))(y-x)\right].\end{aligned}$$

Let $a\to 0^+$ and $b\to 1^-$, then we obtain

$$\begin{aligned}&\frac{1}{2}[s^2(\triangledown_+f(x))(y-x)-(1-s)^2(\triangledown_-f(y))(y-x)]\\ \leq\ &f[(1-s)x+sy]-\int_0^1 f[(1-t)x+ty]dt\\ \leq\ &\frac{1}{2}[s^2(\triangledown_-f((1-s)x+sy))(y-x)-(1-s)^2(\triangledown_+f((1-s)x+sy))(y-x)].\end{aligned}$$

By multiplying the above inequality with -1, we obtain the desired result. The sharpness of the constants will be proven later by considering some particular cases. An alternative proof can be found in Theorem 2.4 of [12]. However, the sharpness of the constants was not considered in that paper. □

The following result provides an improvement for the first Hermite-Hadamard inequality (see also [12]).

Remark 5. One particular case that can be considered is by choosing $s=\frac{1}{2}$ in (13). We obtain

$$\begin{aligned}&\frac{1}{8}\left[\left(\triangledown_+f\left(\frac{x+y}{2}\right)\right)(y-x)-\left(\triangledown_-f\left(\frac{x+y}{2}\right)\right)(y-x)\right]\\ \leq\ &\int_0^1 f[(1-t)x+ty]dt-f\left(\frac{x+y}{2}\right)\\ \leq\ &\frac{1}{8}[(\triangledown_-f(y))(y-x)-(\triangledown_+f(x))(y-x)],\end{aligned}\tag{14}$$

which provides bounds for the distance between the first two terms in the Hermite-Hadamard integral inequality (3). The constant $\frac{1}{8}$ is sharp (the proof follows by a particular case which will be proven later).

4. Applications for Semi-Inner Products

Let $(X, \|\cdot\|)$ be a normed linear space. We obtain the following inequalities for the semi-inner products $\langle\cdot,\cdot\rangle_s$ and $\langle\cdot,\cdot\rangle_i$.

Proposition 6. *Let $I_k : 0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ be a division of the interval $[0,1]$ and α_i $(i = 0,\ldots,k+1)$ be $k+2$ points such that $\alpha_0 = 0$, $\alpha_i \in [s_{i-1}, s_i]$, $(i = 1,\ldots,k)$ and $\alpha_{k+1} = 1$. Assume that $1 \le p < \infty$. Then*

$$\begin{aligned} (15)\quad & \frac{1}{2}p\sum_{i=0}^{k-1}\|(1-\alpha_{i+1})x+\alpha_{i+1}y\|^{p-2}[(s_{i+1}-\alpha_{i+1})^2\langle y-x,(1-\alpha_{i+1})x+\alpha_{i+1}y\rangle_s \\ & -(\alpha_{i+1}-s_i)^2\langle y-x,(1-\alpha_{i+1})x+\alpha_{i+1}y\rangle_i] \\ \le\ & \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\|(1-s_i)x+s_iy\|^p-\int_0^1\|(1-t)x+ty\|^p dt \\ \le\ & \frac{1}{2}p\sum_{i=0}^{k-1}[(s_{i+1}-\alpha_{i+1})^2\|(1-s_{i+1})x+s_{i+1}y\|^{p-2}\langle y-x,(1-s_{i+1})x+s_{i+1}y\rangle_i \\ & -(\alpha_{i+1}-s_i)^2\|(1-s_i)x+s_iy\|^{p-2}\langle y-x,(1-s_i)x+s_iy\rangle_s], \end{aligned}$$

holds for any $x, y \in X$ whenever $p \ge 2$; otherwise, it holds for linearly independent $x, y \in X$.
The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. Apply Theorem 1 to the convex function $f_p(x) = \|x\|^p$, where $x \in X$, and $1 \le p < \infty$ (see (4) and (5)). The sharpness of the constants will be proven later by considering some particular cases. □

Corollary 7. *Let x and y be any two vectors in X, $\sigma \in (0,1)$ and $1 \le p < \infty$. Then*

$$\begin{aligned} & \frac{1}{2}p\|(1-\sigma)x+\sigma y\|^{p-2}[(1-\sigma)^2\langle y-x,(1-\sigma)x+\sigma y\rangle_s \\ & -\sigma^2\langle y-x,(1-\sigma)x+\sigma y\rangle_i] \\ (16)\quad \le\ & (1-\sigma)\|x\|^p+\sigma\|y\|^p-\int_0^1\|(1-t)x+ty\|^p dt \\ \le\ & \frac{1}{2}p[(1-\sigma)^2\|y\|^{p-2}\langle y-x,y\rangle_i-\sigma^2\|x\|^{p-2}\langle y-x,x\rangle_s], \end{aligned}$$

holds for any $x, y \in X$ whenever $p \ge 2$; otherwise, it holds for linearly independent $x, y \in X$.
The constant $\frac{1}{2}$ is sharp in both inequalities.
We also have two particular cases that are of interest, namely

$$\begin{aligned} & (1-\sigma)^2\langle y-x,(1-\sigma)x+\sigma y\rangle_s-\sigma^2\langle y-x,(1-\sigma)x+\sigma y\rangle_i \\ (17)\quad \le\ & (1-\sigma)\|x\|^2+\sigma\|y\|^2-\int_0^1\|(1-t)x+ty\|^2 dt \\ \le\ & (1-\sigma)^2\langle y-x,y\rangle_i-\sigma^2\langle y-x,x\rangle_s, \end{aligned}$$

for any $x, y \in X$ and

$$\begin{aligned}(18)\quad &\frac{1}{2}\left[(1-\sigma)^2\left\langle y-x, \frac{(1-\sigma)x+\sigma y}{\|(1-\sigma)x+\sigma y\|}\right\rangle_s - \sigma^2\left\langle y-x, \frac{(1-\sigma)x+\sigma y}{\|(1-\sigma)x+\sigma y\|}\right\rangle_i\right] \\ \leq\ & (1-\sigma)\|x\| + \sigma\|y\| - \int_0^1 \|(1-t)x+ty\|dt \\ \leq\ & \frac{1}{2}\left[(1-\sigma)^2\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \sigma^2\left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right],\end{aligned}$$

for any linearly independent $x, y \in X$. The constants in (17) and (18) are sharp.

Proof. Choose $k = 1$, $s_0 = \alpha_0 = 0$, $\alpha_1 = \sigma \in (0,1)$, and $s_1 = \alpha_2 = 1$ in Proposition 6. As an alternative proof, this result can be obtained by choosing $f(x) = \|x\|^p$, $(1 \leq p < \infty)$ and $s = \sigma$ in Corollary 2. Take $p = 2$ and $p = 1$ in (16) to obtain (17) and (18) (see also [13, Proposition 3.1 and Proposition 3.2]). The sharpness of the constants will be proven later by considering some particular cases (in [13], the sharpness of the constants was not considered). □

Corollary 8. *Let x and y be any two vectors in X, $\sigma \in (0,1)$ and $1 \leq p < \infty$. Then*

$$\begin{aligned}&\frac{1}{2}p\|(1-\sigma)x+\sigma y\|^{p-2}[(1-\sigma)^2\langle y-x, (1-\sigma)x+\sigma y\rangle_s \\ &-\sigma^2\langle y-x, (1-\sigma)x+\sigma y\rangle_i] \\ (19)\quad \leq\ & \int_0^1 \|(1-t)x+ty\|^p dt - \|(1-\sigma)x+\sigma y\|^p \\ \leq\ & \frac{1}{2}p[(1-\sigma)^2\|y\|^{p-2}\langle y-x, y\rangle_i - \sigma^2\|x\|^{p-2}\langle y-x, x\rangle_s],\end{aligned}$$

holds for any $x, y \in X$ whenever $p \geq 2$; otherwise, it holds for linearly independent $x, y \in X$.
The constant $\frac{1}{2}$ is sharp in both inequalities.
We also have two following particular cases of interest

$$\begin{aligned}&(1-\sigma)^2\langle y-x, (1-\sigma)x+\sigma y\rangle_s - \sigma^2\langle y-x, (1-\sigma)x+\sigma y\rangle_i \\ (20)\quad \leq\ & \int_0^1 \|(1-t)x+ty\|^2 dt - \|(1-\sigma)x+\sigma y\|^2 \\ \leq\ & (1-\sigma)^2\langle y-x, y\rangle_i - \sigma^2\langle y-x, x\rangle_s,\end{aligned}$$

for any $x, y \in X$ and

$$\begin{aligned}(21)\quad &\frac{1}{2}\left[(1-\sigma)^2\left\langle y-x, \frac{(1-\sigma)x+\sigma y}{\|(1-\sigma)x+\sigma y\|}\right\rangle_s - \sigma^2\left\langle y-x, \frac{(1-\sigma)x+\sigma y}{\|(1-\sigma)x+\sigma y\|}\right\rangle_i\right] \\ \leq\ & \int_0^1 \|(1-t)x+ty\|dt - \|(1-\sigma)x+\sigma y\| \\ \leq\ & \frac{1}{2}\left[(1-\sigma)^2\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \sigma^2\left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right],\end{aligned}$$

for any linearly independent $x, y \in X$. The constants in (20) and (21) are sharp.

Proof. Choose $k = 2$ in Proposition 6, then perform a similar steps as in the proof of Corollary 4 for $s_0 = 0 < s_1 = \sigma < 1 = s_2$, $\alpha_0 = 0$, $\alpha_1 = a \in (0, \sigma]$, $\alpha_2 = b \in [\sigma, 1)$ and $\alpha_3 = 1$. As an alternative proof, this result can be obtained by choosing

$f(x) = \|x\|^p$, $(1 \le p < \infty)$ and $s = \sigma$ in Corollary 4. Take $p = 2$ and $p = 1$ in (19) to obtain (20) and (21) (see also [12, Proposition 3.1 and Proposition 3.2]). The sharpness of the constants will be proven later by considering some particular cases (in [12], the sharpness of the constants was not considered). □

5. Some Particular Cases and the Best Constants

The following cases follow from the previous section and provide an improvement for the Hermite-Hadamard inequalities (6), (7) and (8). Some of the results have been obtained before in [12] and [13], but the sharpness of the constants was not considered. Here, we provide the proof for the sharpness of the constants.

Proposition 9. *Let $(X, \|\cdot\|)$ be a normed linear space and $1 \le p < \infty$. Then,*

$$\begin{aligned} 0 &\le \frac{1}{8}p\left\|\frac{y+x}{2}\right\|^{p-2}\left[\left\langle y-x, \frac{y+x}{2}\right\rangle_s - \left\langle y-x, \frac{y+x}{2}\right\rangle_i\right] \\ &\le \frac{\|x\|^p + \|y\|^p}{2} - \int_0^1 \|(1-t)x + ty\|^p dt \\ &\le \frac{1}{8}p[\|y\|^{p-2}\langle y-x, y\rangle_i - \|x\|^{p-2}\langle y-x, x\rangle_s], \end{aligned} \tag{22}$$

holds for any $x, y \in X$ whenever $p \ge 2$; otherwise, it holds for linearly independent $x, y \in X$. The above inequality provides bounds for the distance between the last two terms in (6).
The constant $\frac{1}{8}$ is sharp.
In particular, we have

$$\begin{aligned} 0 &\le \frac{1}{8}[\langle y-x, y+x\rangle_s - \langle y-x, y+x\rangle_i] \\ &\le \frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x + ty\|^2 dt \\ &\le \frac{1}{4}[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s], \end{aligned} \tag{23}$$

for any $x, y \in X$ and

$$\begin{aligned} 0 &\le \frac{1}{8}\left[\left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_s - \left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_i\right] \\ &\le \frac{\|x\| + \|y\|}{2} - \int_0^1 \|(1-t)x + ty\| dt \\ &\le \frac{1}{8}\left[\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right], \end{aligned} \tag{24}$$

for any linearly independent $x, y \in X$.
The constants in (23) and (24) are sharp.

Proof. We obtain (22) by taking $\sigma = \frac{1}{2}$ in (16). We may also obtain (22) by taking $f(x) = \|x\|^p$ $(1 \le p < \infty)$ in Remark 3. Then, (23) and (24) follow by taking $p = 2$ and $p = 1$, respectively, in (22). Note that we may also obtain (23) from (17) and (24) from (18) by letting $\sigma = \frac{1}{2}$. The sharpness of the constants in (22) would follow by the sharpness of the constants in (23) and (24) as its particular cases.

We will now prove the sharpness of the constants in (23). Assume that the above inequality holds for constants $A, B > 0$ instead of $\frac{1}{8}$ and $\frac{1}{4}$ respectively, that is,

$$\begin{aligned} 0 &\leq A[\langle y-x, y+x\rangle_s - \langle y-x, y+x\rangle_i] \\ &\leq \frac{\|x\|^2+\|y\|^2}{2} - \int_0^1 \|(1-t)x+ty\|^2 dt \\ &\leq B[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s]. \end{aligned}$$

Note that in the space $(l^1, \|\cdot\|_1)$, we have the following semi-inner products for any x, y (see [15, 19])

$$\langle x, y\rangle_{s(i)} = \|y\|_1 \left(\sum_{y_i \neq 0} \frac{y_i}{|y_i|} x_i \pm \sum_{y_i = 0} |x_i| \right).$$

Taking $(X, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_1)$, we have the inequality

$$\begin{aligned} & 2A\|y+x\|_1 \sum_{y_i+x_i=0} |y_i - x_i| \\ \leq\ & \frac{\|x\|_1^2+\|y\|_1^2}{2} - \int_0^1 \|(1-t)x+ty\|_1^2 dt \\ \leq\ & B\left[\|y\|_1 \left(\sum_{y_i\neq 0} \frac{y_i}{|y_i|}(y_i - x_i) - \sum_{y_i=0} |y_i - x_i| \right) \right. \\ & \left. -\|x\|_1 \left(\sum_{x_i \neq 0} \frac{x_i}{|x_i|}(y_i - x_i) + \sum_{x_i = 0} |y_i - x_i| \right) \right]. \end{aligned}$$

Take $x = \left(-\frac{1}{n}, n\right)$ and $y = \left(\frac{1}{n}, n\right)$, for any $n \in \mathbb{N}$, then we have the following

$$8A \leq \frac{3n^2+2}{3n^2} \leq 4B\left(\frac{1}{n^2}+1\right),$$

from the previous inequality. Taking $n \to \infty$, we get

$$8A \leq 1 \leq 4B,$$

that is, $A \leq \frac{1}{8}$ in the first inequality, and $B \geq \frac{1}{4}$ in the second inequality. Thus, both the constants $\frac{1}{8}$ and $\frac{1}{4}$ are sharp in the first and second inequality respectively. This implies that the constants in (9), (11), (12), (15), (16), (17), and (22) are sharp.

Now, we will prove the sharpness of the constants in (24). Assume that the above inequality holds for constants $C, D > 0$ instead of $\frac{1}{8}$, that is

$$\begin{aligned} 0 &\leq C\left[\left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|} \right\rangle_s - \left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|} \right\rangle_i \right] \\ &\leq \frac{\|x\|+\|y\|}{2} - \int_0^1 \|(1-t)x+ty\| dt \\ &\leq D\left[\left\langle y-x, \frac{y}{\|y\|} \right\rangle_i - \left\langle y-x, \frac{x}{\|x\|} \right\rangle_s \right]. \end{aligned}$$

Take $(X, \|\cdot\| = (\mathbb{R}^2, \|\cdot\|_1)$, we have the following inequality

$$\begin{aligned} 0 &\leq 2C \sum_{y_i+x_i=0} |y_i - x_i| \\ &\leq \frac{\|x\|_1 + \|y\|_1}{2} - \int_0^1 \|(1-t)x + ty\|_1 dt \\ &\leq D\left[\sum_{y_i\neq 0} \frac{y_i}{|y_i|}(y_i - x_i) - \sum_{y_i=0} |y_i - x_i| - \sum_{x_i\neq 0} \frac{x_i}{|x_i|}(y_i - x_i) - \sum_{x_i=0} |y_i - x_i|\right], \end{aligned}$$

for any linearly independent x and y.

Now, take $x = (1, 0)$ and $y = (-1, 1)$. Clearly x and y are linearly independent, therefore the above inequality holds for these vectors. We have

$$4C \leq \frac{1}{2} \leq 4D,$$

that is, $C \leq \frac{1}{8}$ in the first inequality and $D \geq \frac{1}{8}$ in the second inequality. Thus, the constant $\frac{1}{8}$ is sharp in both inequalities. This implies that the constants in (18) are also sharp. □

Remark 10 (The case of inner product spaces)**.** Let X be an inner product space, with the inner product $\langle \cdot, \cdot \rangle$, in Proposition 9. Then,

$$\begin{aligned} 0 &\leq \frac{\|x\|^p + \|y\|^p}{2} - \int_0^1 \|(1-t)x + ty\|^p dt \\ &\leq \frac{1}{8} p \langle y - x, y\|y\|^{p-2} - x\|x\|^{p-2} \rangle, \end{aligned} \tag{25}$$

holds for any $x, y \in X$ whenever $p \geq 2$; otherwise, it holds for nonzero $x, y \in X$. Particularly, for $p = 2$, we have

$$\begin{aligned} 0 &\leq \frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x + ty\|^2 dt \\ &\leq \frac{1}{4}[\langle y - x, y \rangle - \langle y - x, x \rangle] = \frac{1}{4}\|y - x\|^2, \end{aligned}$$

for any $x, y \in X$. The constant $\frac{1}{4}$ is not the best possible constant in this case, since we always have

$$\frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x + ty\|^2 dt = \frac{1}{6}\|y - x\|^2.$$

If $p = 1$, then

$$\begin{aligned} 0 &\leq \frac{\|x\| + \|y\|}{2} - \int_0^1 \|(1-t)x + ty\| dt \\ &\leq \frac{1}{8}\left\langle y - x, \frac{y}{\|y\|} - \frac{x}{\|x\|} \right\rangle, \end{aligned}$$

for any nonzero $x, y \in X$. We obtain a nontrivial equality by choosing $X = \mathbb{R}$ and multiplication for its inner product (which induces the absolute value for its norm), $x = 1$ and $y = -1$. Thus, the constant $\frac{1}{8}$ is sharp.

Conjecture 11. We conjecture that the constant $\frac{1}{8}$ in (25) is not sharp for any $p > 1$. Utilizing Maple for the real-valued functions

$$F_p(x,y) := \frac{|x|^p + |y|^p}{2} - \int_0^1 |(1-t)x + ty|^p dt,$$
$$G_p(x,y) := \frac{1}{8}p(y-x)(y|y|^{p-2} - x|x|^{p-2}),$$

for $(x,y) \in \mathbb{R}^2$, we observe that for several values of $p > 1$, the equation $F_p(x,y) = G_p(x,y) = k \neq 0$ has no solution in $\mathbb{R}^2$ (see Figure 1 for the plot of these functions with the choice of $p = 3$). Therefore, the constant $\frac{1}{8}$ is not sharp for these values of p, since we have no nontrivial equality. However, we do not have an analytical proof for this claim.

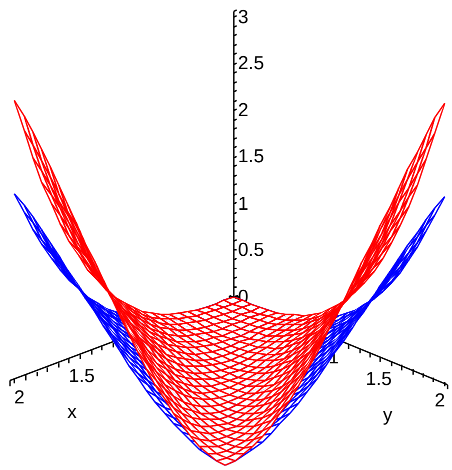

Figure 1. Plot of F_3 and G_3

Proposition 12. *Let $(X, \|\cdot\|)$ be a normed linear space and $1 \leq p < \infty$. Then*

$$\begin{aligned} 0 &\leq \frac{1}{8}p\left\|\frac{y+x}{2}\right\|^{p-2}\left[\left\langle y-x, \frac{y+x}{2}\right\rangle_s - \left\langle y-x, \frac{y+x}{2}\right\rangle_i\right] \\ &\leq \int_0^1 \|(1-t)x+ty\|^p dt - \left\|\frac{x+y}{2}\right\|^p \\ &\leq \frac{1}{8}p[\|y\|^{p-2}\langle y-x, y\rangle_i - \|x\|^{p-2}\langle y-x, x\rangle_s], \end{aligned} \tag{26}$$

holds for any $x, y \in X$ whenever $p \geq 2$; otherwise, it holds for linearly independent $x, y \in X$. The above inequality provides bounds for the distance between the first two terms in (6).
The constant $\frac{1}{8}$ is sharp.

In particular, we have

$$
\begin{aligned}
0 &\leq \frac{1}{8}[\langle y-x, y+x\rangle_s - \langle y-x, y+x\rangle_i] \\
&\leq \int_0^1 \|(1-t)x+ty\|^2 dt - \left\|\frac{x+y}{2}\right\|^2 \qquad (27)\\
&\leq \frac{1}{4}[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s],
\end{aligned}
$$

for any $x, y \in X$ *and*

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[\left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_s - \left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_i\right] \\
&\leq \int_0^1 \|(1-t)x+ty\| dt - \left\|\frac{x+y}{2}\right\| \qquad (28)\\
&\leq \frac{1}{8}\left[\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right],
\end{aligned}
$$

for any linearly independent $x, y \in X$.
The constants in (27) and (28) are sharp.

Proof. We obtain (26) by taking $\sigma = \frac{1}{2}$ in (19). We may also obtain (26) by taking $f(x) = \|x\|^p \quad (1 \leq p < \infty)$ in Remark 5. Then, (27) and (28) follow by taking $p = 2$ and $p = 1$, respectively, in (26). Note that we may also obtain (27) from (20) and (28) from (21) by letting $\sigma = \frac{1}{2}$. The sharpness of the constants in (26) would follow by the sharpness of the constants in (27) and (28) as its particular cases.

We will now prove the sharpness of the constants in (27). Assume that the above inequality holds for constants $E, F > 0$ instead of $\frac{1}{8}$ and $\frac{1}{4}$ respectively, that is

$$
\begin{aligned}
0 &\leq E[\langle y-x, y+x\rangle_s - \langle y-x, y+x\rangle_i] \\
&\leq \int_0^1 \|(1-t)x+ty\|^2 dt - \left\|\frac{x+y}{2}\right\|^2 \\
&\leq F[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s].
\end{aligned}
$$

Now, take $(X, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_1)$, we obtain

$$
\begin{aligned}
&2E\|y+x\|_1 \sum_{y_i+x_i=0} |y_i - x_i| \\
&\leq \int_0^1 \|(1-t)x+ty\|_1^2 dt - \left\|\frac{x+y}{2}\right\|_1^2 \\
&\leq F\left[\|y\|_1\left(\sum_{y_i\neq 0} \frac{y_i}{|y_i|}(y_i - x_i) - \sum_{y_i=0}|y_i - x_i|\right)\right. \\
&\qquad \left. -\|x\|_1\left(\sum_{x_i\neq 0} \frac{x_i}{|x_i|}(y_i - x_i) + \sum_{x_i=0}|y_i - x_i|\right)\right].
\end{aligned}
$$

Choose $x = \left(-\frac{1}{n}, n\right)$ and $y = \left(\frac{1}{n}, n\right)$, for any $n \in \mathbb{N}$ and we have the following

$$
8E \leq \frac{3n^2+1}{3n^2} \leq 4F\left(\frac{1}{n^2} + 1\right).
$$

Taking $n \to \infty$, we get

$$8E \le 1 \le 4F,$$

that is, $E \le \frac{1}{8}$ in the first inequality, and $F \ge \frac{1}{4}$ in the second inequality. Thus, both the constants $\frac{1}{8}$ and $\frac{1}{4}$ are sharp in the first and second inequality respectively. This implies that the constants in (9), (13), (14), (15), (19), (20), and (26) are sharp.

Now, we will prove the sharpness of the constants in (28). Assume that the inequality holds for $G, H > 0$ instead of $\frac{1}{8}$, that is,

$$\begin{aligned} 0 &\le G\left[\left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_s - \left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_i\right] \\ &\le \int_0^1 \|(1-t)x+ty\|dt - \left\|\frac{x+y}{2}\right\| \\ &\le H\left[\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right]. \end{aligned}$$

Again, take $(X, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_1)$, we have the following inequality

$$\begin{aligned} 0 &\le 2G\sum_{y_i+x_i=0}|y_i-x_i| \\ &\le \int_0^1 \|(1-t)x+ty\|_1 dt - \left\|\frac{x+y}{2}\right\|_1 \\ &\le H\left[\sum_{y_i\neq 0}\frac{y_i}{|y_i|}(y_i-x_i) - \sum_{y_i=0}|y_i-x_i| - \sum_{x_i\neq 0}\frac{x_i}{|x_i|}(y_i-x_i) - \sum_{x_i=0}|y_i-x_i|\right], \end{aligned}$$

for any linearly independent $x, y \in X$.

Choose $x = (1,0)$ and $y = (-1,1)$ and we obtain

$$4G \le \frac{1}{2} \le 4H,$$

that is, $G \le \frac{1}{8}$ in the first inequality and $H \ge \frac{1}{8}$ in the second inequality. Therefore the constant $\frac{1}{8}$ is sharp in both inequalities. This implies that the constants in (21) are also sharp. □

Remark 13 (The case of inner product spaces)**.** Let X be an inner product space, with the inner product $\langle\cdot,\cdot\rangle$, in Proposition 12. Then,

$$\begin{aligned} 0 &\le \int_0^1 \|(1-t)x+ty\|^p dt - \left\|\frac{x+y}{2}\right\|^p \\ &\le \frac{1}{8}p\langle y-x, y\|y\|^{p-2} - x\|x\|^{p-2}\rangle, \end{aligned} \tag{29}$$

holds for any $x, y \in X$ whenever $p \ge 2$; otherwise, it holds for nonzero $x, y \in X$. Particularly, for $p = 2$, we have

$$\begin{aligned} 0 &\le \int_0^1 \|(1-t)x+ty\|^2 dt - \left\|\frac{x+y}{2}\right\|^2 \\ &\le \frac{1}{4}[\langle y-x, y\rangle - \langle y-x, x\rangle] = \frac{1}{4}\|y-x\|^2, \end{aligned}$$

for any $x, y \in X$. The constant $\frac{1}{4}$ is not the best possible constant in this case, since we always have

$$\int_0^1 \|(1-t)x + ty\|^2 dt - \left\| \frac{x+y}{2} \right\|^2 = \frac{1}{12} \|y - x\|^2.$$

If $p = 1$, we have

$$\begin{aligned} 0 &\le \int_0^1 \|(1-t)x + ty\| dt - \left\| \frac{x+y}{2} \right\| \\ &\le \frac{1}{8} \left\langle y - x, \frac{y}{\|y\|} - \frac{x}{\|x\|} \right\rangle, \end{aligned}$$

for any nonzero $x, y \in X$. By choosing $X = \mathbb{R}$ and multiplication for its inner product (which induces the absolute value for its norm), $x = 1$ and $y = -1$, we obtain a nontrivial equality. Thus, the constant $\frac{1}{8}$ is sharp.

Conjecture 14. We conjecture that the constant $\frac{1}{8}$ in (29) is not sharp for any $p > 1$. Utilizing Maple for the real-valued functions

$$\begin{aligned} \Phi_p(x, y) &:= \int_0^1 |(1-t)x + ty|^p dt - \left| \frac{x+y}{2} \right|^p, \\ \Psi_p(x, y) &:= \frac{1}{8} p(y - x)(y|y|^{p-2} - x|x|^{p-2}), \end{aligned}$$

for $(x, y) \in \mathbb{R}^2$, we observe that for several values of $p > 1$, the equation $\Phi_p(x, y) = \Psi_p(x, y) = k \neq 0$ has no solution in $\mathbb{R}^2$ (see Figure 2 for the plot of these functions with the choice of $p = 3$). Therefore, the constant $\frac{1}{8}$ is not sharp for these values of p, since we have no nontrivial equality. However, we do not have an analytical proof for this claim.

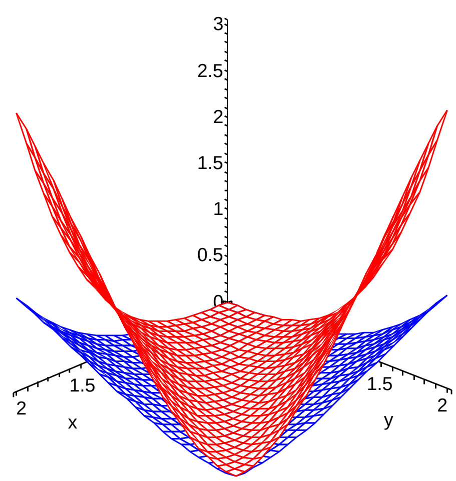

FIGURE 2. Plot of Φ_3 and Ψ_3

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School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City MC, Victoria 8001, Australia

E-mail address: eder.kikianty@research.vu.edu.au

E-mail address: Sever.Dragomir@vu.edu.au

E-mail address: Pietro.Cerone@vu.edu.au