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# A WEIGHTED GEOMETRIC INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. A new weighted geometric inequality is established by Klamkin's polar moment of inertia inequality and the inversion transformation, some interesting applications of this result are given, and some conjectures which verified by computer are also mentioned

## 1. Introduction

In 1975, M.S.Klamkin [1] established the following inequality: Let  $ABC$  be an arbitrary triangle of sides  $a, b, c$ , and let  $P$  be an arbitrary point in space, the distances of  $P$  from the vertices  $A, B, C$  are  $R_1, R_2, R_3$ .  $x, y, z$  are real numbers, then

$$(1.1) \quad (x + y + z)(xR_1^2 + yR_2^2 + zR_3^2) \geq yza^2 + zxb^2 + xyc^2,$$

with equality if and only if  $P$  lies the plane of  $\triangle ABC$  and  $x : y : z = \vec{S}_{\triangle PBC} : \vec{S}_{\triangle PCA} : \vec{S}_{\triangle PAB}$  ( $\vec{S}_{\triangle PCA}$  denote the algebra area, etc.)

Inequality (1.1) is called the polar moment of inertia inequality, it is one of the most important inequality for the triangle, there exist many consequences and applications for this one, see [1]-[5]. In this paper, we will apply Klamkin' inequality (1.1) and the inversion transformation to deduce a new weighted geometric inequality, then we discuss applications of our results. In addition, we also pose some conjectures.

## 2. Main Result

In order to prove our new results, first of all, we give the following lemma.

**Lemma** Let  $ABC$  be an arbitrary triangle, and let  $P$  be an arbitrary point in the plane of the triangle  $ABC$ , if the following inequality:

$$(2.1) \quad f(a, b, c, R_1, R_2, R_3) \geq 0.$$

holds, then we have the dual inequality:

$$(2.2) \quad f(aR_1, bR_2, cR_3, R_2R_3, R_3R_1, R_1R_2) \geq 0.$$

Indeed, the above conclusion can be deduced by using inversion transformation, see[6] or [3], [7].

Now, We state and prove main result.

**Theorem** Let  $x, y, z$  be positive real numbers, then for any triangle  $ABC$  and arbitrary point  $P$  in the plane of  $\triangle ABC$  holds the following inequality:

$$(2.3) \quad \frac{R_1^2}{x} + \frac{R_2^2}{y} + \frac{R_3^2}{z} \geq \frac{aR_1 + bR_2 + cR_3}{\sqrt{yz + zx + xy}},$$

with equality if and only if  $\triangle ABC$  is acute-angled,  $P$  coincide with its orthocenter and  $x : y : z = \cot A : \cot B : \cot C$ .

*Proof.* If  $P$  coincide with one of the vertices of  $\triangle ABC$ , for example  $P \equiv A$ , then  $PA = 0, PB = c, PC = b$ , (2.3) becomes trivial inequality and we easily to know it is holds true. In this case, equality in (2.3) obviously cannot occur.

Next assume  $P$  does not coincide with the vertices.

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If  $x, y, z$  are positive real numbers, then by the polar moment of inertia inequality (1.1) we have

$$(xR_1^2 + yR_2^2 + zR_3^2) \left( \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} \right) \geq \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}.$$

On the other hand, from Cauchy-Schwarz inequality we get

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z},$$

with equality if and only if  $x : y : z = a : b : c$ .

Combining these two above inequalities, for any positive real numbers  $x, y, z$ , the following inequality holds:

$$(2.4) \quad (xR_1^2 + yR_2^2 + zR_3^2) \left( \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} \right) \geq \frac{(a+b+c)^2}{x+y+z}.$$

and we easily to know that the equality if and only if  $x : y : z = a : b : c$  and  $P$  is the incenter of  $\triangle ABC$ .

Now, applying the inversion transformation in the lemma to inequality (2.4), we obtain

$$[x(R_2R_3)^2 + y(R_3R_1)^2 + z(R_1R_2)^2] \left( \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} \right) \geq \frac{(aR_1 + bR_2 + cR_3)^2}{x+y+z},$$

or equivalently

$$(2.5) \quad \frac{(R_2R_3)^2}{yz} + \frac{(R_3R_1)^2}{zx} + \frac{(R_1R_2)^2}{xy} \geq \left( \frac{aR_1 + bR_2 + cR_3}{x+y+z} \right)^2.$$

where  $x, y, z$  are positive numbers.

For  $x \rightarrow xR_1^2, y \rightarrow yR_2^2, z \rightarrow zR_3^2$ , then holds

$$(2.6) \quad \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} \geq \left( \frac{aR_1 + bR_2 + cR_3}{xR_1^2 + yR_2^2 + zR_3^2} \right)^2.$$

Take again  $x \rightarrow \frac{1}{x}, y \rightarrow \frac{1}{y}, z \rightarrow \frac{1}{z}$ , then we get the inequality (2.3) of the theorem.

Notice that the conclusion in [7]: If equality in (2.1) occurs only  $P$  is the incenter of  $\triangle ABC$ , then equality in (2.2) only  $\triangle ABC$  is acute-angled and  $P$  is its orthocenter. According to this and the condition which occurs equality in (2.4), we easy know equality in (2.3) if and only if  $\triangle ABC$  is acute-angled,  $P$  is its orthocenter and holds

$$(2.7) \quad \frac{R_1}{xa} = \frac{R_2}{yb} = \frac{R_3}{cz}.$$

Since when  $P$  is the orthocenter of the acute triangle  $ABC$ , we have  $R_1 : R_2 : R_3 = \cos A : \cos B : \cos C$ . Hence from (2.7) we have  $x : y : z = \cot A : \cot B : \cot C$  in this case. Thus, there is equality in (2.3) if and only if  $\triangle ABC$  is acute-angled,  $P$  coincide with its orthocenter and  $x/\cot A = y/\cot B = z/\cot C$ . This complete the proof of the Theorem.  $\square$

*Remark 2.1.* If  $P$  does not coincide with the vertices, then inequality (2.4) is equivalent to the following result in [8]:

$$(2.8) \quad x \frac{R_2R_3}{R_1} + y \frac{R_3R_1}{R_2} + z \frac{R_1R_2}{R_3} \geq 2\sqrt{\frac{xyz}{x+y+z}}s,$$

where  $s$  is the semi-perimeter of  $\triangle ABC$ ,  $x, y, z$  are positive real numbers. In [8], the proof of (2.8) without using the polar moment of inertia inequality, So does not start from Klamkin's inequality (1.1) we can deduce the inequality of the theorem.

### 3. Applications of the theorem

Besides the above notations, as usual, let  $R$  and  $r$  denote the radii of the circumcircle and incircle of triangle  $ABC$ , respectively,  $\Delta$  denote the area,  $r_a, r_b, r_c$  denote the radii of excircles. In addition, when point  $P$  lies interior of triangle  $ABC$ , let  $r_1, r_2, r_3$  denote the distances of  $P$  to the sides  $BC, CA, AB$ .

According to the theorem and the well-known inequality for any point  $P$  in the plane

$$(3.1) \quad aR_1 + bR_2 + cR_3 \geq 4\Delta,$$

We get

**Corollary 1** For any point  $P$  in the plane and arbitrary positive numbers  $x, y, z$ , the following inequality holds:

$$(3.2) \quad \frac{R_1^2}{x} + \frac{R_2^2}{y} + \frac{R_3^2}{z} \geq \frac{4\Delta}{\sqrt{yz + zx + xy}},$$

with equality if and only if  $x : y : z = \cot A : \cot B : \cot C$  and  $P$  is the orthocenter of the acute angled triangle  $ABC$ .

*Remark 3.1.* Clearly, (3.2) is equivalent with

$$(3.3) \quad xR_1^2 + yR_2^2 + zR_3^2 \geq 4\sqrt{\frac{xyz}{x+y+z}}\Delta.$$

The above inequality is first given in [9] by Xue-Zhi Yang. The author [10] obtained the following generalization:

$$(3.4) \quad x\left(\frac{a'}{a}\right)^2 + y\left(\frac{b'}{b}\right)^2 + z\left(\frac{c'}{c}\right)^2 \geq 4\sqrt{\frac{xyz}{x+y+z}}\Delta',$$

where  $a', b', c'$  denote the sides of  $\triangle A'B'C'$ ,  $\Delta'$  denote its area.

If, in (2.3) we put  $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$ , note that  $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}$ , we get the result:

**Corollary 2** For arbitrary point  $P$  in the plane of  $\triangle ABC$ , the following inequality holds:

$$(3.5) \quad \frac{aR_1^2 + bR_2^2 + cR_3^2}{aR_1 + bR_2 + cR_3} \geq \sqrt{2Rr},$$

equality holds if and only if the triangle  $ABC$  is equilateral and  $P$  is its center.

*Remark 3.2.* The conditions for equality that the following inequalities of corollary 4-8 are the same in the statement of Corollary 2.

In the theorem, for  $x = \frac{R_1}{a}, y = \frac{R_2}{b}, z = \frac{R_3}{c}$ , after reductions we obtain

**Corollary 3** If  $P$  is arbitrary point which does not coincide with the vertices of  $\triangle ABC$ , then

$$(3.6) \quad \frac{R_2R_3}{bc} + \frac{R_3R_1}{ca} + \frac{R_1R_2}{ab} \geq 1,$$

equality holds if and only if  $\triangle ABC$  is acute-angled and  $P$  is its orthocenter.

Inequality (3.6) first proved by T.Hayashi (see [11] or [3]), the author gave its two generalizations in [12].

Indeed, assume  $P$  does not coincide with the vertices, put  $x \rightarrow \frac{R_1}{xa}, y \rightarrow \frac{R_2}{yb}, z \rightarrow \frac{R_3}{zc}$  in (2.2), then we can get the weighted generalization form of Hayashi inequality:

$$(3.7) \quad \frac{R_2R_3}{yzbc} + \frac{R_3R_1}{zxca} + \frac{R_1R_2}{xyab} \geq \left(\frac{aR_1 + bR_2 + cR_3}{xaR_1 + ybR_2 + zcR_3}\right)^2.$$

For  $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$ , we have

$$(3.8) \quad (R_2R_3 + R_3R_1 + R_1R_2)(R_1 + R_2 + R_3)^2 \geq (aR_1 + bR_2 + cR_3)^2.$$

Applying the inversion transformation of the lemma to the above inequality, then divide both sides by  $R_1R_2R_3$  we get

**Corollary 4** If  $P$  is arbitrary point which does not coincide with the vertices of  $\triangle ABC$ , then

$$(3.9) \quad (R_2R_3 + R_3R_1 + R_1R_2)^2 \left( \frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2} \right) \geq 4s^2.$$

It is not difficult to know that the above inequality is stronger than the following result which the author obtained many years ago:

$$(3.10) \quad \sqrt{\frac{R_2R_3}{R_1}} + \sqrt{\frac{R_3R_1}{R_2}} + \sqrt{\frac{R_1R_2}{R_3}} \geq \sqrt{2\sqrt{3}s}$$

Now, Let  $P$  be an interior point of the triangle  $ABC$ , then we have the well known inequalities(see[13]):

$$aR_1 \geq br_3 + cr_2, bR_2 \geq cr_1 + ar_3, cR_3 \geq ar_2 + br_1,$$

Summing them up, note that  $a + b + c = 2s$  and the identity  $ar_1 + br_2 + cr_3 = 2rs$  we easily get

$$(3.11) \quad aR_1 + bR_2 + cR_3 \geq 2s(r_1 + r_2 + r_3) - 2rs.$$

Multiplying both sides by 2 then adding inequality (3.1) and using  $\Delta = rs$ , then

$$3(aR_1 + bR_2 + cR_3) \geq 4s(r_1 + r_2 + r_3),$$

that is

$$(3.12) \quad \frac{aR_1 + bR_2 + cR_3}{r_1 + r_2 + r_3} \geq \frac{4}{3}s.$$

According to this and the equivalent form (2.5) of inequality (2.3), we get immediately the result:

**Corollary 5** Let  $P$  be an interior point of the triangle  $ABC$ , then

$$(3.13) \quad \frac{(R_2R_3)^2}{r_2r_3} + \frac{(R_3R_1)^2}{r_3r_1} + \frac{(R_1R_2)^2}{r_1r_2} \geq \frac{16}{9}s^2.$$

From inequality (3.8) and (3.12) we infer that

$$(R_2R_3 + R_3R_1 + R_1R_2)(R_1 + R_2 + R_3)^2 \geq \frac{16}{9}s^2(r_1 + r_2 + r_3)^2,$$

Note that again  $3(R_2R_3 + R_3R_1 + R_1R_2) \leq (R_1 + R_2 + R_3)^2$ , we get the following inequality:

**Corollary 6** Let  $P$  be an interior point of triangle  $ABC$ , then

$$(3.14) \quad \frac{(R_1 + R_2 + R_3)^2}{r_1 + r_2 + r_3} \geq \frac{4}{\sqrt{3}}s.$$

Letting  $x = r_a, y = r_b, z = r_c$  in (2.3) and note that identity  $r_b r_c + r_c r_a + r_a r_b = s^2$ , hence

$$(3.15) \quad \frac{R_1^2}{r_a} + \frac{R_2^2}{r_b} + \frac{R_3^2}{r_c} \geq \frac{1}{s}(aR_1 + bR_2 + cR_3).$$

This inequality and (3.12) lead us get the following inequality:

**Corollary 7** Let  $P$  be an interior point of the triangle  $ABC$ , then

$$(3.16) \quad \frac{R_1^2}{r_a} + \frac{R_2^2}{r_b} + \frac{R_3^2}{r_c} \geq \frac{4}{3}(r_1 + r_2 + r_3).$$

Adding (3.1) and (3.11) then dividing both sides by 2, we have

$$(3.17) \quad aR_1 + bR_2 + cR_3 \geq s(r_1 + r_2 + r_3 + r).$$

From this and (3.15), we get again the following inequality which similar to (3.16):

**Corollary 8** Let  $P$  be an interior point of the triangle  $ABC$ , then

$$(3.18) \quad \frac{R_1^2}{r_a} + \frac{R_2^2}{r_b} + \frac{R_3^2}{r_c} \geq r_1 + r_2 + r_3 + r.$$

When  $P$  locates interior of the triangle  $ABC$ , let  $D, E, F$  be the feet of the perpendicular from  $P$  to the sides  $BC, CA, AB$  respectively. Take  $x = ar_1, y = br_2, z = cr_3$  in equivalent form (2.6) of inequality (2.3), then

$$\frac{1}{bcr_2r_3} + \frac{1}{car_3r_1} + \frac{1}{abr_1r_2} \geq \left( \frac{aR_1 + bR_2 + cR_3}{ar_1R_1 + br_2R_2 + cr_3R_3} \right)^2,$$

Using identity  $ar_1 + br_2 + cr_3 = 2\Delta$  and well known identity(see[7]):

$$(3.19) \quad ar_1R_1^2 + br_2R_2^2 + cr_3R_3^2 = 8R^2\Delta_p,$$

(where  $\Delta_p$  is the area of pedal triangle  $DEF$ ) We get

$$abcr_1r_2r_3(aR_1 + bR_2 + cR_3)^2 \leq 64\Delta R^4\Delta_p^2.$$

Let  $s_p, r_p$  denote the semi-perimeter of the triangle  $DEF$  and radius of incircle respectively. Note that  $\Delta_p = r_p s_p, aR_1 + bR_2 + cR_3 = 4Rs_p$ , from the above inequality we obtain the following inequality which established by the author in [14]:

**Corollary 9** Let  $P$  be an interior point of the triangle  $ABC$ , then

$$(3.20) \quad \frac{r_1r_2r_3}{r_p^2} \leq 2R,$$

equality holds if and only if  $P$  is the orthocenter of the triangle  $ABC$ .

It is well known, the inequality related to a triangle and two points is very few. Several years ago, the author guessed the following inequality holds:

$$(3.21) \quad \frac{R_1^2}{d_1} + \frac{R_2^2}{d_2} + \frac{R_3^2}{d_3} \geq 4(r_1 + r_2 + r_3),$$

where  $d_1, d_2, d_3$  denote the distances from an interior point  $Q$  to the sides of  $\triangle ABC$ .

Inequality (3.21) is very interesting, the author have been trying to prove it. In what follows, we will prove the stronger result. To do so, we need a corollary of the following conclusion(see[15]):

Let  $Q$  be an interior point of  $\triangle ABC$ ,  $t_1, t_2, t_3$  denote the bisector of  $\angle BQC, \angle CQA, \angle AQB$  respectively,  $\triangle A'B'C'$  is an arbitrary triangle, then

$$(3.22) \quad t_2t_3 \sin A' + t_3t_1 \sin B' + t_1t_2 \sin C' \leq \frac{1}{2}\Delta,$$

with equality if and only if  $\triangle A'B'C' \sim \triangle ABC$ , and  $Q$  is the circumcentre of  $\triangle ABC$ .

In (3.22), letting  $\triangle ABC$  be equilateral, then we immediately get

$$(3.23) \quad t_2t_3 + t_3t_1 + t_1t_2 \leq \frac{1}{\sqrt{3}}\Delta.$$

From this and the simple inequality  $s^2 \geq 3\sqrt{3}\Delta$ , we have

$$(3.24) \quad t_2t_3 + t_3t_1 + t_1t_2 \leq \frac{1}{9}s^2.$$

According to the inequality (2.3) of the theorem and (3.24), we can see that

$$(3.25) \quad \frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq \frac{3}{s}(aR_1 + bR_2 + cR_3).$$

By using inequality (3.12), we obtain the following stronger of inequality (3.21)

**Corollary 10** Let  $P$  and  $Q$  be two interior points of the  $\triangle ABC$ , then

$$(3.26) \quad \frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq 4(r_1 + r_2 + r_3).$$

with equality if and only if  $\triangle ABC$  is equilateral,  $P$  and  $Q$  both are its center.

Analogously, from inequality (3.17) and inequality (3.25) we get

**Corollary 11** Let  $P$  and  $Q$  be two interior points of  $\triangle ABC$ , then

$$(3.27) \quad \frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq 3(r_1 + r_2 + r_3 + r).$$

with equality if and only if  $\triangle ABC$  is equilateral, and  $P$  and  $Q$  both are its center.

#### 4. Some conjectures

In this section, we will state some conjectures in allusion to the inequality which appeared in this paper.

Inequality (3.8) is equivalent to

$$(4.1) \quad R_2R_3 + R_3R_1 + R_1R_2 \geq \left( \frac{aR_1 + bR_2 + cR_3}{R_1 + R_2 + R_3} \right)^2.$$

Enlightened by this one and the well known inequality:

$$(4.2) \quad R_2R_3 + R_3R_1 + R_1R_2 \geq 4(w_2w_3 + w_3w_1 + w_1w_2),$$

We pose the following

**Conjecture 1** Let  $P$  be an arbitrary interior point of the triangle  $ABC$ , then

$$(4.3) \quad \left( \frac{aR_1 + bR_2 + cR_3}{R_1 + R_2 + R_3} \right)^2 \geq 4(w_2w_3 + w_3w_1 + w_1w_2).$$

We consider the stronger of corollary 5, the author posed these two conjectures:

**Conjecture 2** Let  $P$  be an arbitrary interior point of the triangle  $ABC$ , then

$$(4.4) \quad \frac{(R_2R_3)^2}{w_2w_3} + \frac{(R_3R_1)^2}{w_3w_1} + \frac{(R_1R_2)^2}{w_1w_2} \geq \frac{4}{3}(a^2 + b^2 + c^2).$$

**Conjecture 3** Let  $P$  be an arbitrary interior point of the triangle  $ABC$ , then

$$(4.5) \quad \frac{(R_2R_3)^2}{r_2r_3} + \frac{(R_3R_1)^2}{r_3r_1} + \frac{(R_1R_2)^2}{r_1r_2} \geq 4(R_1^2 + R_2^2 + R_3^2).$$

For the inequality of corollary 6, we guess the following stronger inequality holds:

**Conjecture 4** Let  $P$  be an arbitrary interior point of the triangle  $ABC$ , then

$$(4.6) \quad \frac{R_2R_3 + R_3R_1 + R_1R_2}{r_1 + r_2 + r_3} \geq \frac{4}{3\sqrt{3}}s.$$

On the other hand, for the acute-angled triangle, we pose the following conjecture:

**Conjecture 5** Let  $\triangle ABC$  be acute-angled and  $P$  is an arbitrary point in its interior, then

$$(4.7) \quad \frac{(R_1 + R_2 + R_3)^2}{w_1 + w_2 + w_3} \geq 6R.$$

Two years ago, Xue-Zhi Yang proved the following inequality (in a private communication with the author):

$$(4.8) \quad \frac{(R_1 + R_2 + R_3)^2}{r_1 + r_2 + r_3} \geq 2\sqrt{a^2 + b^2 + c^2}.$$

which is stronger than (3.14). Here, we further put forward the following

**Conjecture 6** Let  $P$  be an arbitrary interior point of the triangle  $ABC$ , then

$$(4.9) \quad \frac{(R_1 + R_2 + R_3)^2}{w_1 + w_2 + w_3} \geq 2\sqrt{a^2 + b^2 + c^2}.$$

In [14], the author point out the following phenomenon(so-called  $r - w$  phenomenon): If holds the inequality for  $r_1, r_2, r_3$ (this inequality can be also includes  $R_1, R_2, R_3$  and otherwise geometric elements), then after changing  $r_1, r_2, r_3$  into  $w_1, w_2, w_3$  respectively, the stronger inequality often holds or often holds for the acute triangle. Conjecture 6 was proposed based on this kind of phenomenon. Analogously, we pose these following four conjectures:

**Conjecture 7** Let  $\triangle ABC$  be acute-angled and  $P$  is an arbitrary point in its interior, then

$$(4.10) \quad \frac{aR_1 + bR_2 + cR_3}{w_1 + w_2 + w_3} \geq \frac{4}{3}s.$$

**Conjecture 8** Let  $\triangle ABC$  be acute-angled and  $P$  is an arbitrary point in its interior, then

$$(4.11) \quad \frac{aR_1 + bR_2 + cR_3}{w_1 + w_2 + w_3 + r} \geq 2s.$$

**Conjecture 9** Let  $P$  and  $Q$  be two interior points of the  $\triangle ABC$ , then

$$(4.12) \quad \frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq 4(w_1 + w_2 + w_3).$$

**Conjecture 10** Let  $P$  and  $Q$  be two interior points of the  $\triangle ABC$ , then

$$(4.13) \quad \frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq 3(w_1 + w_2 + w_3 + r).$$

*Remark 4.1.* If conjecture 7 and 8 are proofed, then we can proof conjecture 9 and 10 valid for the acute triangle  $ABC$ .

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