A New Refinement of Young's Inequality

A NEW REFINEMENT OF YOUNG'S INEQUALITY

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Abstract. In this short note, the well-known Young's inequality is refined by a double inequality.

The original Young's inequality is as follows.

Theorem 1 ([7]). *Let $f(x)$ be a continuous and strictly increasing function on $[0,A]$ for $A>0$. If $f(0)=0$, $a\in[0,A]$ and $b\in[0,f(A)]$, then*

$$\int_0^a f(x)\,\mathrm{d}x+\int_0^b f^{-1}(x)\,\mathrm{d}x\ge ab, \tag{1}$$

where f^{-1} is the inverse function of f. Equality in (1) *is valid if and only if $b=f(a)$.*

The following theorem is a converse of Theorem 1 which was proved in [5].

Theorem 2 ([5]). *If the functions $f(x)$ and $g(x)$ for $x\ge0$ are continuous and strictly increasing with $f(0)=g(0)=0$, $g^{-1}(x)\ge f(x)$ and*

$$\int_0^a f(x)\,\mathrm{d}x+\int_0^b g(x)\,\mathrm{d}x\ge ab \tag{2}$$

for all $a>0$ and $b>0$, then $f=g^{-1}$.

The following reversed version of Young's inequality (1) was obtained in [6].

Theorem 3 ([6]). *Under the assumptions of Theorem 1, inequality*

$$\min\biggl\{1,\frac{b}{f(a)}\biggr\}\int_0^a f(t)\,\mathrm{d}t+\min\biggl\{1,\frac{a}{f^{-1}(b)}\biggr\}\int_0^b f^{-1}(t)\,\mathrm{d}t\le ab, \tag{3}$$

holds. Equality in (3) *is valid if and only if $b=f(a)$.*

For more information on Young's inequality, please refer to [1, pp. 651–653], [2, pp. 48–50], [3, Chapter XIV, pp. 379–389] and the references therein.

In this short note, we would like to refine Young's inequality (1) by a double inequality below.

Theorem 4. *Let $f(x)$ be a continuous, differentiable and strictly increasing function on $[0,A]$ for $A>0$. If $f(0)=0$, $a\in[0,A]$, $b\in[0,f(A)]$ and $f'(x)$ is strictly monotonic on $[0,A]$, then*

$$\frac{m}{2}\bigl[a-f^{-1}(b)\bigr]^2\le\int_0^a f(x)\,\mathrm{d}x+\int_0^b f^{-1}(x)\,\mathrm{d}x-ab\le\frac{M}{2}\bigl[a-f^{-1}(b)\bigr]^2, \tag{4}$$

Date: This paper was finalized on March 6, 2007.
2000 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Young's inequality, refinement, integral.
This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

where $m = \min\{f'(a), f'(f^{-1}(b))\}$ *and* $M = \max\{f'(a), f'(f^{-1}(b))\}$. *Equalities in* (4) *are valid if and only if* $b = f(a)$.

Proof. Changing variable of integration by $x = f(y)$ and integrating by part of the second integral in (4) yields

$$\begin{aligned}\int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(x)\,\mathrm{d}x &= \int_0^a f(x)\,\mathrm{d}x + \int_0^{f^{-1}(b)} y f'(y)\,\mathrm{d}y \\ &= \int_0^a f(x)\,\mathrm{d}x + b f^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\,\mathrm{d}x \\ &= b f^{-1}(b) + \int_{f^{-1}(b)}^a f(x)\,\mathrm{d}x \\ &= ab + \int_{f^{-1}(b)}^a \big[f(x) - b\big]\,\mathrm{d}x.\end{aligned} \tag{5}$$

From the fourth line in (5), it is easy to see that if $f^{-1}(b) = a$ then equalities in (4) hold.

If $f^{-1}(b) < a$, since $f(x)$ is strictly increasing, then $f(x) - b > 0$ for $x \in (f^{-1}(b), a)$. By mean value theorem, it is obtained that there exists $c = c(x)$ satisfying $f^{-1}(b) < c < x \le a$ such that $0 < f(x) - b = [x - f^{-1}(b)]f'(c)$. Further, by virtue of the monotonicity of $f'(x)$ on $[0, A]$, it is revealed that

$$0 < m \triangleq \min\{f'(a), f'(f^{-1}(b))\} < f'(c) < \max\{f'(a), f'(f^{-1}(b))\} \triangleq M.$$

Consequently,

$$0 < m[x - f^{-1}(b)] < f(x) - b < M[x - f^{-1}(b)].$$

As a result,

$$m\int_{f^{-1}(b)}^a \big[x - f^{-1}(b)\big]\,\mathrm{d}x < \int_{f^{-1}(b)}^a \big[f(x) - b\big]\,\mathrm{d}x < M\int_{f^{-1}(b)}^a \big[x - f^{-1}(b)\big]\,\mathrm{d}x$$

which is equivalent to

$$\frac{m}{2}\big[a - f^{-1}(b)\big]^2 < \int_{f^{-1}(b)}^a \big[f(x) - b\big]\,\mathrm{d}x < \frac{M}{2}\big[a - f^{-1}(b)\big]^2. \tag{6}$$

If $f^{-1}(b) > a$, inequalities in (6) can be deduced by a similar argument as above.

Substituting (6) into (5) leads to (4). The proof of Theorem 4 is complete. □

Remark 1. Taking $f(x) = \sqrt[4]{x^4 + 1} - 1$, $a = 3$ and $b = 2$ in Theorem 4 and direct calculation gives

$$\int_0^3 f(x)\,\mathrm{d}x = \int_0^3 \sqrt[4]{x^4 + 1}\,\mathrm{d}x - 3,$$

$$\int_0^2 f^{-1}(x)\,\mathrm{d}x = \int_0^2 \sqrt[4]{(x + 1)^4 - 1}\,\mathrm{d}x = \int_1^3 \sqrt[4]{x^4 - 1}\,\mathrm{d}x,$$

$$f'(x) = \frac{x^3}{\sqrt[4]{(x^4 + 1)^3}}, \quad f'(3) = \frac{27}{\sqrt[4]{82^3}}, \quad f'(f^{-1}(2)) = f'(\sqrt[4]{80}) = \frac{8\sqrt[4]{5^3}}{27},$$

$$m = \frac{2\sqrt[4]{5}}{27}, \quad M = \frac{27}{\sqrt[4]{82^3}}$$

and

$$9+\frac{\sqrt[4]{5}}{27}\left[3-2\sqrt[4]{5}\right]^2<\int_0^3\sqrt[4]{x^4+1}\,\mathrm{d}x+\int_1^3\sqrt[4]{x^4-1}\,\mathrm{d}x<9+\frac{27}{2\sqrt[4]{82^3}}\left[3-2\sqrt[4]{5}\right]^2$$

which can be computed numerically as

$$9.000004792\cdots<\int_0^3\sqrt[4]{x^4+1}\,\mathrm{d}x+\int_1^3\sqrt[4]{x^4-1}\,\mathrm{d}x<9.000042871\cdots.$$

This refines the following double inequality

$$9<\int_0^3\sqrt[4]{x^4+1}\,\mathrm{d}x+\int_1^3\sqrt[4]{x^4-1}\,\mathrm{d}x<9.0001$$

in [4, Problem 3].

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