Some various inequalities Involving Prime Numbers Counting Function

SOME VARIOUS INEQUALITIES INVOLVING PRIME NUMBERS COUNTING FUNCTION

MEHDI HASSANI

ABSTRACT. In this paper we find some inequalities concerning $\pi(x)$ by considering some known inequalities involving it and some inequalities between important means.

1. INTRODUCTION

Suppose that $a > b \geq c > d > 0$, and consider the function

$$f(x) = \frac{a^x - b^x}{c^x - d^x} \qquad (-\infty < x < +\infty).$$

A simple calculation, yields that

$$f(x) = \begin{cases} \frac{a-b}{c-d}\frac{L(c,d)}{L(a,b)} & x = 0, \\ \frac{a-b}{c-d} & x = 1, \\ \frac{a-b}{c-d}\left(\frac{L_{x-1}(a,b)}{L_{x-1}(c,d)}\right)^{x-1} & x \neq 0, 1, \end{cases}$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x}\ln\frac{I(a^x, b^x)}{I(c^x, d^x)} \qquad (x \neq 0),$$

$$f'(0) = \frac{a-b}{c-d}\frac{L(c,d)}{L(a,b)}\ln\frac{G(a,b)}{G(c,d)},$$

in which the notations $L(a,b)$, $L_p(a,b)$ and $G(a,b)$ are well-known means between $a, b > 0$ and we recall them in the following table:

Name	Notation	Definition
arithmetic mean	$A(a,b)$	$\frac{a+b}{2}$
geometric mean	$G(a,b)$	$\sqrt{ab}$
harmonic mean	$H(a,b)$	$\frac{2}{\frac{1}{a}+\frac{1}{b}}$
logarithmic mean	$L(a,b)$	$\begin{cases} a & a = b \\ \frac{a-b}{\ln a - \ln b} & a \neq b \end{cases}$
identric mean	$I(a,b)$	$\begin{cases} a & a = b \\ \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}} & a \neq b \end{cases}$
$p-$logarithmic mean	$L_p(a,b)$	$\begin{cases} a & a = b \\ \left(\frac{a^{p+1}-b^{p+1}}{(p+1)(a-b)}\right)^{\frac{1}{p}} & a \neq b \end{cases}$ $p \neq 0, -1$

1991 *Mathematics Subject Classification.* 11A41, 11N05, 26E60, 26D15.
Key words and phrases. Primes, Distribution of Primes, Means, Inequalities.

In [3] it is shown that f is strictly convex on the real line, and the function

$$g(x) = \ln \frac{a^x - b^x}{c^x - d^x},$$

is strictly convex if $ad - bc > 0$, and is strictly concave if $ad - bc < 0$. Using these facts, some inequalities between important means has been yield.

♢ Considering Theorem 3.1 of [3], for $a > b > c > 0$ we have

$$\exp\left(1 - \frac{L(b,c)}{L(a,b)}\right) < \frac{I(a,b)}{I(b,c)} < \exp\left(\frac{L(a,b)}{L(b,c)} - 1\right), \tag{1.1}$$

and for $a > b > 0$

$$\exp\left(1 - \frac{b}{L(a,b)}\right) < \frac{I(a,b)}{b} < \exp\left(\frac{L(a,b)}{b} - 1\right). \tag{1.2}$$

♢ Considering Theorem 3.2 of [3], for $a > b > c > 0$ we have

$$\frac{L(a,b)}{L(b,c)} > 1 + \ln \frac{G(a,b)}{G(b,c)} > \frac{2a}{a+c}, \tag{1.3}$$

and

$$\frac{L(a,b)}{L(b,c)} > \frac{\ln \frac{G(a,b)}{G(b,c)}}{\ln \frac{I(a,b)}{I(b,c)}}. \tag{1.4}$$

In particular, if $a > b > 0$, then

$$\frac{L(a,b)}{b} > 1 + \frac{1}{2} \ln \frac{a}{b} > \frac{2a}{a+b} > \frac{\ln \frac{a}{b}}{2 \ln \frac{I(a,b)}{b}}. \tag{1.5}$$

♢ Theorem 3.4 of [3]: Suppose $a \geq b \geq c \geq d > 0$. If $ad - bc < 0$, then

$$\frac{H(a,b)}{H(c,d)} > \frac{G(a,b)}{G(c,d)} > \frac{L(a,b)}{L(c,d)} > \frac{I(a,b)}{I(c,d)} > \frac{A(a,b)}{A(c,d)}. \tag{1.6}$$

If $ad - bc > 0$, all inequalities reverse, and if $ad - bc = 0$, all inequalities turn out to be equalities.
Above inequalities are between two or three variables $a > b > 0$ and $a > b > c > 0$ or between four conditionally variables $a \geq b \geq c \geq d > 0$. We have many inequalities concerning prime numbers distribution functions which has above mentioned conditions. So, we can use above mentioned inequalities concerning means to get some new (and maybe complicated) inequalities involving prime numbers counting function. As usual, let $\mathbb{P}$ is the set of all primes and $\pi(x) = \#\mathbb{P} \cap [2, x]$. To get results, we recall some of known inequalities involving $\pi(x)$ at bellow.

♢ Corollary 2.2 of [1]: 1. For real $x \geq 3$ and $x = 2$, we have

$$\frac{\pi(x) + \pi(2x)}{2} \leq \pi(3x) \leq \pi(x) + \pi(2x). \tag{1.7}$$

2. For $x \geq 3$ and $k \in \mathbb{N}$, we have

$$\pi(kx) \leq k\pi(x). \tag{1.8}$$

♢ Proposition 2 of [2]: For every real $x \geq 284.5$, we have

$$\pi(x) + \pi(4x) < \pi(2x) + \pi(3x). \tag{1.9}$$

♢ Proposition 3 with Remark 4 of [2]: For every $n \geq 223$, we have

$$\pi\big(2n + \pi(n)\big) < 2\pi(n). \tag{1.10}$$

Now, we are ready to introduce some various inequalities involving $\pi(x)$.

2. Some Various Inequalities Involving $\pi(x)$

Proposition 2.1. *For every $n \geq 223$, we have*

$$\begin{aligned} \frac{L(2\pi(n), \pi\big(2n+\pi(n)\big))}{\pi\big(2n+\pi(n)\big)} &> 1 + \frac{1}{2}\ln\frac{2\pi(n)}{\pi\big(2n+\pi(n)\big)} > \\ \frac{4\pi(n)}{2\pi(n) + \pi\big(2n+\pi(n)\big)} &> \frac{\ln\frac{2\pi(n)}{\pi\left(2n+\pi(n)\right)}}{2\ln\frac{I(2\pi(n),\pi(2n+\pi(n)))}{\pi(2n+\pi(n))}}, \end{aligned}$$

and

$$e^{\left(1-\frac{\pi(2n+\pi(n))}{L(2\pi(n),\pi(2n+\pi(n)))}\right)} < \frac{I(2\pi(n),\pi(2n+\pi(n)))}{\pi(2n+\pi(n))} < e^{\left(\frac{L(2\pi(n),\pi(2n+\pi(n)))}{\pi(2n+\pi(n))}-1\right)}.$$

Proof. Considering relations (1.5) and (1.2), respectively with (1.10), we obtain all required inequalities. ☐

Proposition 2.2. *For every $n \geq 223$, we have*

$$e^{\left(1-\frac{L(\pi(2n+\pi(n)),\pi(2n))}{L(2\pi(n),\pi(2n+\pi(n)))}\right)} < \frac{I(2\pi(n),\pi(2n+\pi(n)))}{I(\pi(2n+\pi(n)),\pi(2n))} < e^{\left(\frac{L(2\pi(n),\pi(2n+\pi(n)))}{L(\pi(2n+\pi(n)),\pi(2n))}-1\right)},$$

$$\frac{L(2\pi(n),\pi(2n+\pi(n)))}{L(\pi(2n+\pi(n)),\pi(2n))} > 1 + \ln\frac{G(2\pi(n),\pi(2n+\pi(n)))}{G(\pi(2n+\pi(n)),\pi(2n))} > \frac{4\pi(n)}{2\pi(n)+\pi(2n)},$$

and

$$\frac{L(2\pi(n),\pi(2n+\pi(n)))}{L(\pi(2n+\pi(n)),\pi(2n))} > \frac{\ln\frac{G(2\pi(n),\pi(2n+\pi(n)))}{G(\pi(2n+\pi(n)),\pi(2n))}}{\ln\frac{I(2\pi(n),\pi(2n+\pi(n)))}{I(\pi(2n+\pi(n)),\pi(2n))}}.$$

Proof. Considering (1.10), the relation $\pi(2n) < \pi\big(2n+\pi(n)\big) < 2\pi(n)$ holds for every $n \geq 223$ and considering this, with relations (1.1), (1.3) and (1.4) respectively, we get the results. ☐

Proposition 2.3. *For every real $x \geq 284.5$, we have*

$$e^{\left(1-\frac{\pi(x)+\pi(4x)}{L(\pi(2x)+\pi(3x),b)}\right)} < \frac{I(\pi(2x)+\pi(3x),\pi(x)+\pi(4x))}{\pi(x)+\pi(4x)} < e^{\left(\frac{L(\pi(2x)+\pi(3x),\pi(x)+\pi(4x))}{\pi(x)+\pi(4x)}-1\right)},$$

and

$$\begin{aligned} \frac{L(\pi(2x)+\pi(3x),\pi(x)+\pi(4x))}{\pi(x)+\pi(4x)} &> 1 + \frac{1}{2}\ln\frac{\pi(2x)+\pi(3x)}{\pi(x)+\pi(4x)} > \\ \frac{2\pi(2x)+\pi(3x)}{\pi(2x)+\pi(3x)+\pi(x)+\pi(4x)} &> \frac{\ln\frac{\pi(2x)+\pi(3x)}{b}}{2\ln\frac{I(\pi(2x)+\pi(3x),\pi(x)+\pi(4x))}{\pi(x)+\pi(4x)}}. \end{aligned}$$

Proof. Considering relations (1.5) and (1.2), respectively with (1.9), we obtain all required inequalities. ☐

In proof of the next proposition, we will require the following known inequalities concerning $\pi(x)$, due to J. Barkley Rosser and L. Schoenfeld [4], which holds for every $x \geq 67$

$$\frac{x}{\log x - \frac{1}{2}} < \pi(x) < \frac{x}{\log x - \frac{3}{2}}. \tag{2.1}$$

Proposition 2.4. *For every real $x \geq 3$, we have*

$$\frac{H(3\pi(x), \pi(x)+\pi(2x))}{H(\pi(3x), \frac{\pi(x)+\pi(2x)}{2})} > \frac{G(3\pi(x), \pi(x)+\pi(2x))}{G(\pi(3x), \frac{\pi(x)+\pi(2x)}{2})} > \frac{L(3\pi(x), \pi(x)+\pi(2x))}{L(\pi(3x), \frac{\pi(x)+\pi(2x)}{2})} >$$
$$\frac{I(3\pi(x), \pi(x)+\pi(2x))}{I(\pi(3x), \frac{\pi(x)+\pi(2x)}{2})} > \frac{A(3\pi(x), \pi(x)+\pi(2x))}{A(\pi(3x), \frac{\pi(x)+\pi(2x)}{2})}.$$

Proof. Considering relations (1.7) and (1.8), for every real $x \geq 3$, we have $\frac{\pi(x)+\pi(2x)}{2} \leq \pi(3x) \leq \pi(x) + \pi(2x) \leq 3\pi(x)$. Now, let $a = 3\pi(x)$, $b = \pi(x) + \pi(2x)$, $c = \pi(3x)$ and $d = \frac{\pi(x)+\pi(2x)}{2}$, and consider

$$ad - bc = \frac{\pi(x) + \pi(2x)}{2}(3\pi(x) - 2\pi(3x)).$$

Using (2.1), we can get the inequality $3\pi(x) - 2\pi(3x) < 0$ for every $x \geq 67$ and by a simple computation it holds for every $x \geq 3$. So, $ad - bc < 0$ and we obtain the result by considering (1.6). □

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Institute for Advanced, Studies in Basic Sciences, P.O. Box 45195-1159, Zanjan, Iran.

E-mail address: mhassani@iasbs.ac.ir