Some Inequalities for Commutators of Bounded Linear Operators in Hilbert Spaces

Some Inequalities for Commutators of Bounded Linear Operators in Hilbert Spaces

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Abstract. Some new inequalities for commutators that complement and in some instances improve recent results obtained by F. Kittaneh in [**4**] and [**6**] are given.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a separable complex Hilbert space. The commutator of two bounded linear operators A and B is the operator $AB-BA$. For the usual operator norm $\|\cdot\|$ and for any two operators A and B, by using the triangle inequality and the submultiplicativity of the norm, one can state the following inequality:

$$\|AB-BA\| \le 2\cdot\|A\|\,\|B\|\,. \tag{1.1}$$

The constant 2 is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity in general. As mentioned in [**6**], the equality case is realized in (1.1) if, for instance, one takes

$$A=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

If either A or B is a positive operator, then, the following result due to F. Kittaneh [**5**], holds:

$$\|AB-BA\| \le \|A\|\,\|B\|\,. \tag{1.2}$$

The inequality (1.1) is sharp. The equality case is realized if one takes, for instance,

$$A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Moreover, in the case when *both operators* are positive, then

$$\|AB-BA\| \le \frac{1}{2}\cdot\|A\|\,\|B\|\,. \tag{1.3}$$

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This result is also due to F. Kittaneh and has been obtained in [**4**]. The constant $\frac{1}{2}$ is sharp. The equality case can be obtained by choosing the operators

$$A=\begin{bmatrix}1&0\\0&0\end{bmatrix}\qquad\text{and}\qquad B=\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

In the sequel we also need the following notations and definitions.

For the complex numbers α,β and the bounded linear operator T we define the following transform [**3**]

$$C_{\alpha,\beta}(T):=(T^{*}-\overline{\alpha}I)(\beta I-T),\tag{1.4}$$

where T^{*} denotes the adjoint of T.

We list some properties of the transform $C_{\alpha,\beta}(\cdot)$ that are useful in the following:

(i) For any $\alpha,\beta\in\mathbb{C}$ and $T\in B(H)$ we have:

$$C_{\alpha,\beta}(I)=(1-\overline{\alpha})(\beta-1)I,\qquad C_{\alpha,\alpha}(T)=-(\alpha I-T)^{*}(\alpha I-T),$$

$$C_{\alpha,\beta}(\gamma T)=|\gamma|^{2}C_{\frac{\alpha}{\gamma},\frac{\beta}{\gamma}}(T)\quad\text{for each }\gamma\in\mathbb{C}\setminus\{0\},$$

$$[C_{\alpha,\beta}(T)]^{*}=C_{\beta,\alpha}(T)$$

and

$$C_{\overline{\beta},\overline{\alpha}}(T^{*})-C_{\alpha,\beta}(T)=T^{*}T-TT^{*}.$$

(ii) The operator $T\in B(H)$ is normal if and only if $C_{\overline{\beta},\overline{\alpha}}(T^{*})=C_{\alpha,\beta}(T)$ for each $\alpha,\beta\in\mathbb{C}$.

We recall that a bounded linear operator T on the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty,y\rangle\geq 0$ for any $y\in H$.

Utilising the identity

$$\operatorname{Re}\langle C_{\alpha,\beta}(T)x,x\rangle=\operatorname{Re}\langle C_{\beta,\alpha}(T)x,x\rangle=\frac{1}{4}|\beta-\alpha|^{2}-\left\|\left(T-\frac{\alpha+\beta}{2}I\right)x\right\|^{2}\tag{1.5}$$

that holds for any scalars α,β and any vector $x\in H$ with $\|x\|=1$, we can give a simple characterisation result that is useful in the sequel:

Lemma 1. *For $\alpha,\beta\in\mathbb{C}$ and $T\in B(H)$ the following statements are equivalent:*

(i) *The transform $C_{\alpha,\beta}(T)$ (or, equivalently, $C_{\beta,\alpha}(T)$) is accretive;*

(ii) *The transform $C_{\overline{\alpha},\overline{\beta}}(T^{*})$ $\left(\text{or, equivalently, } C_{\overline{\beta},\bar{\alpha}}(T^{*})\right)$ is accretive;*

(iii) *We have the norm inequality*

$$\left\|T-\frac{\alpha+\beta}{2}\cdot I\right\|\leq\frac{1}{2}|\beta-\alpha|\tag{1.6}$$

or, equivalently,

$$\left\|T^{*}-\frac{\bar{\alpha}+\bar{\beta}}{2}\cdot I\right\|\leq\frac{1}{2}|\beta-\alpha|.\tag{1.7}$$

Remark 1. *In order to give examples of operators $T\in B(H)$ and numbers $\alpha,\beta\in\mathbb{C}$ such that the transform $C_{\alpha,\beta}(T)$ is accretive, it suffices to select a bounded linear operator S and the complex numbers z,w $(w\neq 0)$ with the property that $\|S-zI\|\leq|w|$ and, by choosing $T=S$, $\alpha=\frac{1}{2}(z+w)$ and $\beta=\frac{1}{2}(z-w)$ we observe that T satisfies (1.6),i.e., $C_{\alpha,\beta}(T)$ is accretive.*

2. General Inequalities

The following general results may be stated:

THEOREM 1. *For any $T,U\in B(H)$ we have*

$$\|TU-UT\|\leq 2\cdot\min\{\|T\|,\|U\|\}\min\{\|T-U\|,\|T+U\|\}. \tag{2.1}$$

PROOF. Observe that for any $T,U\in B(H)$ we have

$$\begin{aligned}\|TU-UT\| &= \|(T-U)U-U(T-U)\|\\ &\leq 2\|T-U\|\|U\|\end{aligned}$$

and, similarly

$$\|TU-UT\|\leq 2\|T-U\|\|T\|.$$

Utilising these two inequalities, we have

$$\|TU-UT\|\leq 2\cdot\min\{\|T\|,\|U\|\}\|T-U\|. \tag{2.2}$$

By writing (2.2) for $-U$ instead of U we also have

$$\|TU-UT\|\leq 2\cdot\min\{\|T\|,\|U\|\}\|T+U\|. \tag{2.3}$$

Finally, on making use of (2.2) and (2.3), we deduce the desired result (2.1). ∎

The following result for the self-commutator holds:

COROLLARY 1. *For any $T\in B(H)$ we have*

$$\|TT^*-T^*T\|\leq 2\cdot\|T\|\min\{\|T-T^*\|,\|T+T^*\|\}. \tag{2.4}$$

Now if A and B are the Cartesian decomposition of an operator T, i.e., $T=A+iB$, then the following results also holds:

COROLLARY 2. *For any $T\in B(H)$ we have*

$$\|TT^*-T^*T\|\leq 4\cdot\min\{\|A\|,\|B\|\}\min\{\|A-B\|,\|A+B\|\}. \tag{2.5}$$

PROOF. If we write the inequality (2.1) for A and B we have

$$\|AB-BA\|\leq 2\cdot\min\{\|A\|,\|B\|\}\min\{\|A-B\|,\|A+B\|\}.$$

Taking into account that

$$AB-BA=\frac{1}{2i}(TT^*-T^*T) \tag{2.6}$$

we then deduce the desired result (2.5) for the self-commutator. ∎

The following result also holds

THEOREM 2. *Let $T,U\in B(H)$.*

(1) *If $T-U$ is positive, then*

$$\|TU-UT\|\leq\min\{\|T\|,\|U\|\}\|T-U\|. \tag{2.7}$$

(2) *If U (or T) is positive, then*

$$\|TU-UT\|\leq\|U\|\|T-U\|\,(or\ \|T\|\|T-U\|). \tag{2.8}$$

(3) *If U and T are positive then (2.7) is also true.*

Proof. As in the proof of Theorem 1, we have

$$\|TU-UT\| = \|(T-U)\,U-U\,(T-U)\|\,.$$

Applying Kittaneh's inequality (1.2) for the positive operator $T-U$, we then have

$$\|(T-U)\,U-U\,(T-U)\| \leq \|U\|\,\|T-U\|\,. \tag{2.9}$$

Since $\|TU-UT\| = \|(T-U)\,T-T\,(T-U)\|\,,$ then by the same inequality (1.2) we have

$$\|(T-U)\,T-T\,(T-U)\| \leq \|T\|\,\|T-U\|\,. \tag{2.10}$$

Making use of (2.9) and (2.10), we obtain the desired inequality (2.7).

The proof of the second and third part goes likewise and the details are omitted. ∎

We can state the following particular case for the self-commutator:

Corollary 3. *Let $T \in B(H)$ and A, B be its Cartesian decomposition.*

(1) *If $A-B$ is positive, then*

$$\|TT^*-T^*T\| \leq 2\cdot\min\{\|A\|,\|B\|\}\,\|A-B\|\,. \tag{2.11}$$

(2) *If A (or B) is positive, then*

$$\|TT^*-T^*T\| \leq 2\cdot\|A\|\,\|A-B\| \quad (or\ 2\cdot\|B\|\,\|A-B\|)\,.$$

(3) *If A and B are positive then (2.11) is also true.*

The proof follows by Theorem 2 and the identity (2.6). The details are omitted.

When more assumptions on the operators U and T are imposed, then better inequalities can be stated as follows:

Theorem 3. *Let $T, U \in B(H)$ be such that either $T \geq U \geq 0$ or $U \geq T \geq 0$. Then*

$$\begin{aligned}\|TU-UT\| &\leq \frac{1}{2}\cdot\min\{\|T\|,\|U\|\}\,\|T-U\| \\ &\leq \frac{1}{2}\cdot\|T\|\,\|U\|\,.\end{aligned} \tag{2.12}$$

Proof. We give an argument only for the first case.

Utilising Kittaneh's result (1.3) for the positive operators $T-U$ and U, we have

$$\|(T-U)\,U-U\,(T-U)\| \leq \frac{1}{2}\cdot\|U\|\,\|T-U\|\,.$$

The same inequality for the operators $T-U$ and T provides

$$\|(T-U)\,T-T\,(T-U)\| \leq \frac{1}{2}\cdot\|T\|\,\|T-U\|\,.$$

These inequalities and the equality

$$\begin{aligned}\|TU-UT\| &= \|(T-U)\,U-U\,(T-U)\| \\ &= \|(T-U)\,T-T\,(T-U)\|\end{aligned}$$

now produce the first inequality in (2.12)

For the positive operators U and T we know that

$$\|T-U\| \leq \max\{\|T\|,\|U\|\}\,. \tag{2.13}$$

Generalizations and applications of this result can be found for instance in [**1**, p. 280], [**2**], [**7**, pp. 36-37], [**8**] and the references therein.

Now, by the inequality (2.13) we have

$$\begin{aligned}\frac{1}{2}\cdot\min\{\|T\|,\|U\|\}\|T-U\| &\le \frac{1}{2}\cdot\min\{\|T\|,\|U\|\}\max\{\|T\|,\|U\|\}\\ &= \frac{1}{2}\cdot\|T\|\|U\|\end{aligned}$$

and the proof is completed. ∎

Remark 2. *The above inequality (2.12) provides a refinement of Kittaneh's result (1.3) in the case of two positive operators for which, in addition, we assume that one is greater than the other in the operator order.*

The following result for the self-commutator may be stated as well:

Corollary 4. *Let $T\in B(H)$ and A,B be its Cartesian decomposition. If either $A\ge B\ge 0$ or $B\ge A\ge 0$, then*

$$\begin{aligned}\|TT^*-T^*T\| &\le \min\{\|A\|,\|B\|\}\|A-B\|\\ &\le \|A\|\|B\|.\end{aligned} \tag{2.14}$$

3. Other Inequalities

When information about the accretivity of the transform $C_{\alpha,\beta}(\cdot)$ is available, we can state the following result as well:

Theorem 4. *Let $T,U\in B(H)$ and $\alpha,\beta\in\mathbb{K}$ be such that $C_{\alpha,\beta}(T)$ is accretive, then*

$$\|TU-UT\|\le|\beta-\alpha|\|U\|. \tag{3.1}$$

Moreover, if U is positive, then we have the better inequality

$$\|TU-UT\|\le\frac{1}{2}\cdot|\beta-\alpha|\|U\|. \tag{3.2}$$

Proof. Observe that the following equality holds

$$TU-UT=\left(T-\frac{\alpha+\beta}{2}\cdot I\right)U-U\left(T-\frac{\alpha+\beta}{2}\cdot I\right), \tag{3.3}$$

for any $T,U\in B(H)$ and $\alpha,\beta\in\mathbb{K}$.

Since $C_{\alpha,\beta}(T)$ is accretive, then by Lemma 1, we have

$$\left\|T-\frac{\alpha+\beta}{2}\cdot I\right\|\le\frac{1}{2}|\beta-\alpha|. \tag{3.4}$$

Taking the norm in (3.3) and utilizing the triangle inequality we have

$$\begin{aligned}\|TU-UT\| &= \left\|\left(T-\frac{\alpha+\beta}{2}\cdot I\right)U-U\left(T-\frac{\alpha+\beta}{2}\cdot I\right)\right\|\\ &\le 2\cdot\left\|T-\frac{\alpha+\beta}{2}\cdot I\right\|\|U\|,\end{aligned}$$

which together with (3.4) produces (3.1).

Now, on making use of Kittaneh's inequality (1.2) for U positive we also have

$$\left\|\left(T-\frac{\alpha+\beta}{2}\cdot I\right)U-U\left(T-\frac{\alpha+\beta}{2}\cdot I\right)\right\|\le\left\|T-\frac{\alpha+\beta}{2}\cdot I\right\|\|U\|$$

which together with (3.4) yields the desired result (3.2). ∎

We can apply the above result for self-commutators as follows:

Corollary 5. *Let $T \in B(H)$ and A, B be its Cartesian decomposition. If $a, c \in \mathbb{R}$ are such that $C_{a,c}(A)$ is accretive, then*

$$\|TT^* - T^*T\| \leq 2 \cdot |a - c| \, \|B\| .$$

Moreover, if B is positive, then we have the better inequality

$$\|TT^* - T^*T\| \leq |a - c| \, \|B\| .$$

The proof follows by Theorem 4 and the identity (2.6). The details are omitted.

Remark 3. *For applications it is perhaps more convenient to assume that $a \cdot I \leq A \leq c \cdot I$. Then, for any B we have*

$$\|TT^* - T^*T\| \leq 2 \cdot (c - a) \, \|B\| , \tag{3.5}$$

while for positive B the inequality can be improved as follows:

$$\|TT^* - T^*T\| \leq (c - a) \, \|B\| . \tag{3.6}$$

Remark 4. *It is well known that for a self-adjoint operator A and for a positive number a we have $\|A\| \leq a$ if and only if $-a \cdot I \leq A \leq a \cdot I$. This is also equivalent to the condition $\sigma(A) \subseteq [-a, a]$, where $\sigma(A)$ denotes the spectrum of A. Therefore the inequalities (3.5) and (3.6) produce*

$$\|TT^* - T^*T\| \leq 4 \cdot \|A\| \, \|B\|$$

and for B positive,

$$\|TT^* - T^*T\| \leq 2 \cdot \|A\| \, \|B\| ,$$

respectively.

The following lemma may be useful in applications:

Lemma 2. *Let A and B be the Cartesian decomposition of an operator T and $\alpha = a + ib$, $\beta = c + di$ with $a, b, c, d \in \mathbb{R}$. If $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive, then*

$$\left\| T - \frac{\alpha + \beta}{2} \cdot I \right\| \leq \frac{1}{2} |a - c| + \frac{1}{2} |b - d| .$$

Proof. Since $C_{a,c}(A)$ and $C_{b,d}(A)$ are accretive, then by Lemma 1 we have

$$\left\| A - \frac{a + c}{2} \cdot I \right\| \leq \frac{1}{2} |a - c| \quad \text{and} \quad \left\| B - \frac{b + d}{2} \cdot I \right\| \leq \frac{1}{2} |b - d| .$$

By the triangle inequality we have

$$\begin{aligned} \left\| T - \frac{\alpha + \beta}{2} \cdot I \right\| &= \left\| A + iB - \frac{a + c}{2} \cdot I - \frac{b + d}{2} \cdot iI \right\| \\ &\leq \left\| A - \frac{a + c}{2} \cdot I \right\| + \left\| B - \frac{b + d}{2} \cdot I \right\| \\ &\leq \frac{1}{2} |a - c| + \frac{1}{2} |b - d| \end{aligned}$$

and the lemma is proved. ∎

Utilizing this lemma and a similar argument to the one from the proof of Theorem 4, we may state the following result as well:

Theorem 5. *Let A and B be the Cartesian decomposition of an operator T and $a,b,c,d\in\mathbb{R}$ so that $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive. Then for any $U\in B(H)$ we have the inequality*

$$\|TU-UT\|\leq(|a-c|+|b-d|)\|U\|. \tag{3.7}$$

Moreover, if U is positive, then we have the better inequality

$$\|TU-UT\|\leq\frac{1}{2}(|a-c|+|b-d|)\|U\|. \tag{3.8}$$

The following particular case also holds:

Corollary 6. *Let A and B be the Cartesian decomposition of an operator T and $a,b,c,d\in\mathbb{R}$ so that $a\cdot I\leq A\leq c\cdot I$ and $b\cdot I\leq B\leq d\cdot I$. Then for any $U\in B(H)$ we have the inequality*

$$\|TU-UT\|\leq(c+d-a-b)\|U\|. \tag{3.9}$$

Moreover, if U is positive, then we have the better inequality

$$\|TU-UT\|\leq\frac{1}{2}(c+d-a-b)\|U\|. \tag{3.10}$$

Proof. Since $a\cdot I\leq A\leq c\cdot I$ and $b\cdot I\leq B\leq d\cdot I$, then obviously $(c\cdot I-A)(A-a\cdot I)$ is positive and *a fortiori* $C_{a,c}(A)$ is accretive. The same applies for $C_{b,d}(B)$, and by Theorem 5 we deduce the desired result. ∎

Remark 5. *On making use of Remark 4, we can state the following inequalities:*

$$\|TU-UT\|\leq(\|T+T^*\|+\|T-T^*\|)\|U\|$$

for any $U\in B(H)$ and

$$\|TU-UT\|\leq\frac{1}{2}(\|T+T^*\|+\|T-T^*\|)\|U\|$$

for U a positive operator.

The following particular result may be stated as well:

Corollary 7. *Let A, B be the Cartesian decomposition of T and C, D the Cartesian decomposition of U. If $a,b,c,d,m,n,p,q\in\mathbb{R}$ are such that $C_{a,c}(A)$, $C_{b,d}(B)$, $C_{m,p}(C)$ and $C_{n,q}(D)$ are accretive, then*

$$\|TU-UT\|\leq\frac{1}{2}(|a-c|+|b-d|)(|m-p|+|n-q|). \tag{3.11}$$

Proof. Utilising (3.7) we have

$$\begin{aligned}&\|TU-UT\|\\&=\left\|T\left(U-\left(\frac{m+n}{2}+\frac{p+q}{2}\cdot i\right)\cdot I\right)-\left(U-\left(\frac{m+n}{2}+\frac{p+q}{2}\cdot i\right)\cdot I\right)T\right\|\\&\leq(|a-c|+|b-d|)\left\|U-\left(\frac{m+n}{2}+\frac{p+q}{2}\cdot i\right)\cdot I\right\|\\&\leq\frac{1}{2}(|a-c|+|b-d|)(|m-p|+|n-q|),\end{aligned}$$

where for the last inequality we have used Lemma 2 for the operator U. ∎

Remark 6. *Let A and B be the Cartesian decomposition of T and C, D the Cartesian decomposition of U. If $a,b,c,d,m,n,p,q\in\mathbb{R}$ are so that $a\cdot I\leq A\leq c\cdot I$, $b\cdot I\leq B\leq d\cdot I$, $m\cdot I\leq C\leq p\cdot I$ and $n\cdot I\leq D\leq q\cdot I$, then, by Corollary 7, we get the inequality*

$$\|TU-UT\|\leq\frac{1}{2}\left(c+d-a-b\right)\left(p+q-m-n\right). \tag{3.12}$$

This result has been obtained in a different way by F. Kittaneh in [**6**]. *Therefore the Corollary 7 can be regarded as a generalization of Kittaneh's result (3.12) in the case when the transforms $C_{a,c}(A)$, $C_{b,d}(B)$, $C_{m,p}(C)$ and $C_{n,q}(D)$ are accretive.*

Remark 7. *On utilizing the inequality (3.12) we can also state that*

$$\|TU-UT\|\leq\frac{1}{2}\left(\|T+T^{*}\|+\|T-T^{*}\|\right)\left(\|U+U^{*}\|+\|U-U^{*}\|\right),$$

for any $T,U\in B(H)$.

Finally for the section, we can state the following result as well:

Theorem 6. *Let $T,U\in B(H)$ and $\alpha,\beta,\gamma,\delta\in\mathbb{K}$ be such that $C_{\alpha,\beta}(T)$ and $C_{\gamma,\delta}(U)$ are accretive, then*

$$\|TU-UT\|\leq\frac{1}{2}\left|\beta-\alpha\right|\left|\gamma-\delta\right|. \tag{3.13}$$

Proof. Observe that the following identity holds

$$TU-UT=\left(T-\frac{\beta+\alpha}{2}\cdot I\right)\left(U-\frac{\gamma+\delta}{2}\cdot I\right)-\left(U-\frac{\gamma+\delta}{2}\cdot I\right)\left(T-\frac{\beta+\alpha}{2}\cdot I\right), \tag{3.14}$$

for each $T,U\in B(H)$ and $\alpha,\beta,\gamma,\delta\in\mathbb{K}$.

Taking the norm, using its properties and utilizing the fact that $C_{\alpha,\beta}(T)$ and $C_{\gamma,\delta}(T)$ are accretive produces the required inequality (3.13). ∎

Remark 8. *On making use of the identity (3.14) and Lemma 2, one can easily re-obtain the result from Corollary 7. The details are omitted.*

The following particular result for self-commutators may be stated as well:

Corollary 8. *Let A, B be the Cartesian decomposition of T. If $a,b,c,d\in\mathbb{R}$ are such that $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive, then*

$$\|TT^{*}-T^{*}T\|\leq\left|a-c\right|\left|b-d\right|.$$

In particular, if $a\cdot I\leq A\leq c\cdot I$, $b\cdot I\leq B\leq d\cdot I$, then (see also [**6**]*)*

$$\|TT^{*}-T^{*}T\|\leq\left(c-a\right)\left(d-b\right).$$

4. Inequalities for Normal Operators

The following result improving Lemma 2 for the case of normal operators holds:

Lemma 3. *Let T be a normal operator and A,B its Cartesian decomposition. If $a,b,c,d\in\mathbb{R}$ are such that $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive and if $\alpha:=a+ib$, $\beta:=c+di$, then $C_{\alpha,\beta}(T)$ is accretive as well.*

PROOF. It is well known that for a normal operator T with the Cartesian decomposition A, B we have

$$TT^* = A^2 + B^2 \quad \text{and} \quad \|T\|^2 = \left\|A^2 + B^2\right\|.$$

Moreover, for any complex number z the operator $T - z \cdot I$ is also normal.

Now, since T is normal, then $T - \frac{\alpha+\beta}{2} \cdot I$ is normal and

$$\operatorname{Re}\left(T - \frac{\alpha+\beta}{2} \cdot I\right) = A - \frac{a+c}{2} \cdot I, \qquad \operatorname{Im}\left(T - \frac{\alpha+\beta}{2} \cdot I\right) = B - \frac{b+d}{2} \cdot I.$$

Due to the fact that $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive, by Lemma 1 we have the inequalities

$$\left\|A - \frac{a+c}{2} \cdot I\right\| \leq \frac{1}{2}|c-a| \quad \text{and} \quad \left\|B - \frac{b+d}{2} \cdot I\right\| \leq \frac{1}{2}|d-b|$$

and then

$$\begin{aligned}\left\|T - \frac{\alpha+\beta}{2} \cdot I\right\|^2 &= \left\|\left(A - \frac{a+c}{2} \cdot I\right)^2 + \left(B - \frac{b+d}{2} \cdot I\right)^2\right\| \\ &\leq \left\|A - \frac{a+c}{2} \cdot I\right\|^2 + \left\|B - \frac{b+d}{2} \cdot I\right\|^2 \\ &\leq \frac{1}{4}\left[(c-a)^2 + (d-b)^2\right] = \frac{1}{4}|\beta - \alpha|^2,\end{aligned}$$

which, by Lemma 1 is equivalent with the fact that $C_{\alpha,\beta}(T)$ is accretive. ∎

The following result for commutators when one of the operators is normal may be stated:

THEOREM 7. *Let T be a normal operator and A, B its Cartesian decomposition. If $a, b, c, d \in \mathbb{R}$ are such that $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive, then for any $U \in B(H)$ we have the inequality*

$$\|TU - UT\| \leq \sqrt{(c-a)^2 + (d-b)^2}\,\|U\|. \tag{4.1}$$

Moreover, if U is positive, then we have the better inequality

$$\|TU - UT\| \leq \frac{1}{2} \cdot \sqrt{(c-a)^2 + (d-b)^2}\,\|U\|. \tag{4.2}$$

The proof follows from Theorem 4 via Lemma 3 and the details are omitted.

COROLLARY 9. *Let A and B be the Cartesian decomposition of a normal operator T and $a, b, c, d \in \mathbb{R}$ such that $a \cdot I \leq A \leq c \cdot I$ and $b \cdot I \leq B \leq d \cdot I$. Then for any $U \in B(H)$ we have the inequality (4.1). Moreover, if U is positive, then we have the better inequality (4.2).*

REMARK 9. *The inequality (4.1), under the assumptions from Corollary 9, has been obtained in* [**6**] *for unitarily invariant norms. However, for the case of usual operator norms the result in Theorem 7 is more general since the fact that $a \cdot I \leq A \leq c \cdot I$ implies that $C_{a,c}(A)$ is accretive, but the converse is obviously not true in general.*

Remark 10. *On making use of an argument similar to the one in Remark 4, we deduce from (4.1) and (4.2) the following inequalities:*

$$\|TU - UT\| \leq 2 \cdot \sqrt{\|T + T^*\|^2 + \|T - T^*\|^2} \cdot \|U\|,$$

for any $U \in B(H)$ and

$$\|TU - UT\| \leq \sqrt{\|T + T^*\|^2 + \|T - T^*\|^2} \cdot \|U\|,$$

for any positive operator U, respectively.

The following result for commutators of two normal operators may be stated as well:

Theorem 8. *Let T and U be two normal operators and A, B be the Cartesian decomposition of T while C, D are the Cartesian decomposition of U. If $a, b, c, d, m, n, p, q \in \mathbb{R}$ are such that $C_{a,c}(A)$, $C_{b,d}(B)$, $C_{m,p}(C)$ and $C_{n,q}(D)$ are accretive, then*

$$\|TU - UT\| \leq \frac{1}{2} \cdot \sqrt{(c-a)^2 + (d-b)^2} \sqrt{(p-m)^2 + (q-n)^2}. \tag{4.3}$$

The proof follows from Theorem 6 via Lemma 3 and the details are omitted.

As a particular case we can obtain the following result obtained by Kittaneh in [**6**]:

Corollary 10. *Assume that T, U, A, B, C, D are as in Theorem 8. If $a, b, c, d, m, n, p, q \in \mathbb{R}$ are such that $a \cdot I \leq A \leq c \cdot I$, $b \cdot I \leq B \leq d \cdot I$, $m \cdot I \leq C \leq p \cdot I$ and $n \cdot I \leq D \leq q \cdot I$, then we have the inequality (4.3).*

Similar inequalities to the ones in Remark 10 can be stated, but the details are omitted. For other results in which the real and imaginary parts of the involved operators are supposed to be positive, see the paper [**6**].

References

[1] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.
[2] R. Bhatia and F. Kittaneh, On the singular values of a product of operators, *SIAM J. Matrix Anal. Appl.*, **11** (1990), 272-277.
[3] S.S. Dragomir, New inequalities of the Kantorovich type for bounded linear operators in Hilbert spaces, *Linear Algebra Appl.* (in press).
[4] F. Kittaneh, Inequalities for commutators of positive operators, *J. Funct. Anal.*, **250** (2007), 132-143.
[5] F. Kittaneh, Norm inequalities for commutators of positive operators and applications, *Math. Z.,* **258**(4) (2008), 845-849.
[6] F. Kittaneh, Norm inequalities for commutators of Hilbert space operators, *submitted.*
[7] X. Zhan, *Matrix Inequalities*, Springer, Berlin, 2002.
[8] X. Zhan, On some matrix inequalities, *Linear Algebra Appl.*, **376** (2004), 299-303.

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