Norm and Numerical Radius Inequalities for Two Linear Operators in Hilbert Spaces

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Abstract. Some inequalities for the norm and numerical radius of two bounded linear operators in Hilbert spaces are established.

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [**6**, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, \ x \in H, \ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [**6**, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x \rangle|, \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [**6**, p. 9]:

Theorem 1 (Equivalent norm). *For any $T \in B(H)$ one has*

$$w(T) \leq \|T\| \leq 2w(T). \tag{1.2}$$

For other results on numerical radius, see [**7**], Chapter 11. For some recent and interesting results concerning inequalities for the numerical radius, see [**8**], [**9**] and [**1**].

We recall some classical results involving the numerical radius of two linear operators A, B.

The following general result for the product of two operators holds [**6**, p. 37]:

Theorem 2. *If A, B are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then*

$$w(AB) \leq 4w(A)w(B). \tag{1.3}$$

In the case that $AB = BA$, then

$$w(AB) \leq 2w(A)w(B). \tag{1.4}$$

The following results are also well known [**6**, p. 38].

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Theorem 3. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \leq w(B). \tag{1.5}$$

If A is an isometry and $AB = BA$, then (1.5) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$.
The following result holds [**6**, p. 38].

Theorem 4 (Double commute). *If the operators A and B double commute, then*

$$w(AB) \leq w(B) \|A\|. \tag{1.6}$$

As a consequence of the above, we have [**6**, p. 39]:

Corollary 1. *Let A be a normal operator commuting with B. Then*

$$w(AB) \leq w(A) w(B). \tag{1.7}$$

For other results and historical comments on the above see [**6**, p. 39–41]. For more results on the numerical radius, see [**7**].
In the recent survey paper [**2**] we provided other inequalities for the numerical radius of the product of two operators. We list here some of the results:

Theorem 5. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then*

$$\left\| \frac{A^*A + B^*B}{2} \right\| \leq w(B^*A) + \frac{1}{2} \|A - B\|^2 \tag{1.8}$$

and

$$\left\| \frac{A + B}{2} \right\|^2 \leq \frac{1}{2} \left[\left\| \frac{A^*A + B^*B}{2} \right\| + w(B^*A) \right], \tag{1.9}$$

respectively.

If more information regarding one operator is available, then the following results may be stated as well:

Theorem 6. *Let $A, B : H \to H$ be two bounded linear operators on H and B is invertible such that, for a given $r > 0$,*

$$\|A - B\| \leq r. \tag{1.10}$$

Then

$$\|A\| \leq \left\| B^{-1} \right\| \left[w(B^*A) + \frac{1}{2} r^2 \right] \tag{1.11}$$

and

$$(0 \leq) \|A\| \|B\| - w(B^*A) \leq \frac{1}{2} r^2 + \frac{\|B\|^2 \left\| B^{-1} \right\|^2 - 1}{2 \left\| B^{-1} \right\|^2}, \tag{1.12}$$

respectively.

Motivated by the results outlined above, it is the main aim of the present paper to establish other inequalities for the composite operator BA under suitable assumptions for the transform $C_{\cdot,\cdot}(\cdot)$ (see (2.1) below) of the operators involved. The transform $C_{\cdot,\cdot}(\cdot)$ has been recently introduced in the literature by the author (see [**3**]) in order to provide various generalizations for the operator version of the Kantorovich famous inequality obtained by Greub and Rheinboldt in [**5**]. Some elementary properties of this transform will be provided at the beginning of the next section.

2. Norm & Numerical Radius Inequalities

For the complex numbers α, β and the bounded linear operator T we define the following transform (see [**3**]):

$$C_{\alpha,\beta}(T) := (T^* - \overline{\alpha} I)(\beta I - T), \tag{2.1}$$

where by T^* we denote the adjoint of T.

We list some properties of the transform $C_{\alpha,\beta}(\cdot)$ that are of interest:

(i) For any $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ we have:

$$C_{\alpha,\beta}(I) = (1-\overline{\alpha})(\beta - 1) I, \qquad C_{\alpha,\alpha}(T) = -(\alpha I - T)^* (\alpha I - T), \tag{2.2}$$

$$C_{\alpha,\beta}(\gamma T) = |\gamma|^2 C_{\frac{\alpha}{\gamma},\frac{\beta}{\gamma}}(T) \quad \text{for each } \gamma \in \mathbb{C}\setminus\{0\}, \tag{2.3}$$

$$[C_{\alpha,\beta}(T)]^* = C_{\beta,\alpha}(T) \tag{2.4}$$

and

$$C_{\overline{\beta},\overline{\alpha}}(T^*) - C_{\alpha,\beta}(T) = T^*T - TT^*. \tag{2.5}$$

(ii) The operator $T \in B(H)$ is normal if and only if $C_{\overline{\beta},\overline{\alpha}}(T^*) = C_{\alpha,\beta}(T)$ for each $\alpha, \beta \in \mathbb{C}$.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\operatorname{Re}\langle C_{\alpha,\beta}(T)x, x\rangle = \operatorname{Re}\langle C_{\beta,\alpha}(T)x, x\rangle = \frac{1}{4}|\beta - \alpha|^2 - \left\|\left(T - \frac{\alpha+\beta}{2} I\right)x\right\|^2 \tag{2.6}$$

that holds for any scalars α, β and any vector $x \in H$ with $\|x\| = 1$ we can give a simple characterization result that is useful in the following:

Lemma 1. *For $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ the following statements are equivalent:*

(i) The transform $C_{\alpha,\beta}(T)$ (or, equivalently, $C_{\beta,\alpha}(T)$) is accretive;

(ii) The transform $C_{\overline{\alpha},\overline{\beta}}(T^*)$ $\Big($or, equivalently, $C_{\overline{\beta},\bar{\alpha}}(T^*)\Big)$ is accretive;

(iii) We have the norm inequality

$$\left\|T - \frac{\alpha+\beta}{2}\cdot I\right\| \leq \frac{1}{2}|\beta - \alpha| \tag{2.7}$$

or, equivalently,

$$\left\|T^* - \frac{\bar{\alpha}+\bar{\beta}}{2}\cdot I\right\| \leq \frac{1}{2}|\beta - \alpha|. \tag{2.8}$$

Remark 1. *In order to give examples of operators $T \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha,\beta}(T)$ is accretive, it suffices to select a bounded linear operator S and the complex numbers z, w with the property that $\|S - zI\| \leq |w|$ and, by choosing $T = S$, $\alpha = \frac{1}{2}(z + w)$ and $\beta = \frac{1}{2}(z - w)$ we observe that T satisfies (2.7), i.e., $C_{\alpha,\beta}(T)$ is accretive.*

In the recent paper [**4**], the following Grüss type result in comparing the quantities $w(BA)$ and $w(A)\,w(B)$ has been given:

Theorem 7. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then*

$$w(BA) \leq w(A)\,w(B) + \frac{1}{4}|\beta - \alpha|\,|\gamma - \delta|. \tag{2.9}$$

Another similar result obtained in [**4**] is the following one

Theorem 8. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that $\operatorname{Re}(\beta\overline{\alpha}) > 0, \operatorname{Re}(\delta\overline{\gamma}) > 0$ and the transforms $C_{\alpha,\beta}(A), C_{\gamma,\delta}(B)$ are accretive, then*

$$\frac{w(BA)}{w(A)\,w(B)} \leq 1 + \frac{1}{4} \cdot \frac{|\beta - \alpha|\,|\delta - \gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{1/2}} \tag{2.10}$$

and

$$\begin{aligned} w(BA) &\leq w(A)\,w(B) \\ &+ \left[\left(|\alpha + \beta| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{1/2}\right)\left(|\delta + \gamma| - 2\left[\operatorname{Re}(\delta\overline{\gamma})\right]^{1/2}\right)\right]^{1/2}[w(A)\,w(B)]^{1/2}, \end{aligned} \tag{2.11}$$

respectively.

In the light of the above results it is then natural to compare the quantities $\|AB\|$ and $w(A)\,w(B) + w(A)\|B\| + \|A\|\,w(B)$ provided that some information about the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are available, where $\alpha, \beta, \gamma, \delta \in \mathbb{K}$.

Theorem 9. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then*

$$\|BA\| \leq w(A)\,w(B) + w(A)\|B\| + \|A\|\,w(B) + \frac{1}{4}|\beta - \alpha|\,|\gamma - \delta|. \tag{2.12}$$

Proof. Since $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then, on making use of Lemma 1 we have that $\left\|Ax - \frac{\alpha+\beta}{2}x\right\| \leq \frac{1}{2}|\beta - \alpha|$ and $\left\|B^*x - \frac{\bar{\gamma}+\bar{\delta}}{2}x\right\| \leq \frac{1}{2}\left|\bar{\gamma} - \bar{\delta}\right|$, for any $x \in H, \|x\| = 1$.

Utilizing the Schwarz inequality we may write that

$$\begin{aligned} &|\langle Ax - \langle Ax, x\rangle x, B^*y - \langle B^*y, y\rangle y\rangle| \\ &\qquad\leq \|Ax - \langle Ax, x\rangle x\|\,\|B^*y - \langle B^*y, y\rangle y\|, \end{aligned} \tag{2.13}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Since for any vectors $u, f \in H$ with $\|f\| = 1$ we have $\|u - \langle u, f\rangle f\| = \inf_{\mu\in\mathbb{K}}\|u - \mu f\|$, then obviously

$$\|Ax - \langle Ax, x\rangle x\| \leq \left\|Ax - \frac{\alpha + \beta}{2}x\right\| \leq \frac{1}{2}|\beta - \alpha|$$

and

$$\|B^*y - \langle B^*y, y\rangle y\| \le \left\|B^*y - \frac{\bar{\gamma}+\bar{\delta}}{2}y\right\| \le \frac{1}{2}|\gamma-\delta|$$

producing the inequality

$$\|Ax - \langle Ax, x\rangle x\| \|B^*y - \langle B^*y, y\rangle y\| \le \frac{1}{4}|\beta-\alpha||\gamma-\delta|. \tag{2.14}$$

Now, observe that

$$\begin{aligned} & \langle Ax - \langle Ax, x\rangle x, B^*y - \langle B^*y, y\rangle y\rangle \\ = \; & \langle BAx, y\rangle + \langle Ax, x\rangle \langle By, y\rangle \langle x, y\rangle - \langle Ax, x\rangle \langle Bx, y\rangle - \langle Ax, y\rangle \langle By, y\rangle, \end{aligned}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Taking the modulus in the equality and utilizing its properties we have successively

$$\begin{aligned} & |\langle Ax - \langle Ax, x\rangle x, B^*y - \langle B^*y, y\rangle y\rangle| \\ \ge \; & |\langle BAx, y\rangle| - |\langle Ax, x\rangle \langle Bx, y\rangle + \langle Ax, y\rangle \langle By, y\rangle - \langle Ax, x\rangle \langle By, y\rangle \langle x, y\rangle| \\ \ge \; & |\langle BAx, y\rangle| - |\langle Ax, x\rangle \langle Bx, y\rangle| - |\langle Ax, y\rangle \langle By, y\rangle| - |\langle Ax, x\rangle \langle By, y\rangle \langle x, y\rangle| \end{aligned}$$

which is equivalent with

$$\begin{aligned} & |\langle Ax - \langle Ax, x\rangle x, B^*y - \langle B^*y, y\rangle y\rangle| \\ & + |\langle Ax, x\rangle \langle Bx, y\rangle| + |\langle Ax, y\rangle \langle By, y\rangle| + |\langle Ax, x\rangle \langle By, y\rangle \langle x, y\rangle| \\ \ge \; & |\langle BAx, y\rangle|, \end{aligned} \tag{2.15}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Finally, on making use of the inequalities (2.13)-(2.15) we can state that

$$\begin{aligned} & \frac{1}{4}|\beta-\alpha||\gamma-\delta| \\ & + |\langle Ax, x\rangle \langle Bx, y\rangle| + |\langle Ax, y\rangle \langle By, y\rangle| + |\langle Ax, x\rangle \langle By, y\rangle \langle x, y\rangle| \\ & \ge |\langle BAx, y\rangle|, \end{aligned} \tag{2.16}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Taking the supremum in (2.16) over $\|x\| = \|y\| = 1$ and noticing that

$$\sup_{\|x\|=1} |\langle Ax, x\rangle| = w(A), \quad \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle| = \|A\|, \quad \sup_{\|y\|=1} |\langle By, y\rangle| = w(B),$$

$$\sup_{\|x\|=\|y\|=1} |\langle Bx, y\rangle| = \|B\|, \quad \sup_{\|x\|=\|y\|=1} |\langle x, y\rangle| = 1 \text{ and } \sup_{\|x\|=\|y\|=1} |\langle BAx, y\rangle| = \|BA\|,$$

we deduce the desired result (2.12). ∎

Remark 2. *It is an open problem whether or not the constant $\frac{1}{4}$ is best possible in the inequality (2.12).*

A different approach is consider in the following result:

Theorem 10. *With the assumptions from Theorem 9 we have the inequality*

$$\|BA\| \le w(A)\|B\| + \frac{1}{4}|\beta-\alpha|(|\gamma+\delta| + |\gamma-\delta|). \tag{2.17}$$

Proof. By the Schwarz inequality and taking into account the assumptions for the operators A and B we may state that

$$\begin{aligned}(2.18)\quad & \left|\left\langle Ax-\langle Ax,x\rangle x, B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle\right| \leq \|Ax-\langle Ax,x\rangle x\| \left\|B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\| \\ \leq & \left\|Ax-\frac{\alpha+\beta}{2}x\right\| \left\|B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\| \leq \frac{1}{4}|\beta-\alpha||\gamma-\delta|,\end{aligned}$$

for any $x, y \in H$, with $\|x\|=\|y\|=1$.

Now, since

$$\begin{aligned}& \left\langle Ax-\langle Ax,x\rangle x, B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle \\ = & \langle BAx,y\rangle-\langle Ax,x\rangle\langle Bx,y\rangle-\frac{\gamma+\delta}{2}\langle Ax-\langle Ax,x\rangle x,y\rangle,\end{aligned}$$

on taking the modulus in this equality we have

$$\begin{aligned}(2.19)\quad & \left|\left\langle Ax-\langle Ax,x\rangle x, B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle\right| \\ \geq & |\langle BAx,y\rangle|-|\langle Ax,x\rangle\langle Bx,y\rangle|-\left|\frac{\gamma+\delta}{2}\right||\langle Ax-\langle Ax,x\rangle x,y\rangle|,\end{aligned}$$

for any $x, y \in H$, with $\|x\|=\|y\|=1$.

On making use of (2.18) and (2.19) we get

$$\begin{aligned}(2.20)\quad & |\langle BAx,y\rangle| \\ \leq & |\langle Ax,x\rangle\langle Bx,y\rangle|+\left|\frac{\gamma+\delta}{2}\right||\langle Ax-\langle Ax,x\rangle x,y\rangle|+\frac{1}{4}|\beta-\alpha||\gamma-\delta| \\ \leq & |\langle Ax,x\rangle\langle Bx,y\rangle|+\left|\frac{\gamma+\delta}{2}\right|\left\|Ax-\frac{\alpha+\beta}{2}x\right\|+\frac{1}{4}|\beta-\alpha||\gamma-\delta| \\ \leq & |\langle Ax,x\rangle\langle Bx,y\rangle|+\frac{1}{4}|\beta-\alpha|(|\gamma+\delta|+|\gamma-\delta|),\end{aligned}$$

for any $x, y \in H$, with $\|x\|=\|y\|=1$.

Taking the supremum over $\|x\|=\|y\|=1$ in (2.20) we deduce the desired inequality (2.17). ∎

In a similar manner we can state the following results as well:

Theorem 11. *With the assumptions from Theorem 9 we have the inequality*

$$(2.21)\quad \|BA\| \leq w(A)\|B\|+\frac{1}{2}|\gamma+\delta|(w(A)+\|A\|)+\frac{1}{4}|\beta-\alpha||\gamma-\delta|.$$

Indeed, we observe that

$$\begin{aligned}& \left\langle Ax-\langle Ax,x\rangle x, B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle \\ = & \langle BAx,y\rangle-\langle Ax,x\rangle\langle Bx,y\rangle-\frac{\gamma+\delta}{2}\langle Ax,y\rangle+\frac{\gamma+\delta}{2}\langle Ax,x\rangle\langle x,y\rangle\end{aligned}$$

which produces the inequality

$$\begin{aligned}
&\left|\left\langle Ax - \langle Ax, x\rangle x, B^*y - \frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle\right| + |\langle Ax, x\rangle \langle Bx, y\rangle| \\
&+ \left|\frac{\gamma+\delta}{2}\right| |\langle Ax, y\rangle| + \left|\frac{\gamma+\delta}{2}\right| |\langle Ax, x\rangle| \, |\langle x, y\rangle| \\
\geq\ & |\langle BAx, y\rangle| ,
\end{aligned}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

On utilizing the same argument as in the proof of the above theorem, we get the desired result (2.21). The details are omitted.

3. Other Norm Inequalities

The following result concerning an upper bound for the norm of the operator product may be stated.

THEOREM 12. *With the assumptions from Theorem 9 we have the inequality*

$$\begin{aligned}
&\|BA\| \\
\leq\ & \frac{1}{4}|\beta-\alpha| \, |\gamma-\delta| \\
&+ \left\|\frac{\alpha+\beta}{2}\cdot B + \frac{\gamma+\delta}{2}\cdot A - \frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\cdot I\right\| \\
\leq\ & \frac{1}{4}|\beta-\alpha| \, |\gamma-\delta| \\
&+ \min\left\{\left|\frac{\alpha+\beta}{2}\right| \left(\|B\| + \frac{1}{2}|\beta-\alpha|\right), \left|\frac{\gamma+\delta}{2}\right| \left(\|A\| + \frac{1}{2}|\gamma-\delta|\right)\right\}.
\end{aligned} \tag{3.1}$$

PROOF. By Schwarz inequality and utilizing the assumptions about A and B we have

$$\begin{aligned}
&\left|\left\langle Ax - \frac{\alpha+\beta}{2}x, B^*y - \frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle\right| \\
\leq\ & \left\|Ax - \frac{\alpha+\beta}{2}x\right\| \left\|B^*y - \frac{\bar{\gamma}+\bar{\delta}}{2}y\right\| \\
\leq\ & \frac{1}{4}|\beta-\alpha| \, |\gamma-\delta| ,
\end{aligned} \tag{3.2}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Also, the following identity is of interest in itself

$$\begin{aligned}
&\left\langle Ax - \frac{\alpha+\beta}{2}x, B^*y - \frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle \\
=\ & \langle BAx, y\rangle + \frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\langle x, y\rangle - \frac{\alpha+\beta}{2}\langle Bx, y\rangle - \frac{\gamma+\delta}{2}\langle Ax, y\rangle ,
\end{aligned} \tag{3.3}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This identity gives

$$\begin{aligned} & \left\langle Ax-\frac{\alpha+\beta}{2}x, B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle \\ & +\left\langle \frac{\alpha+\beta}{2}\cdot Bx+\frac{\gamma+\delta}{2}\cdot Ax-\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}x, y\right\rangle \\ = & \left\langle BAx, y\right\rangle, \end{aligned}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Taking the modulus and utilizing (3.2) we get

$$\begin{aligned} \left|\left\langle BAx, y\right\rangle\right| \leq & \left|\left\langle Ax-\frac{\alpha+\beta}{2}x, B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle\right| \\ & +\left|\left\langle \frac{\alpha+\beta}{2}\cdot Bx+\frac{\gamma+\delta}{2}\cdot Ax-\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}x, y\right\rangle\right| \\ \leq & \frac{1}{4}\left|\beta-\alpha\right|\left|\gamma-\delta\right| \\ & +\left\|\frac{\alpha+\beta}{2}\cdot Bx+\frac{\gamma+\delta}{2}\cdot Ax-\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}x\right\|, \end{aligned}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

Finally, taking the supremum over $\|x\| = \|y\| = 1$ we deduce the first part of the desired inequality (3.1). The second part is obvious by the triangle inequality and by the assumptions on A and B. ∎

The following particular case also holds

Corollary 2. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ is accretive. Then*

$$\begin{aligned} \left\|A^{2}\right\| \leq & \frac{1}{4}\left|\beta-\alpha\right|^{2}+\left|\frac{\alpha+\beta}{2}\right|\left\|2\cdot A-\frac{\alpha+\beta}{2}\cdot I\right\| \\ & \left(\leq \frac{1}{4}\left|\beta-\alpha\right|^{2}+\left|\frac{\alpha+\beta}{2}\right|\left(\|A\|+\frac{1}{2}\left|\beta-\alpha\right|\right)\right) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \|A\|^{2} \leq & \frac{1}{4}\left|\beta-\alpha\right|^{2} \\ & +\left\|\frac{\bar{\alpha}+\bar{\beta}}{2}\cdot A^{*}+\frac{\alpha+\beta}{2}\cdot A-\left|\frac{\alpha+\beta}{2}\right|^{2}\cdot I\right\| \\ & \left(\leq \frac{1}{4}\left|\beta-\alpha\right|^{2}+\left|\frac{\alpha+\beta}{2}\right|\left(\|A\|+\frac{1}{2}\left|\beta-\alpha\right|\right)\right), \end{aligned} \tag{3.5}$$

respectively.

The following result provides an approximation for the operator product in terms of some simpler quantities:

Theorem 13. *With the assumptions from Theorem 9 we have the inequality*

$$\left\|BA-\frac{\alpha+\beta}{2}\cdot B-\frac{\gamma+\delta}{2}\cdot A+\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\cdot I\right\| \leq \frac{1}{4}\left|\beta-\alpha\right|\left|\gamma-\delta\right|. \tag{3.6}$$

PROOF. The identity (3.3) can written in an equivalent form as

$$\left\langle Ax-\frac{\alpha+\beta}{2}x, B^{*}y-\frac{\bar{\gamma}+\bar{\delta}}{2}y\right\rangle \tag{3.7}$$

$$= \left\langle \left(BA-\frac{\alpha+\beta}{2}\cdot B-\frac{\gamma+\delta}{2}\cdot A+\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\cdot I\right)x,y\right\rangle, \tag{3.8}$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$.

Taking the modulus and making use of the inequality (3.2) we get

$$\begin{aligned}&\left|\left\langle \left(BA-\frac{\alpha+\beta}{2}\cdot B-\frac{\gamma+\delta}{2}\cdot A+\frac{\alpha+\beta}{2}\cdot\frac{\gamma+\delta}{2}\cdot I\right)x,y\right\rangle\right|\\ \leq\ &\frac{1}{4}\left|\beta-\alpha\right|\left|\gamma-\delta\right|,\end{aligned}$$

for any $x,y\in H$, with $\|x\|=\|y\|=1$, which implies the desired result (3.6). ∎

COROLLARY 3. *Let $A\in B(H)$ and $\alpha,\beta\in\mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ is accretive, then*

$$\left\|A^{2}-(\alpha+\beta)\cdot A+\left(\frac{\alpha+\beta}{2}\right)^{2}\cdot I\right\|\leq\frac{1}{4}\left|\beta-\alpha\right|^{2} \tag{3.9}$$

and

$$\left\|A^{*}A-\frac{\alpha+\beta}{2}\cdot A^{*}-\frac{\bar{\alpha}+\bar{\beta}}{2}\cdot A+\left|\frac{\alpha+\beta}{2}\right|^{2}\cdot I\right\|\leq\frac{1}{4}\left|\beta-\alpha\right|^{2}, \tag{3.10}$$

respectively.

REMARK 3. *It is an open problem whether or not the constant $\frac{1}{4}$ is best possible in either of the inequalities (3.6), (3.9) or (3.10) above.*

The following theorem provides an approximating for the operator $\frac{1}{2}(U^{*}U+UU^{*})$ when some information about the real or imaginary part of the operator U are given.

We recall that $U=\operatorname{Re}(U)+i\operatorname{Im}(U)$, i.e., $\operatorname{Re}(U)=\frac{1}{2}(U+U^{*})$ and $\operatorname{Im}(U)=\frac{1}{2i}(U-U^{*})$. For the simplicity, we denote with A the real part of U and with B its imaginary part.

THEOREM 14. *Suppose that $a,b,c,d\in\mathbb{R}$ are so that $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive. Denote $\alpha:=a+ib$ and $\beta:=c+id\in\mathbb{C}$, then*

$$\begin{aligned}&\left\|\frac{1}{2}(U^{*}U+UU^{*})-\frac{\bar{\alpha}+\bar{\beta}}{2}\cdot U-\frac{\alpha+\beta}{2}\cdot U^{*}+\left|\frac{\alpha+\beta}{2}\right|^{2}\cdot I\right\|\\ \leq\ &\frac{1}{4}\left|\alpha-\beta\right|^{2}.\end{aligned} \tag{3.11}$$

PROOF. It is well known that for any operator T with $T=C+iD$ we have

$$\frac{1}{2}(T^{*}T+TT^{*})=C^{2}+D^{2}. \tag{3.12}$$

For any $z\in\mathbb{C}$ we also have the identity

$$\begin{aligned}&\frac{1}{2}\left[(U-zI)(U^{*}-\bar{z}I)+(U^{*}-\bar{z}I)(U-zI)\right]\\ =\ &\frac{1}{2}(U^{*}U+UU^{*})-\bar{z}\cdot U-z\cdot U^{*}+\left|z\right|^{2}\cdot I.\end{aligned} \tag{3.13}$$

For $z=\frac{\alpha+\beta}{2}$ we observe that

$$\operatorname{Re}(U-zI)=A-\frac{a+c}{2}\cdot I \text{ and } \operatorname{Im}(U-zI)=B-\frac{b+d}{2}\cdot I$$

and utilizing the identities (3.12) and (3.13) we deduce

$$\begin{aligned}
&\left\|\frac{1}{2}\left(U^{*}U+UU^{*}\right)-\bar{z}\cdot U-z\cdot U^{*}+|z|^{2}\cdot I.\right\| \\
= &\left\|\left(A-\frac{a+c}{2}\cdot I\right)^{2}+\left(B-\frac{b+d}{2}\cdot I\right)^{2}\right\| \\
\leq &\left\|A-\frac{a+c}{2}\cdot I\right\|^{2}+\left\|B-\frac{b+d}{2}\cdot I\right\|^{2} \\
\leq &\frac{1}{4}\left[(c-a)^{2}+(d-b)^{2}\right]=\frac{1}{4}|\alpha-\beta|^{2},
\end{aligned}$$

where for the last inequality we have used the fact that $C_{a,c}(A)$ and $C_{b,d}(B)$ are accretive. ∎

Remark 4. *It is an open problem whether or not the constant* $\frac{1}{4}$ *is best possible in (3.11).*

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