Hermite-Hadamard's Inequality and the p-HH-Norm on the Cartesian Product of Two Copies of a Normed Space

HERMITE-HADAMARD'S INEQUALITY AND THE p-HH-NORM ON THE CARTESIAN PRODUCT OF TWO COPIES OF A NORMED SPACE

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Abstract. The Cartesian product of two copies of a normed space is naturally equipped with the well-known p-norm. In this paper, another notion of norm is introduced, and will be called the p-HH-norm. This norm is an extension of the generalised logarithmic mean and is connected to the p-norm by the Hermite-Hadamard's inequality. The Cartesian product space (with respect to both norms) is complete, when the (original) normed space is. A proof for the completeness of the p-HH-norm via Ostrowski's inequality is provided. This space is embedded as a subspace of the well-known Lebesgue-Bochner function space (as a closed subspace, when the norm is a Banach norm). Consequently, its geometrical properties are inherited from those of Lebesgue-Bochner space. An explicit expression of the superior (inferior) semi-inner product associated to both norms is considered and used to provide alternative proofs for the smoothness and reflexivity of this space.

1. Introduction

We recall the classical Hermite-Hadamard's inequality for any convex function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ (see, for instance, [15]):

$$(b-a)f\left(\frac{a+b}{2}\right) \le \int_a^b f(t)dt \le (b-a)\left[\frac{f(a)+f(b)}{2}\right]. \tag{1.1}$$

Recently, the Hermite-Hadamard's inequality has been extended for convex functions in linear spaces (see, for instance, [11, 12]). To be precise, let $\mathbf{X}$ be a linear space over $\mathbb{R}$, x and y be two distinct vectors in $\mathbf{X}$, and define the segment $[x, y] := \{(1-t)x + ty,\ t \in [0, 1]\}$. Let $f : [x, y] \to \mathbb{R}$ be a convex function, then the following Hermite-Hadamard integral inequality (see [11, p. 2], [12, p. 2], [15, p. 78], and [27, p. 103–105]) is obtained from (1.1):

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x+ty]dt \le \frac{f(x)+f(y)}{2}. \tag{1.2}$$

When $\mathbf{X}$ is equipped by a norm, namely $\|\cdot\|$, then for any $p \ge 1$, we have the following norm inequality (see [15, p. 79] and [27, p. 106]):

$$\left\|\frac{x+y}{2}\right\| \le \left(\int_0^1 \|(1-t)x+ty\|^p dt\right)^{\frac{1}{p}} \le \frac{1}{2^{\frac{1}{p}}}\left(\|x\|^p + \|y\|^p\right)^{\frac{1}{p}}, \tag{1.3}$$

by (1.2) and the convexity of $f(x) = \|x\|^p$ $(x \in \mathbf{X},\ p \ge 1)$.

We are interested in investigating the Cartesian product of two copies of a normed linear space $(\mathbf{X}, \|\cdot\|)$, where the addition and scalar multiplication are defined in the usual way. The Cartesian product space $\mathbf{X}^2$ is naturally equipped by the well-known p-norm, i.e. $\|(x, y)\|_p := (\|x\|^p + \|y\|^p)^{\frac{1}{p}}$ for any (x, y) in $\mathbf{X}^2$. Previous results regarding the Cartesian product of Banach

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spaces have been considered in [8,20,25], where the results were stated in a more general setting, i.e. the Banach-valued sequence space $l^p(\mathbf{X})$.

In this paper, we prove that the quantity $\left(\int_0^1 \|(1-t)x+ty\|^p dt\right)^{\frac{1}{p}}$ is a norm of (x,y) in $\mathbf{X}^2$ (which will be called the p-HH-norm). We observe that, when $\mathbf{X}$ is the field of real numbers, this quantity is the generalised logarithmic mean of two positive numbers x and y. This observation rises due to the fact that $\int_0^1 \|(1-t)x+ty\|^p dt$ is the integral mean of $\|\cdot\|^p$ on segment $[x,y]$. Therefore, the p-HH-norm extends the generalised logarithmic mean to a more general setting of normed spaces. We also remark that inequality (1.3) gives a relation among the value of the function $\|\cdot\|^p$ at the midpoint of segment $[x,y]$, its integral mean (i.e. the p-HH-norm of (x,y) in $\mathbf{X}^2$), and the p-norm of (x,y) in $\mathbf{X}^2$.

When $\mathbf{X}$ is a Banach space, $\mathbf{X}^2$ with respect to the p-norm is also complete, due to the previous results of the Banach-valued sequence space $l^p(\mathbf{X})$ (see, for example, [8]). In this paper, we show that the space $\mathbf{X}^2$ together with the p-HH-norm is also complete, whenever $\mathbf{X}$ is. It implies that these norms are equivalent in $\mathbf{X}^2$. Thus, they induce the same topology. Moreover, we are able to prove the equivalency of these norms on $\mathbf{X}^2$, via Ostrowski's inequality for absolutely continuous function on segment in normed spaces [18]. This fact provides an alternative proof for the completeness of the p-HH-norm on $\mathbf{X}^2$, via equivalency of both norms.

We are also interested in investigating the geometrical properties of $\mathbf{X}^2$ with respect to both norms. For this purpose, we recall the well-known Lebesgue-Bochner function space $L^p([0,1],\mathbf{X})$ $(1 \le p \le \infty)$, i.e. the space of functions f defined on the interval $[0,1]$, which take values in the normed space $\mathbf{X}$, where $\int_0^1 \|f(t)\|^p dt$ is finite [3]. This space is a normed space together with the norm $\|f\|_{L^p} = \left(\int_0^1 \|f(t)\|^p dt\right)^{\frac{1}{p}}$ and is a Banach space whenever $\mathbf{X}$ is (see [2,16]). For $1 < p < \infty$, some particular geometrical properties (i.e. strict convexity, uniform convexity, smoothness, Fréchet smoothness, and reflexivity) of $L^p([0,1],\mathbf{X})$ are implied by those of $\mathbf{X}$ (see [2,3,9,10,19,22] for references).

We examine that the Cartesian product space $\mathbf{X}^2$, with respect to both p-norm and p-HH-norm, is embedded as a subspace of the Lebesgue-Bochner space $L^p([0,1],\mathbf{X})$. In particular, when $\mathbf{X}$ is a Banach space, the Cartesian product space can be embedded as a closed subspace of $L^p([0,1],\mathbf{X})$. As a consequence, the geometrical properties of the Cartesian product space are inherited from the Lebesgue-Bochner space.

In this paper, we also provide an explicit expression of the superior (inferior) semi-inner product in $\mathbf{X}^2$ associated to both norms. By using the semi-inner product, we provide alternative proofs for the smoothness and the reflexivity of $\mathbf{X}^2$. Although the proofs are simpler by considering the embedding, we keep the representation of semi-inner product for further research on the orthogonality concepts that can be considered in the Cartesian product space.

2. Definitions, notations, and preliminary results

All definitions, notations, and related properties, which are used in the paper, are described in this section for references. Throughout this paper, we assume that all linear spaces are over the field of real numbers. We also denote $(\mathbf{X},\|\cdot\|)$ as a normed space and $(\mathbf{B},\|\cdot\|)$ as a Banach space. Unless mentioned otherwise, the measure that we consider in this paper is in Lebesgue sense, and we denote $m(E)$ as the Lebesgue measure of a subset E of $\mathbb{R}$. We also denote $\bar{\mathbb{R}}$ for the extended real numbers.

2.1. **Geometrical properties of Banach space.** In any normed space $\mathbf{X}$, the norm $\|\cdot\|$ is right (left)-Gâteaux differentiable at $x \in \mathbf{X} \setminus \{0\}$, i.e. the following limits

$$(\nabla_{+(-)}\|\cdot\|(x))(y) := \lim_{t\to 0^{+(-)}} \frac{\|x+ty\| - \|x\|}{t}$$

exist for all $y \in \mathbf{X}$ (see [23, p. 483–485] for the proof)*. The norm $\|\cdot\|$ is Gâteaux differentiable at $x \in \mathbf{X} \setminus \{0\}$ if and only if $(\nabla_+\|\cdot\|(x))(y) = (\nabla_-\|\cdot\|(x))(y)$, for all $y \in \mathbf{X}$. A normed linear space $(\mathbf{X}, \|\cdot\|)$ is said to be *smooth* if and only if the norm $\|\cdot\|$ is Gâteaux differentiable on $\mathbf{X}\setminus\{0\}$. The norm $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ is said to be *Fréchet differentiable* at $x \in \mathbf{X}$ if and only if there exists a continuous linear functional G_x on $\mathbf{X}$ such that

$$\lim_{\|h\|\to 0} \frac{|\|x+th\| - \|x\| - G_x(h)|}{\|h\|} = 0.$$

When this property holds for any $x \in \mathbf{X}$, then the normed space is said to be *Fréchet smooth* (see [19, p. 230] and [23, p. 504] for references).

The function $f_0(\cdot) = \frac{1}{2}\|\cdot\|^2$ on $\mathbf{X}$ is convex and therefore, the following limits

$$\langle x, y\rangle_{s(i)} := (\nabla_{+(-)} f_0(y))(x) = \lim_{t\to 0^+(-)} \frac{\|y+tx\|^2 - \|y\|^2}{2t}$$

exist for any $x, y \in \mathbf{X}$ and are called the *superior (inferior) semi-inner products* (s.i.p.) associated to the norm $\|\cdot\|$ [13, p. 27] (for their further properties, see [13, p. 27–39]). The following identity [13, p. 43] gives a relationship between the s.i.p. and the Gâteaux lateral (one-sided) derivatives of the given norm:

$$\langle x, y\rangle_{s(i)} = \|y\|(\nabla_{+(-)}\|\cdot\|(y))(x), \quad \text{for all } x, y \in \mathbf{X}, \text{ where } y \neq 0. \tag{2.1}$$

Due to the convexity of f_0, we have in general

$$\langle x, y\rangle_i \le \langle x, y\rangle_s, \text{ for all } x, y \in \mathbf{X}. \tag{2.2}$$

The following provides a necessary and sufficient condition for a normed space to be smooth (see also [11, p. 2], [12, p. 2], and [14, p. 338]).

Proposition 1. *Equality holds in (2.2) if and only if* $\mathbf{X}$ *is smooth.*

We remark that every subspace of a (Fréchet) smooth normed space is itself a (Fréchet) smooth space [23, p. 488]. Note that Fréchet differentiability implies Gâteaux differentiability [23, p. 504], but not conversely. As an example (this example is due to Sova [30], with remark by Gieraltowska-Kedzierska and Van Vleck [17]), the mapping $f : L^1[0,\pi] \to \mathbb{R}$ defined by $f(x) = \int_0^\pi \sin x(t)dt$ is everywhere Gâteaux differentiable, but nowhere Fréchet differentiable.

A normed space is reflexive whenever it is isomorphic to its bidual. It implies that any reflexive normed space is always complete, by the completeness of the dual space [23, p. 99]. Thus, the completeness is necessary for a normed space to be a reflexive space. The incomplete reflexive normed space is defined by the reflexivity of its completion [23, p. 99]. We also note that a Banach space $\mathbf{B}$ is reflexive if and only if the dual space $\mathbf{B}^*$ is (see [23, p. 104] for the proof). Every closed subspace of a reflexive normed space is reflexive (proof can be found in [23, p. 104]).

A normed space that is isomorphic to a reflexive space is itself reflexive. Moreover, a Banach space is reflexive if it is an image of a reflexive space under a bounded linear operator, regardless of whether it is an isomorphism or not [23, p. 105]. The following lemma is a direct consequence of this fact:

*In some literature, the quantities $(\nabla_{+(-)}\|\cdot\|(x))(y)$ are denoted by $\tau_\pm(x,y)$, and are called the tangent functionals (see, for example, [13, p. 43]).

Lemma 1. *Let* $(\mathbf{B}, \|\cdot\|)$ *be a reflexive Banach space. If there exists a norm* $|||\cdot|||$ *on* $\mathbf{B}$ *which is equivalent to* $\|\cdot\|$*, then* $(\mathbf{B}, |||\cdot|||)$ *is also reflexive.*

Proof. Since $\|\cdot\|$ and $|||\cdot|||$ are equivalent, the identity operator, considered as a linear operator from $(\mathbf{B}, \|\cdot\|)$ onto $(\mathbf{B}, |||\cdot|||)$, is bounded. Therefore $(\mathbf{B}, |||\cdot|||)$ is reflexive, since $(\mathbf{B}, \|\cdot\|)$ is. □

The following result is a natural generalisation of the Riesz representation theorem and is used to characterise the reflexivity of a Banach space (see [13, p. 150] for the complete proof):

Proposition 2. *Let* $\langle\cdot,\cdot\rangle_{s(i)}$ *be the superior (inferior) s.i.p. associated to the norm* $\|\cdot\|$ *on a Banach space* $\mathbf{B}$*. Then,* $\mathbf{B}$ *is reflexive if and only if for every continuous linear functional* f *on* $\mathbf{B}$ *there exists an element* u *in* $\mathbf{B}$ *such that*

$$\langle x,u\rangle_i \le f(x) \le \langle x,u\rangle_s, \quad for\ all\ x \in \mathbf{B},\ and\ \|f\| = \|u\|.$$

The strict convexity (or rotundity) can be intuitively described as the condition where any nontrivial straight line segments, whose endpoints lie in the unit sphere, has its midpoint in the interior of the closed unit ball (see [23, p. 441]). The notion of uniform convexity deals with the question of how far the midpoint (of such segment) into the interior of the closed unit ball is (see [23, p. 441-442]). The formal definitions can be stated as follows:

Definition 1. Let $S_{\mathbf{X}}$ be the unit sphere in $\mathbf{X}$, that is, $S_{\mathbf{X}} := \{x \in \mathbf{X} : \|x\| = 1\}$. Then,

(1) $\mathbf{X}$ will be called *strictly convex* if for every $x, y \in S_{\mathbf{X}}$ with $x \neq y$, we have $\|\lambda x + (1-\lambda)y\| < 1$, for all $\lambda \in (0,1)$;
(2) $\mathbf{X}$ is *uniformly convex* if for any positive ϵ, there exists a positive δ depending on ϵ such that $\left\|\frac{x+y}{2}\right\| \le 1-\delta$ whenever $x, y \in S_{\mathbf{X}}$ and $\|x-y\| > \epsilon$.

Proposition 3. *The strict (uniform) convexity of a normed space is inherited by its subspaces.*[†]

2.2. **Lebesgue-Bochner function spaces.** A definition of Lebesgue integral for functions on an interval of real numbers to a Banach space $(\mathbf{B}, \|\cdot\|)$ has been given by Bochner in [2], which is now referred to as the Bochner integral. Bochner introduced a generalisation of Lebesgue function space L^p as follows: the space $L^p([0,1], \mathbf{B})$ is the class of functions f defined on the interval $[0,1]$, with values in $\mathbf{B}$ for which the norm $\|f\|_{L^p} := \left(\int_0^1 \|f(t)\|^p dt\right)^{\frac{1}{p}}$ is finite [3, p. 914]. With this definition of norm, $L^p([0,1], \mathbf{B})$ is a Banach space (for references, see [2, 16]). This space is called the Lebesgue-Bochner (or sometimes, Bochner) function space (see, for example, [29]).

Lemma 2. *Let* $1 < p < \infty$*. The space* $L^p([0,1], \mathbf{B})$ *is a smooth (Fréchet smooth) Banach space whenever* $\mathbf{B}$ *is.*[‡]

Lemma 3. *Let* $1 < p < \infty$*. The space* $L^p([0,1], \mathbf{B})$ *is a reflexive Banach space if* $\mathbf{B}$ *is.*

Bochner in [3, p. 930] stated that if $\mathbf{B}$ and its dual $\mathbf{B}^*$ are of (D)-property (i.e. any function of bounded variation is differentiable almost everywhere [3, p. 914–915]) and $\mathbf{B}$ is reflexive, then $L^p([0,1], \mathbf{B})$ is reflexive. However, further studies have proven that these conditions could be reduced to a simpler one. The argument is as follows: any reflexive space has the *Radon-Nikodym property*, i.e. every absolutely continuous Banach-valued function is differentiable almost everywhere [1, p. 20]. Hence, any function of bounded variation is differentiable almost everywhere. By the fact that $\mathbf{B}$ is reflexive if and only if $\mathbf{B}^*$ is, we conclude that the reflexivity of $\mathbf{B}$ is the only condition required such that $L^p([0,1], \mathbf{B})$ is reflexive.

[†]See [23, p. 436] and [23, p. 454] for the proof.
[‡]Proof can be found in [22, p. 233-237, 404].

Lemma 4. *Let $1 < p < \infty$. The space $L^p([0,1],\mathbf{B})$ is a strictly (uniformly) convex Banach space, whenever $\mathbf{B}$ is.*

The proof is implied by the strict (uniform) convexity of $l^p(\mathbf{B})$ (for the complete proof, see [9,10]). Note that $l^p(\mathbf{B}) := \{(x_n) : x_n \in \mathbf{B}, \sum_n \|x_n\|^p < \infty\}$. The proof for $L^p([0,1],\mathbf{B})$ follows by the embedding argument similar to Clarkson's argument in [8], which can be briefly stated as follows: consider a step function on a partition of $[0,1]$ into equal parts. Such function can be identified as of $l^p(\mathbf{B})$. Since the set of all step functions on $[0,1]$ is a dense set in $L^p([0,1],\mathbf{B})$ [33, p. 132], and by the continuity of the norm, each function can be "identified" by an element in $l^p(\mathbf{B})$.

3. Power mean and generalised logarithmic mean

In this section, we summarise the definitions and basic properties of power mean and generalised logarithmic mean, which have close relations to the norms that can be defined on the Cartesian space $\mathbb{R}^2$.

The *logarithmic mean* [7, p. 615] of two positive numbers x and y is defined by:

$$L(x,y) = \begin{cases} \frac{x-y}{\log(x)-\log(y)}, & x \neq y; \\ x, & x = y. \end{cases}$$

The logarithmic mean L is symmetric, homogeneous in x and y, and continuous at $x = y$ [7, p. 615]. Lin in [21, p. 879] mentioned the use of logarithmic mean in some practical problems, such as in heat transfer and fluid mechanics.

In [31, p. 88], Stolarsky mentioned the matter of understanding why $L(x,y)$ is a mean, i.e. it is *internal*: $\min\{x,y\} \leq L(x,y) \leq \max\{x,y\}$. Stolarsky considered the mean value theorem for differentiable functions f

$$\frac{f(x)-f(y)}{x-y} = f'(u), \quad x \neq y,$$

where u is strictly between x and y, and derived that if $f(x) = \log x$, then $u = L(x,y)$. This motivates us to 'create new means' by varying the function f. One of the function that was considered in [31] is $f(x) = x^p$ ($p \in \mathbb{R}$, $p \neq 0,1$). This is later known as the generalisation of logarithmic mean (see [4, p. 385], [31, p. 88–90], and [32, p. 545] for references).

Definition 2. Let $p \in \bar{\mathbb{R}}$, $x,y > 0$, and $x \neq y$. The *generalised logarithmic mean* of order p of x and y is defined by

$$\mathfrak{L}^{[p]}(x,y) = \begin{cases} \left[\frac{1}{p+1}\left(\frac{y^{p+1}-x^{p+1}}{y-x}\right)\right]^{\frac{1}{p}}, & \text{if } p \neq -1,0,\pm\infty; \\ \frac{y-x}{\log y - \log x}, & \text{if } p = -1; \\ \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}, & \text{if } p = 0; \\ \max\{x,y\}, & \text{if } p = +\infty; \\ \min\{x,y\}, & \text{if } p = -\infty, \end{cases} \tag{3.1}$$

and $\mathfrak{L}^{[p]}(x,x) = x$.

This mean is homogeneous and symmetric [4, p. 385], so in particular there is no loss in generality by assuming that $0 < x < y$. Note that the generalised logarithmic mean is related to the other well-known means. We summarised the relations as follows:

(1) $\mathfrak{L}^{[-1]}(x,y) = L(x,y)$, the logarithmic mean of x and y;
(2) $\mathfrak{L}^{[0]}(x,y) = I(x,y)$, the *identric mean* of x and y;
(3) $\mathfrak{L}^{[1]}(x,y) = \frac{x+y}{2} = A(x,y)$, the *arithmetic mean* of x and y;
(4) $\mathfrak{L}^{[-2]}(x,y) = \sqrt{xy} = G(x,y)$, the *geometric mean* of x and y;

(5) $\mathfrak{L}^{[2]}(x,y) = \sqrt{\frac{1}{3}(x^2+xy+y^2)} = Q(x,y,G(x,y))$, the *quadratic mean* of x, y and $G(x,y)$.

Proposition 4. *Let $\mathfrak{L}^{[p]}$ $(p \in \bar{\mathbb{R}})$ be the generalised logarithmic mean, as defined in Definition 2. Then,*

(1) *if $0 < x \le y$ and $-\infty \le r < s \le \infty$, then $x \le \mathfrak{L}^{[r]}(x,y) \le \mathfrak{L}^{[s]}(x,y) \le y$, with equality if and only if $x = y$ (in particular, the generalised logarithmic mean is strictly internal);*
(2) *for all $p \in \bar{\mathbb{R}}$, $\mathfrak{L}^{[p]}(x,y)$ is strictly increasing both as function of x and y.*

Proof of Proposition 4 can be found in [4, p. 387] (see [31, p. 88-90] for an alternative proof of part (1)). By the fact that $\mathfrak{L}^{[p]}$ is increasing as a function of p on $\bar{\mathbb{R}}$, we have the following inequalities

$$G(x,y) \le L(x,y) \le I(x,y) \le A(x,y), \quad \text{for all } x,y \in \mathbb{R}. \tag{3.2}$$

For further properties of logarithmic mean and its relationship with the other means, we refer to [4–7, 21, 26, 28, 31, 32].

The following definition (see [21, p. 879–880] and [28, p. 19–20]) is a generalisation of the root mean square (the quadratic mean [21, p. 879]):

Definition 3. Let x and y be two positive numbers and $p \in \mathbb{R}$. The *power mean* of x and y is defined by

$$M_p = M_p(x,y) = \begin{cases} \left(\dfrac{x^p+y^p}{2}\right)^{\frac{1}{p}}, & p \neq 0; \\ G(x,y), & p = 0. \end{cases} \tag{3.3}$$

The basic properties of power mean can be summarised as follows [31, p. 88]:

Proposition 5. *Let M_p $(p \in \mathbb{R})$ be the power mean. Then, the following holds for any positive numbers x and y:*

(1) $\min\{x,y\} \le M_p(x,y) \le \max\{x,y\}$ *(internal);*
(2) *$M_p(x,y)$ is continuous in p;*
(3) *$M_p(x,y) \le M_q(x,y)$ if $p \le q$;*
(4) *$M_0(x,y) = G(x,y)$ and $M_1(x,y) = A(x,y)$.*

We refer to [4, 21, 26, 28, 32] for further properties of power means.

In [6, p. 36], Carlson suggested the following inequalities which involve the power mean and the generalised logarithmic mean (by considering an inequality for certain hypergeometric functions):

$$\left(\frac{x+y}{2}\right)^p \le \frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)} \le \frac{x^p+y^p}{2}, \quad x,y>0,\ x \neq y \text{ and, } p \ge 1. \tag{3.4}$$

Another way to verify the inequalities (3.4) is by considering the normed space $(\mathbb{R}, |\cdot|)$ in the Hermite-Hadamard's inequality (1.3).

4. The Cartesian product of two normed spaces

4.1. **The p-norm.** We are interested in investigating the Cartesian product space $\mathbf{X}^2 = \mathbf{X} \times \mathbf{X} := \{(x,y) : x,y \in \mathbf{X}\}$, where the addition and scalar multiplication are defined in the usual way. The 'standard' way of constructing norm on the Cartesian product space $\mathbf{X}^2$ is to define $\|(x,y)\| = \varphi(\|x\|,\|y\|)$, $(x,y \in \mathbf{X})$ via some functions φ on $\mathbb{R}^2$ [20, p. 35–36]. One common example of φ (see [8, p. 397–398], [20, p. 36], and [25, p. 142]) is

$$\varphi(a,b) = (a^p+b^p)^{\frac{1}{p}},\ a,b \in \mathbb{R},\ 1 \le p \le \infty.$$

Definition 4. The p-norm on $\mathbf{X}^2$ is defined by

$$\|(x,y)\|_p := \begin{cases} (\|x\|^p + \|y\|^p)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \max\{\|x\|, \|y\|\}, & p = \infty, \end{cases}$$

for any $(x,y) \in \mathbf{X}^2$.

This norm is symmetric in the sense that $\|(x,y)\|_p = \|(y,x)\|_p$ for any $(x,y) \in \mathbf{X}^2$.

Remark 1. If $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, then $\mathbf{X}^2 = l_2^p = \{(x_n)_{n=1}^{\infty} \in l^p : x_n = 0, \forall n > 2\}$. Furthermore, if $x, y > 0$, then

$$\|(x,y)\|_p = (x^p + y^p)^{\frac{1}{p}} = 2^{\frac{1}{p}} M_p(x,y),$$

where M_p is the power mean (as defined in Section 3).

Lemma 5. *All p-norms $(1 \le p \le \infty)$ are equivalent in $\mathbf{X}^2$. Furthermore, we have the following inequalities for $1 < p \le \infty$:*

$$\|(x,y)\|_p \le \|(x,y)\|_1 \le 2\|(x,y)\|_p.$$

Lemma 6. *The space $(\mathbf{B}^2, \|(\cdot,\cdot)\|_p)$ $(1 \le p \le \infty)$ is a Banach space, when $\mathbf{B}$ is.*

The proof of Lemma 5 and Lemma 6 can be derived by the similar statements that we have for the l_2^p spaces (see, for example, [8, p. 397–398]).

Proposition 6. *The p-norm is decreasing as a function of p on $[1, \infty]$, that is, for any $1 \le r < s \le \infty$ and $(x,y) \in \mathbf{X}^2$, we have*

$$\max\{\|x\|, \|y\|\} \le \|(x,y)\|_s = (\|x\|^s + \|y\|^s)^{\frac{1}{s}} \le (\|x\|^r + \|y\|^r)^{\frac{1}{r}} = \|(x,y)\|_r. \tag{4.1}$$

Proof. The first part of inequalities (4.1) follows by Lemma 5. We have the following inequality for any $1 \le r < s \le \infty$ (see [4, p. 186]):

$$(a^s + b^s)^{\frac{1}{s}} \le (a^r + b^r)^{\frac{1}{r}},$$

for any real numbers $a, b > 0$. Choose $a = \|x\|$ and $b = \|y\|$ to obtain the desired result. □

4.2. **The p-HH-norm: definition and example.** Define the quantity

$$\|(x,y)\|_{p-HH} := \begin{cases} \left(\int_0^1 \|(1-t)x + ty\|^p dt\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty; \\ \sup\limits_{t \in [0,1]} \|(1-t)x + ty\|, & \text{if } p = \infty, \end{cases} \tag{4.2}$$

for any $x, y \in \mathbf{X}$. The integral is finite by the Hermite-Hadamard's inequality, i.e.

$$\int_0^1 \|(1-t)x + ty\|^p dt \le \frac{\|x\|^p + \|y\|^p}{2} = \frac{\|(x,y)\|_p^p}{2} < \infty, \quad \text{for any } x, y \in \mathbf{X}. \tag{4.3}$$

Note that $\|(\cdot,\cdot)\|_{p-HH}$ is symmetric, that is, $\|(x,y)\|_{p-HH} = \|(y,x)\|_{p-HH}$ for all $x, y \in \mathbf{X}$.

Remark 2. Consider the function

$$f(t) = \|(1-t)x + ty\|, \ t \in [0,1], \ x, y \in \mathbf{X}.$$

Since it is continuous and convex on $[0,1]$, the supremum of f on $[0,1]$ is exactly its maximum, and is attained at one of the endpoints. In other words, for any $x, y \in \mathbf{X}$

$$\|(x,y)\|_{\infty-HH} = \sup_{t \in [0,1]} \|(1-t)x + ty\| = \max\{\|x\|, \|y\|\} = \|(x,y)\|_{\infty}.$$

Thus, $\|(\cdot,\cdot)\|_{\infty-HH}$ defines a norm. We will not distinguish $\|(\cdot,\cdot)\|_{\infty-HH}$ from $\|(\cdot,\cdot)\|_{\infty}$, and refer to them as $\|(\cdot,\cdot)\|_{\infty}$.

Lemma 7. *The space* $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ $(1 \le p < \infty)$ *is a normed linear space.*

Proof. The homogeneity of the norm follows directly by definition. The triangle inequality follows from the Minkowski's inequality for $(L_p([0,1],\mathbf{X}), \|\cdot\|_{L^p})$ [16, p. 120]. The nonnegativity of the norm is derived from the definition. Now, if $(x,y) = (0,0)$, then $\|(1-t)x + ty\| = 0$ $(t \in [0,1])$, therefore, $\|(x,y)\|_{p-HH} = 0$. Conversely, let $x, y \in \mathbf{X}$ such that $\|(x,y)\|_{p-HH} = 0$. Since

$$0 \le \left\| \frac{x+y}{2} \right\| \le \left(\int_0^1 \|(1-t)x + ty\|^p dt \right)^{\frac{1}{p}} = 0,$$

we have $\left\|\frac{x+y}{2}\right\| = 0$. Thus, $x = -y$ and

$$\left(\int_0^1 \|(1-t)x + ty\|^p dt \right)^{\frac{1}{p}} = \left(\int_0^1 |2t-1|^p \|y\|^p dt \right)^{\frac{1}{p}} = \|y\| \left(\frac{1}{p+1} \right)^{\frac{1}{p}}. \tag{4.4}$$

Since $\|(x,y)\|_{p-HH} = 0$ and $\frac{1}{p+1} \neq 0$, $\|y\| = 0$ by (4.4), which implies that $x = y = 0$. □

Remark 3 (Special case)**.** Note that if the norm $\|\cdot\|$ on $\mathbf{X}$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then

$$\|(x,y)\|_{2-HH}^2 = \int_0^1 \|(1-t)x + ty\|^2 dt = \frac{1}{3}\left(\|x\|^2 + \langle x,y\rangle + \|y\|^2\right). \tag{4.5}$$

Remark 4 (Case of $p < 1$)**.** Although it is possible to define the quantity in (4.2) for $p < 1$, we are only interested in the case where $p \ge 1$, since $\|(\cdot,\cdot)\|_{p-HH}$ does not define a norm on $\mathbf{X}^2$ for $p < 1$. For example, we consider the normed space $(\mathbb{R}, |\cdot|)$. Thus, for any $(x,y) \in \mathbb{R}^2$ and $p < 1$, we have

$$|(x,y)|_{p-HH} = \left(\int_0^1 |(1-t)x + ty|^p dt \right)^{\frac{1}{p}}.$$

We claim that $|(\cdot,\cdot)|_{p-HH}$ is not a norm on $\mathbf{R}^2$. To verify this, choose $(x,y) = (1,0)$ and $(u,v) = (0,1)$, then consider the following cases:
Case 1: $p \in (-1,1)$**.** We have

$$|(x,y)|_{p-HH} + |(u,v)|_{p-HH} = 2(p+1)^{-\frac{1}{p}} \quad \text{and} \quad |(x,y)+(u,v)|_{p-HH} = 1.$$

We claim that $(p+1)^{-\frac{1}{p}} < \frac{1}{2}$ for any $p \in (-1,1)$. Thus,

$$|(x,y)|_{p-HH} + |(u,v)|_{p-HH} = 2(p+1)^{-\frac{1}{p}} < 1 = |(x,y)+(u,v)|_{p-HH},$$

which fails the triangle inequality.

Proof of claim. Define $f(p) = (p+1)^{-\frac{1}{p}}$ for $p \in (-1,1) \setminus \{0\}$ and $f(0) = e^{-1}$. By Proposition 4, we have for any $a > 0$ and $-1 \le r < s \le 1$ $(r, s \neq 0)$:

$$\mathfrak{L}^{[r]}(a,1) = \left[\frac{1}{r+1} \left(\frac{1-a^{r+1}}{1-a} \right) \right]^{\frac{1}{r}} < \left[\frac{1}{s+1} \left(\frac{1-a^{s+1}}{1-a} \right) \right]^{\frac{1}{s}} = \mathfrak{L}^{[s]}(a,1),$$

by the definition of the generalised logarithmic mean. By taking $a \to 0^+$, we get

$$(r+1)^{-\frac{1}{r}} < (s+1)^{-\frac{1}{s}},$$

which shows that f is strictly increasing on $(-1,1) \setminus \{0\}$. Since $\lim_{p\to 0}(p+1)^{-\frac{1}{p}} = e^{-1}$, f is continuous at $p = 0$ (thus, continuous on $(-1,1)$), which implies that f is strictly increasing on $(-1,1)$. Thus, $\sup_{p\in(-1,1)}(p+1)^{-\frac{1}{p}} = \lim_{p\to 1^-}(p+1)^{-\frac{1}{p}} = \frac{1}{2}$. Thus, $(p+1)^{-\frac{1}{p}} < \frac{1}{2}$ for all $p \in (-1,1)$. □

Case 2: $p \in (-\infty, -1)$**.** We have

$$|(x,y)|_{p-HH}^{p} = \int_0^1 (1-t)^p dt \to \infty, \text{ and } |(u,v)|_{p-HH}^{p} = \int_0^1 t^p dt \to \infty.$$

Since $p < 0$, we get $|(x,y)|_{p-HH} \to 0$ and $|(u,v)|_{p-HH} \to 0$, which imply that

$$|(x,y)|_{p-HH} + |(u,v)|_{p-HH} \to 0. \tag{4.6}$$

We also have $|(x,y)+(u,v)|_{p-HH} = 1$. By (4.6), we can find $\epsilon > 0$ such that

$$0 < |(x,y)|_{p-HH} + |(u,v)|_{p-HH} < \epsilon < 1 = |(x,y)+(u,v)|_{p-HH},$$

which fails the triangle inequality.

Example 1 (Real numbers: $\mathbb{R}$)**.** In $\mathbb{R}^2$, we have the following norm:

$$|(x,y)|_{p-HH} := \left(\int_0^1 |(1-t)x + ty|^p dt\right)^{\frac{1}{p}}, \; p \geq 1,$$

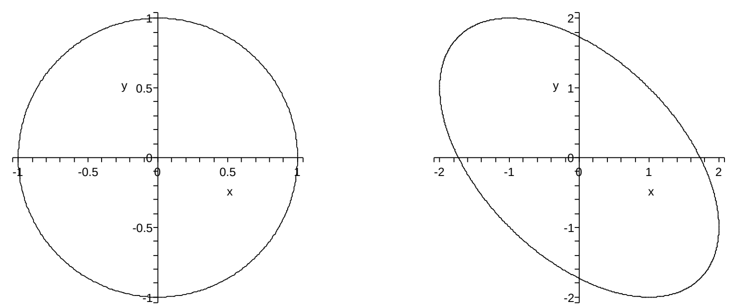
for any $(x,y) \in \mathbb{R}^2$. Note that for $x = y$, $|(x,y)|_{p-HH} = |x|$, so we may assume $x \neq y$ and without loss of generality (since p-HH-norm is symmetric), $x < y$. Therefore,

$$|(x,y)|_{p-HH} = \begin{cases} \left[\frac{1}{p+1}\left(\frac{y^{p+1}-x^{p+1}}{y-x}\right)\right]^{\frac{1}{p}}, & \text{if } x,y \geq 0; \\ \left[\frac{1}{p+1}\left(\frac{(-x)^{p+1}+y^{p+1}}{y-x}\right)\right]^{\frac{1}{p}}, & \text{if } x < 0 \text{ and } y \geq 0; \\ \left[\frac{1}{p+1}\left(\frac{(-x)^{p+1}-(-y)^{p+1}}{y-x}\right)\right]^{\frac{1}{p}}, & \text{if } x,y < 0. \end{cases} \tag{4.7}$$

Particularly for $p = 2$, we have the following for any $(x,y) \in \mathbb{R}^2$:

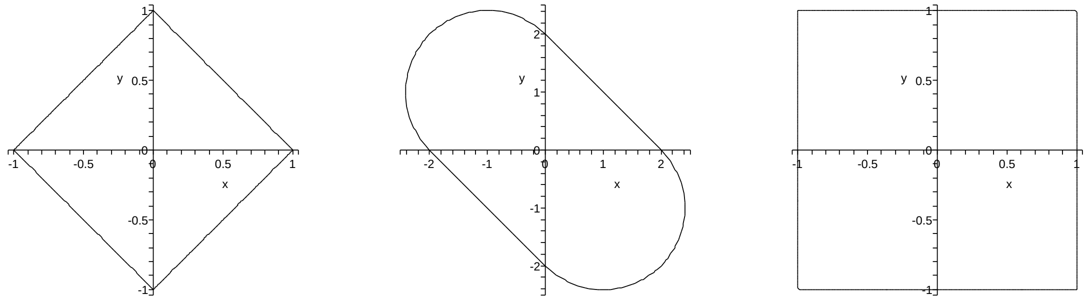
$$|(x,y)|_2 = (x^2+y^2)^{\frac{1}{2}} \quad \text{and} \quad |(x,y)|_{2-HH} = \frac{1}{\sqrt{3}}\left(x^2+xy+y^2\right)^{\frac{1}{2}}.$$

The unit circle $\{(x,y) \in \mathbb{R}^2 | |(x,y)|_2 = 1\}$ is the usual Euclidean circle, while the unit circle $\{(x,y) \in \mathbb{R}^2 | |(x,y)|_{2-HH} = 1\}$ is an ellipse in Euclidean space (see Figure 1). Figure 2 shows the unit circle for 1-norm, 1-HH-norm, and ∞-norm.



(a) Unit circle in $(\mathbb{R}^2, |(\cdot,\cdot)|_2)$ (b) Unit circle in $(\mathbb{R}^2, |(\cdot,\cdot)|_{2-HH})$

FIGURE 1



(a) Unit circle of $(\mathbb{R}^2, |(\cdot,\cdot)|_1)$ (b) Unit circle of $(\mathbb{R}^2, |(\cdot,\cdot)|_{1-HH})$ (c) Unit circle of $(\mathbb{R}^2, |(\cdot,\cdot)|_\infty)$

FIGURE 2

4.3. **Relation with the generalised logarithmic mean.** Particularly, in the field of real numbers, it follows that $|(x,y)|_{p-HH} = \mathfrak{L}^{[p]}(x,y)$ $(1 \le p \le \infty)$ for $x, y > 0$, i.e. the generalised logarithmic mean. As described in Section 3, the reason why $\mathfrak{L}^{[p]}$ is a mean, rises from the mean value theorem for differentiation. The similar reason can also be extracted from the mean value theorem for integration which would agree with the fact that $\|(x,y)\|_{p-HH}^p$ is the integral mean of $\|\cdot\|^p$ on the segment $[x,y]$.

Let us recall the mean value theorem for integration: if $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function, then there exists a point $c \in (a,b)$ such that

$$\int_a^b g(t)dt = g(c)(b-a).$$

If we consider the continuous function $g(t) = t^p$ for any $p \ge 1$ on the interval $[x,y]$, where $x, y > 0$ and $x \ne y$, then there exists a point $s \in (x,y)$ such that

$$\int_x^y t^p dt = s^p(y-x),$$

which implies that

$$s = \left(\frac{1}{y-x}\int_x^y t^p dt\right)^{\frac{1}{p}} = \left(\int_0^1 |(1-t)x + ty|^p dt\right)^{\frac{1}{p}} = \left[\frac{1}{p+1}\left(\frac{y^{p+1}-x^{p+1}}{y-x}\right)\right]^{\frac{1}{p}},$$

that is, $s = |(x,y)|_{p-HH}$.

Therefore, the p-HH-norm extends the generalised logarithmic mean to normed linear space setting. The monotonicity remains to hold in this extension. The following result [4, p. 375–376] will be used to prove the monotonicity of the p-HH-norm as a function of p on $[1,\infty]$.

Proposition 7. *Let $f : I = [a,b] \to \mathbb{R}$, $f \in L^p[a,b]$ $(-\infty \le p \le \infty)$, $f \ge 0$ almost everywhere on I, and $f > 0$ almost everywhere on I if $p < 0$. The p-th power mean of f on $[a,b]$, which is defined by*

$$\mathfrak{M}_{[a,b]}^{[p]}(f) = \left(\frac{1}{b-a}\int_a^b f(x)^p dx\right)^{\frac{1}{p}},$$

is increasing on $\mathbb{R}$, that is, if $-\infty \le r < s \le \infty$, then,

$$\mathfrak{M}_{[a,b]}^{[r]}(f) \le \mathfrak{M}_{[a,b]}^{[s]}(f).$$

Corollary 1. *The p-HH-norm is monotonically increasing as a function of p on $[1,\infty]$, that is, for any $1 \le r < s \le \infty$ and $x, y \in \mathbf{X}$, we have*

$$\|(x,y)\|_{r-HH} \le \|(x,y)\|_{s-HH}.$$

Proof. Consider the non-negative function $f(t) = \|(1-t)x + ty\|$ on $[0,1]$. By the Hermite-Hadamard's inequality (1.3), we conclude that $f \in L^p[0,1]$ for $1 \le p \le \infty$. We obtain the desired result by applying Proposition 7 to f for $1 \le p \le \infty$. □

Remark 5. By Proposition 6 and Corollary 1, we have the following inequalities

$$\|(x,y)\|_{1-HH} \le \|(x,y)\|_{2-HH} \le \cdots \le \|(x,y)\|_{\infty} \le \cdots \le \|(x,y)\|_{2} \le \|(x,y)\|_{1},$$

for any $(x,y) \in \mathbf{X}^2$.

5. Completeness of $(\mathbf{B}^2, \|(\cdot,\cdot)\|_{p-HH})$

The main result of this section can be stated as follows:

Theorem 1. *The space $(\mathbf{B}^2, \|(\cdot,\cdot)\|_{p-HH})$ $(1 \le p < \infty)$ is a Banach space, when $\mathbf{B}$ is.*

Proof. Let $(X_n)_{n=1}^{\infty} = ((x_n, y_n))_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbf{B}^2$ and $\varepsilon > 0$. Then, there exists an $N = N(\varepsilon) \in \mathbb{N}$, such that

$$\|(x_n,y_n) - (x_m,y_m)\|_{p-HH} < \varepsilon, \quad \text{for all } n, m \ge N.$$

Observe that for any n and m, we have

$$\begin{aligned} \|(x_n,y_n) - (x_m,y_m)\|_{p-HH} &= \left(\int_0^1 \|(1-t)(x_n - x_m) + t(y_n - y_m)\|^p dt\right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \|(1-t)x_n + ty_n - [(1-t)x_m + ty_m]\|^p dt\right)^{\frac{1}{p}}. \end{aligned} \tag{5.1}$$

Define $f_n : [0,1] \to \mathbf{B}$, where $f_n(t) = (1-t)x_n + ty_n$, then f_n is measurable and integrable by the Hermite-Hadamard's inequality. Thus, $f_n \in L^p([0,1],\mathbf{B})$, and (5.1) gives us $\|f_n - f_m\|_{L^p} < \varepsilon$ for any $n, m \ge N(\varepsilon)$. Therefore, (f_n) is a Cauchy sequence in $L^p([0,1],\mathbf{B})$. By the completeness of $L^p([0,1],\mathbf{B})$ [16, p. 146], $(f_n)_{n=1}^{\infty}$ converges in norm to a function $f \in L^p([0,1],\mathbf{B})$. It implies that $f_n \to f$ in measure on $[0,1]$ [16, p. 122]. We claim that there are at least two distinct points $t_1, t_2 \in [0,1]$, such that $f_n(t_1) \to f(t_1)$ and $f_n(t_2) \to f(t_2)$ in $\mathbf{B}$. Suppose that the claim is false, so there is at most one point of convergence, say $t_0 \in [0,1]$. So, for any $t \in [0,1] \setminus \{t_0\}$, we can find a $\delta = \delta(t) > 0$, such that for any $K \in \mathbb{N}$, we have

$$n \ge K \text{ and } \|f_n(t) - f(t)\| \ge \delta.$$

By taking $\delta_0 = \sup\{\delta(t) : t \in [0,1] \setminus \{t_0\}\} > 0$, then for any $K \in \mathbb{N}$ and $n \ge K$, we have

$$m(t : t \in [0,1] | \|f_n(t) - f(t)\| \ge \delta_0) = m([0,1] \setminus \{t_0\}) = m([0,1]) = 1,$$

since $m(\{t_0\}) = 0$. It implies that

$$\lim_{n\to\infty} m(t : t \in [0,1] | \|f_n(t) - f(t)\| \ge \delta_0) = 1,$$

that is, f_n is not convergent in measure to f.

If $t_1 = 0$ and $t_2 = 1$, then there exist $x_0, y_0 \in \mathbf{B}$ such that

$$x_n = f_n(0) \to f(0) = x_0 \text{ and } y_n = f_n(1) \to f(1) = y_0,$$

i.e. $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converge to x_0 and y_0, respectively. If $t_1, t_2 \in (0,1)$, then

$$(1-t_1)x_n + t_1 y_n \to f(t_1), \tag{5.2}$$

$$\text{and} \quad (1-t_2)x_n + t_2 y_n \to f(t_2). \tag{5.3}$$

Multiply (5.2) by t_2, (5.3) with $-t_1$ and add up the obtained sequences, we get

$$(t_2 - t_1)x_n \to t_2 f(t_1) - t_1 f(t_2).$$

Since $t_2 - t_1 \neq 0$, we have

$$x_n \to \frac{t_2 f(t_1) - t_1 f(t_2)}{(t_2 - t_1)} =: x_0,$$

which shows that $(x_n)_{n=1}^{\infty}$ converges to $x_0 \in \mathbf{B}$. Now, multiply (5.2) by $(1-t_2)$, (5.3) by $-(1-t_1)$ and add up the obtained sequences, we have

$$(t_1 - t_2)y_n \to (1 - t_2)f(t_1) - (1 - t_1)f(t_2).$$

Again, since $t_1 - t_2 \neq 0$, we have

$$y_n \to \frac{(1 - t_2)f(t_1) - (1 - t_1)f(t_2)}{(t_1 - t_2)} =: y_0,$$

which shows that $(y_n)_{n=1}^{\infty}$ converges to $y_0 \in \mathbf{B}$.

Now, we have the fact that $x_n \to x_0$ and $y_n \to y_0$. Then, for the given $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\|x_n - x_0\| < \frac{\varepsilon}{2}, \quad \text{for all } n \geq N_1, \text{ and } \|y_n - y_0\| < \frac{\varepsilon}{2}, \quad \text{for all } n \geq N_2.$$

Choose $N_0 = \max\{N_1, N_2\}$, then for all $n \geq N_0$, we have

$$\begin{aligned} \|(x_n, y_n) - (x_0, y_0)\|_{p-HH} &= \left(\int_0^1 \|(1-t)(x_n - x_0) + t(y_n - y_0)\|^p dt\right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 \|(1-t)(x_n - x_0)\|^p dt\right)^{\frac{1}{p}} + \left(\int_0^1 \|t(y_n - y_0)\|^p dt\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} (\|(x_n - x_0)\| + \|(y_n - y_0)\|) \\ &\leq \|(x_n - x_0)\| + \|(y_n - y_0)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $(x_n, y_n) \to (x_0, y_0)$ in $(\mathbf{B}^2, \|(\cdot,\cdot)\|_{p-HH})$. □

We are able to prove Theorem 1 via Ostrowski's inequality for absolutely continuous function on segment in normed spaces (see [18]). Before we state the proof, recall the following results:

Proposition 8. *Let $(\mathbf{B}, \|\cdot\|)$ be a Banach space. If there exists a norm $|||\cdot|||$ on $\mathbf{B}$ which equivalent to $\|\cdot\|$, then $(\mathbf{B}, |||\cdot|||)$ is also a Banach space.*

Proof. Consider the identity operator from $(\mathbf{B}, \|\cdot\|)$ onto $(\mathbf{B}, |||\cdot|||)$. It is linear and bijective. Since the two norms are equivalent, the identity operator is bounded. Therefore, it is an isomorphism. It implies that its range, i.e. the space $(\mathbf{B}, |||\cdot|||)$ is also a Banach space (see [23, p. 31]). □

Lemma 8. *For any $x, y \in \mathbf{X}$, we have the following inequality*

$$\frac{\|x\| + \|y\|}{4} \leq \int_0^1 \|(1-t)x + ty\| dt \leq \frac{\|x\| + \|y\|}{2}. \tag{5.4}$$

The constants $\frac{1}{4}$ and $\frac{1}{2}$ are sharp.

Proof. Recall the following refinement of the Hermite-Hadamard's inequality [18, p. 15]

$$0 \leq \frac{\|x\| + \|y\|}{2} - \int_0^1 \|(1-t)x + ty\| dt \leq \frac{1}{4}\|y - x\|. \tag{5.5}$$

By triangle inequality, we have

$$\frac{\|x\| + \|y\|}{2} - \int_0^1 \|(1-t)x + ty\| dt \leq \frac{1}{4}\|y - x\| \leq \frac{\|x\| + \|y\|}{4},$$

or equivalently,

$$\frac{\|x\| + \|y\|}{4} \leq \int_0^1 \|(1-t)x + ty\| dt.$$

The proof is completed by the second part of the Hermite-Hadamard's inequality. Now, we will prove the sharpness of both constants. Suppose that the first inequality holds for a constant $A > 0$ instead of $\frac{1}{4}$, i.e.

$$A(\|x\| + \|y\|) \leq \int_0^1 \|(1-t)x + ty\| dt.$$

Choose $(\mathbf{B}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $x = 1$, and $y = -1$ to obtain $2A \leq \frac{1}{2}$. Thus, $A \leq \frac{1}{4}$.

On the other hand, suppose that the second inequality holds for a constant $B > 0$ instead of $\frac{1}{2}$, i.e.

$$\int_0^1 \|(1-t)x + ty\| dt \leq B(\|x\| + \|y\|).$$

Choose $(\mathbf{B}, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_{l^1})$, $x = (1,0)$, and $y = (0,1)$ to obtain $\int_0^1 (|t| + |1-t|) dt \leq 2B$. Since $0 \leq t \leq 1$, we have $B \geq \frac{1}{2}$. □

Corollary 2. *The p-norm and p-HH-norm are equivalent in $\mathbf{X}^2$, for any $p \geq 1$.*

Proof. It is enough to prove the equivalency for finite p's. The case for $p = 1$ is stated in Lemma 8. By Remark 5 and inequality (5.4), we have the following for any $p > 1$

$$\frac{1}{4}\|(x,y)\|_p \leq \frac{1}{4}\|(x,y)\|_1 \leq \|(x,y)\|_{1-HH} \leq \|(x,y)\|_{p-HH}.$$

The proof is completed by the Hermite-Hadamard's inequality, i.e.

$$\|(x,y)\|_{p-HH} \leq \frac{1}{2^{\frac{1}{p}}}\|(x,y)\|_p.$$

□

Alternative proof for Theorem 1. If $\mathbf{B}$ is a Banach space, then $(\mathbf{B}^2, \|(\cdot,\cdot)\|_p)$ is also complete. Since both p-norm and p-HH-norm are equivalent in $\mathbf{B}^2$, the p-HH-norm is a Banach norm in $\mathbf{B}^2$ by Proposition 8. □

Remark 6. We have for any $p \geq 1$ and any $(x,y) \in \mathbf{X}^2$,

$$\frac{1}{4}\|(x,y)\|_p \leq \|(x,y)\|_{p-HH} \leq \frac{1}{2^{\frac{1}{p}}}\|(x,y)\|_p. \tag{5.6}$$

The constant $\frac{1}{2^{\frac{1}{p}}}$ is sharp. For simplicity, we write,

$$\|(x,y)\|_{p-HH}^p \leq \frac{1}{2}\|(x,y)\|_p^p.$$

Suppose that the above inequality holds for a constant $C > 0$ instead of $\frac{1}{2}$, that is,

$$\int_0^1 \|(1-t)x + ty\|^p dt \leq C(\|x\|^p + \|y\|^p).$$

Choose $(\mathbf{B}, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_{l^1})$, $x = (1,0)$, and $y = (0,1)$ to obtain $\int_0^1 (|t| + |1-t|)^p dt \le 2C$. Since $0 \le t \le 1$, we have $C \ge \frac{1}{2}$.

On the other hand, the constant $\frac{1}{4}$ in (5.6) is not always sharp for any $p > 1$. The following proposition provides an example for the statement.

Proposition 9. *Let $(\mathbf{H}, \langle\cdot,\cdot\rangle)$ be an inner product space, then*

$$\frac{1}{6}\|(x,y)\|_2^2 \le \|(x,y)\|_{2-HH}^2 \le \frac{1}{2}\|(x,y)\|_2^2. \tag{5.7}$$

The constants $\frac{1}{6}$ and $\frac{1}{2}$ are sharp.

Proof. For any $(x,y) \in \mathbf{H}^2$, we have

$$\|(x,y)\|_{2-HH}^2 = \int_0^1 \|(1-t)x + ty\|^2 dt \le \frac{\|x\|^2 + \|y\|^2}{2} = \frac{1}{2}\|(x,y)\|_2^2,$$

by the second part of the Hermite-Hadamard's inequality. Observe that

$$0 \le \|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2.$$

By adding $\|x\|^2 + \|y\|^2$ to this inequality, we obtain $\|x\|^2 + \|y\|^2 \le 2(\|x\|^2 + \langle x,y\rangle + \|y\|^2)$, which is equivalent to

$$\frac{1}{6}(\|x\|^2 + \|y\|^2) \le \frac{1}{3}(\|x\|^2 + \langle x,y\rangle + \|y\|^2) = \|(x,y)\|_{2-HH}^2,$$

by (4.5). Thus, we have

$$\frac{1}{6}\|(x,y)\|_2^2 \le \|(x,y)\|_{2-HH}^2 \le \frac{1}{2}\|(x,y)\|_2^2.$$

Now, we will prove the sharpness of the constants. Suppose that the first inequality holds for a constant $D > 0$, that is,

$$D\|(x,y)\|_2^2 \le \|(x,y)\|_{2-HH}^2.$$

By choosing $(\mathbf{H}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $x = 1$, $y = -1$, we obtain $D \le \frac{1}{6}$. Therefore the constant $\frac{1}{6}$ is sharp. Now suppose that the second inequality holds for a constant $E > 0$, that is,

$$\|(x,y)\|_{2-HH}^2 \le E\|(x,y)\|_2^2.$$

By choosing $(\mathbf{H}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $x = 1$, $y = 1$, we obtain $E \ge \frac{1}{2}$. Therefore the constant $\frac{1}{2}$ is sharp. □

The following lemma provides a better constant for the first part of inequality (5.6), in the case of real numbers.

Lemma 9. *Let $x, y \in \mathbb{R}$. For any $p \ge 1$, we have*

$$\frac{1}{2}\left(\frac{|x|^p + |y|^p}{p+1}\right) \le \int_0^1 |(1-t)x + ty|^p dt.$$

The constant $\frac{1}{2}$ is sharp.

Proof. The proof is trivial, when $x = y$, so, we assume that $x \ne y$. Without loss of generality, we assume that $x < y$.

Case 1: $x, y > 0$. We have

$$\int_0^1 |(1-t)x + ty|^p dt = \frac{1}{p+1}\left(\frac{y^{p+1} - x^{p+1}}{y - x}\right).$$

Since $0 < x < y$, we have $x^p < y^p$ and

$$x^{p+1} + x^p y = x^p(y + x) < y^p(y + x) = y^{p+1} + xy^p,$$

or equivalently,

$$x^p y - xy^p < y^{p+1} - x^{p+1}.$$

Therefore, we have

$$\frac{(x^p + y^p)}{2} = \frac{(x^p + y^p)(y - x)}{2(y - x)} = \frac{y^{p+1} - x^{p+1} + x^p y - xy^p}{2(y - x)} \leq \frac{y^{p+1} - x^{p+1}}{(y - x)},$$

which shows that

$$\frac{(x^p + y^p)}{2(p + 1)} \leq \frac{y^{p+1} - x^{p+1}}{(p + 1)(y - x)}.$$

Case 2: $x, y < 0$. Note that

$$\int_0^1 |(1 - t)x + ty|^p dt = \frac{1}{p + 1}\left(\frac{(-x)^{p+1} - (-y)^{p+1}}{y - x}\right) = \frac{1}{p + 1}\left(\frac{(-x)^{p+1} - (-y)^{p+1}}{-x - (-y)}\right).$$

We choose $u = -x > 0$ and $v = -y > 0$; thus, the conclusion follows by Case 1.
Case 3: $x < 0$ and $y > 0$. Note that

$$\int_0^1 |(1 - t)x + ty|^p dt = \frac{1}{p + 1}\left(\frac{(-x)^{p+1} + y^{p+1}}{y - x}\right).$$

We provide the proof for the subcase where $0 < -x < y$ (as for the subcase where $0 < y < -x$ can be proven in a similar way). We have $(-x)^p < y^p$ and

$$(-x)^p y - (-x)^{p+1} = (-x)^p(y - (-x)) < y^p(y - (-x)) = y^{p+1} - (-x)y^p,$$

or equivalently,

$$(-x)^p y + (-x)y^p < (-x)^{p+1} + y^{p+1}.$$

Therefore,

$$\frac{((-x)^p + y^p)}{2} = \frac{((-x)^p + y^p)(y - x)}{2(y - x)} = \frac{(-x)^{p+1} + y^{p+1} + (-x)^p y + (-x)y^p}{2(y - x)} \leq \frac{(-x)^{p+1} + y^{p+1}}{(y - x)},$$

which shows that

$$\frac{((-x)^p + y^p)}{2(p + 1)} \leq \frac{1}{p + 1}\left(\frac{(-x)^{p+1} + y^{p+1}}{y - x}\right).$$

We will prove the sharpness of the constant. First, let us assume that the inequality holds for a constant $F > 0$ instead of $\frac{1}{2}$, i.e.

$$F\left(\frac{|x|^p + |y|^p}{p + 1}\right) \leq \int_0^1 |(1 - t)x + ty|^p dt.$$

Now, choose $x = 1$ and $y = -1$, therefore, we have

$$\frac{2F}{p + 1} \leq \int_0^1 |2t - 1|^p dt = \frac{1}{p + 1},$$

which implies that $F \leq \frac{1}{2}$. □

Conjecture 1. *For any $x, y \in \mathbf{X}$ and $p > 1$, is*

$$\frac{\|x\|^p + \|y\|^p}{2(p + 1)} \leq \int_0^1 \|(1 - t)x + ty\|^p dt \ ?$$

If it is, is $\frac{1}{2(p+1)}$ the best constant for each $p > 1$?

6. The Cartesian product of two inner product spaces

Proposition 10. *Let* $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ *be an inner product space, then* $\|(\cdot,\cdot)\|_2$ *is induced by an inner-product in* $\mathbf{X}^2$*, namely*

$$\langle (x,y),(u,v)\rangle = \langle x,u\rangle + \langle y,v\rangle,$$

for any (x,y) *and* (u,v) *in* $\mathbf{X}^2$*. Furthermore,* $(\mathbf{H}^2, \langle(\cdot,\cdot),(\cdot,\cdot)\rangle)$ *is a Hilbert space, when* $\mathbf{H}$ *is.*

The proof can be established by showing that the parallelogram law holds in $(\mathbf{X}^2, \|(\cdot,\cdot)\|_2)$, which is implied by that of $(\mathbf{X}, \|\cdot\|)$ and its inner product.

Remark 7. (1) For general $1 \le p < \infty$, the norm $\|(\cdot,\cdot)\|_p$ in $\mathbf{H}^2$ does not induce an inner-product. For example, in any inner product space $\mathbf{H}$, with the norm $\|\cdot\|$, take $x, v \in \mathbf{H}$, where $x, v \neq 0$ and $y = u = 0$. Then, for any $p \neq 2$, we have

$$\begin{aligned} \|(x,y)+(u,v)\|_p^2 + \|(x,y)-(u,v)\|_p^2 &= 2(\|x\|^p + \|v\|^p)^{\frac{2}{p}} \\ &\neq 2(\|x\|^2 + \|v\|^2) = 2\left(\|(x,y)\|_p^2 + \|(u,v)\|_p^2\right). \end{aligned}$$

(2) In general Banach space, the 2-norm is not a Hilbertian norm. To verify this, let $(\mathbf{B}, \|\cdot\|)$ be a Banach space, $x, u \in \mathbf{B}$, $x, u \neq 0$, $y = x$, and $v = u$. Then,

$$\begin{aligned} \|(x,y)+(u,v)\|_2^2 + \|(x,y)-(u,v)\|_2^2 &= 2(\|x+u\|^2 + \|x-u\|^2) \\ &\neq 4\left(\|x\|^2 + \|u\|^2\right) = 2\left(\|(x,y)\|_2^2 + \|(u,v)\|_2^2\right), \end{aligned}$$

unless $\mathbf{B}$ is a Hilbert space.

Proposition 11. *Let* $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ *be an inner-product space, then* $\|(\cdot,\cdot)\|_{2-HH}$ *is a Hilbertian norm in* $\mathbf{X}^2$*, namely*

$$\langle (x,y),(u,v)\rangle_{HH} = \frac{1}{6}(2\langle x,u\rangle + 2\langle y,v\rangle + \langle x,v\rangle + \langle u,y\rangle).$$

Furthermore, $(\mathbf{H}^2, \langle(\cdot,\cdot),(\cdot,\cdot)\rangle_{HH})$ *is a Hilbert space, when* $\mathbf{H}$ *is.*

Again, we can prove it by showing that the parallelogram law holds in $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{2-HH})$ and apply the polarisation identity to get the explicit expression of its inner product.

Remark 8. (1) For general $1 \le p < \infty$, the norm $\|(\cdot,\cdot)\|_{p-HH}$ in $\mathbf{H}^2$ does not induce an inner-product. To verify this, let $\mathbf{H}$ be any inner product space with the norm $\|\cdot\|$, $0 \neq x \in \mathbf{H}$, $y = u = 0$, and $v = x$. Then, for $p \neq 2$,

$$\begin{aligned} \|(x,y)+(u,v)\|_{p-HH}^2 + \|(x,y)-(u,v)\|_{p-HH}^2 &= \|x\|^2 \left[1 + \left(\int_0^1 |1-2t|^p \, dt\right)^{\frac{2}{p}}\right] \\ &= \|x\|^2 \left[1 + \left(\frac{1}{p+1}\right)^{\frac{2}{p}}\right] \\ &\neq 4\|x\|^2 \left(\frac{1}{p+1}\right)^{\frac{2}{p}} \\ &= 2\left(\|(x,y)\|_{p-HH}^2 + \|(u,v)\|_{p-HH}^2\right). \end{aligned}$$

(2) In general Banach space, the 2-HH-norm is not a Hilbertian norm. To verify this, let $(\mathbf{B}, \|\cdot\|)$ be a Banach space, $x, u \in \mathbf{B}$ where $x, u \neq 0$, $y = x$, and $v = u$. Then,

$$\begin{aligned} \|(x,y)+(u,v)\|_{2-HH}^2 + \|(x,y)-(u,v)\|_{2-HH}^2 &= \|x+u\|^2 + \|x-u\|^2 \\ &\neq 2\left(\|x\|^2 + \|u\|^2\right) \\ &= 2\left(\|(x,y)\|_{2-HH}^2 + \|(u,v)\|_{2-HH}^2\right), \end{aligned}$$

unless $\mathbf{B}$ is a Hilbert space.

7. Embedding of $\mathbf{B}^2$ in $L^p([0,1],\mathbf{B})$

In this section, we show that the spaces $(\mathbf{B}^2, \|(\cdot,\cdot)\|_p)$ and $(\mathbf{B}^2, \|(\cdot,\cdot)\|_{p-HH})$ can be embedded as closed subspaces of $L^p([0,1],\mathbf{B})$. Thus, it allows us to identify $\mathbf{B}^2$ as a closed subspace of $L^p([0,1],\mathbf{B})$.

7.1. **Embedding of $(\mathbf{B^2}, \|(\cdot,\cdot)\|_\mathbf{p})$ in $\mathbf{L^p([0,1],B)}$.** Consider the mapping Φ on $(\mathbf{X}^2, \|(\cdot,\cdot)\|_p)$ which takes values in $L^p([0,1],\mathbf{X})$, where $\Phi(x,y) = f_{x,y}$, with

$$f_{x,y}(t) = \begin{cases} 2^{\frac{1}{p}}x, & t \in [0,\frac{1}{2}); \\ 2^{\frac{1}{p}}y, & t \in (\frac{1}{2},1]. \end{cases}$$

Theorem 2. *By the above notations, the mapping Φ is an embedding, i.e. a homeomorphism from $(\mathbf{X}^2, \|(\cdot,\cdot)\|_p)$ onto $\Phi(\mathbf{X}^2) \subset L^p([0,1],\mathbf{X})$. Furthermore, if $\mathbf{B}$ is a Banach space, then $\Phi(\mathbf{B}^2)$ is a closed subspace of $L^p([0,1],\mathbf{B})$.*

Proof. By definition, Φ is a linear transformation and also an injective mapping. Let $(x,y),(u,v) \in \mathbf{X}^2$, then

$$\begin{aligned} \|\Phi(x,y)\|_{L^p} = \|f_{x,y}\|_{L^p} &= \left(\int_0^1 \|f_{x,y}(t)\|^p dt\right)^{\frac{1}{p}} \\ &= \left(\int_0^{\frac{1}{2}} \left\|2^{\frac{1}{p}}x\right\|^p dt + \int_{\frac{1}{2}}^1 \left\|2^{\frac{1}{p}}y\right\|^p dt\right)^{\frac{1}{p}} \\ &= (\|x\|^p + \|y\|^p)^{\frac{1}{p}} = \|(x,y)\|_p, \end{aligned}$$

which implies that Φ preserves norm. Thus, it is an isometry isomorphism onto its image $\Phi(\mathbf{X}^2)$. Therefore, it is a homeomorphism onto its image, i.e. an embedding.

Now, let $\mathbf{B}$ be a Banach space and f be a limit point of $\Phi(\mathbf{B}^2)$. We want to show that $f \in \Phi(\mathbf{B}^2)$. Let $\varepsilon_n = \frac{1}{n}$, for any $n \in \mathbb{N}$, then we can find $f_n \in \Phi(\mathbf{B}^2)$, where

$$\|f - f_n\|_{L^p} < \frac{1}{n},\ n \in \mathbb{N},$$

since f is a limit point of $\Phi(\mathbf{B}^2)$. We claim that $\{f_n\}$ is a Cauchy sequence in $L^p([0,1],\mathbf{B})$. Therefore, $\lim\limits_{n\to\infty} f_n = f$, in $L^p([0,1],\mathbf{B})$.

Proof of claim. Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\frac{2}{\varepsilon} < N$. Then, without loss of generality, for $n > m \geq N$, we have

$$\|f_n - f_m\|_{L^p} \leq \|f_n - f\|_{L^p} + \|f - f_m\|_{L^p} < \frac{1}{n} + \frac{1}{m} < \frac{2}{m} \leq \frac{2}{N} < \varepsilon,$$

which shows that $\{f_n\}$ is a Cauchy sequence in $L^p([0,1],\mathbf{B})$. □

For any $n \in \mathbb{N}$, we can find $(u_n,v_n) \in \mathbf{B}^2$, associated to f_n (since $f_n \in \Phi(\mathbf{B}^2)$), such that $f_n = \Phi(u_n,v_n)$. Since Φ is an isometry isomorphism, $\{(u_n,v_n)\}$ is also a Cauchy sequence in $\mathbf{B}^2$. Since $(\mathbf{B}^2, \|(\cdot,\cdot)\|_p)$ is also a Banach space (by Lemma 6), therefore, $\{(u_n,v_n)\}$ has a limit in $(\mathbf{B}^2, \|(\cdot,\cdot)\|_p)$, namely (u,v). By the continuity of Φ (note that it is a homeomorphism), we conclude that

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \Phi(u_n,v_n) = \Phi(u,v),$$

and by the uniqueness of limit, $f = \Phi(u,v)$, that is, $f \in \Phi(\mathbf{B}^2)$. Therefore, $\Phi(\mathbf{B}^2)$ is a closed subspace of $L^p([0,1],\mathbf{B})$. □

7.2. **Embedding of $(\mathbf{B^2}, \|(\cdot,\cdot)\|_{\mathbf{p-HH}})$ in $\mathbf{L^p([0,1],B)}$.** Now, consider a mapping Ψ on $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ to $L^p([0,1],\mathbf{X})$, defined by $\Psi(x,y) = g_{x,y}$, where $g_{x,y}(t) := (1-t)x + ty$, $t \in [0,1]$. It is easy to verify that $g_{x,y}$ is measurable. The integrability follows from the Hermite-Hadamard integral inequality.

Theorem 3. *By the above notations, the mapping Ψ is an embedding, i.e. a homeomorphism from $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ onto $\Psi(\mathbf{X}^2) \subset L^p([0,1],\mathbf{X})$. Furthermore, if $\mathbf{B}$ is a Banach space, then $\Psi(\mathbf{B}^2)$ is a closed subspace of $L^p([0,1],\mathbf{B})$.*

Proof. By definition, Ψ is a linear transformation and also an injective mapping. Let $(x,y),(u,v) \in \mathbf{X}^2$, then

$$\|\Psi(x,y)\|_{L^p} = \|g_{x,y}\|_{L^p} = \left(\int_0^1 \|g_{x,y}(t)\|^p dt\right)^{\frac{1}{p}} = \left(\int_0^1 \|(1-t)x+ty\|^p dt\right)^{\frac{1}{p}} = \|(x,y)\|_{p-HH},$$

which implies that Ψ preserves norm. Thus, it is an isometry isomorphism onto its image $\Psi(\mathbf{X}^2)$. Therefore, it is a homeomorphism onto its image, i.e. an embedding. The last part of this theorem can be proven in a similar way to that of Theorem 2. □

8. Semi-inner products

In this section, we present the superior (inferior) semi-inner product in $\mathbf{X}^2$ associated to both p-norm and p-HH-norm in an explicit form.

8.1. **Semi-inner products in $\mathbf{X^2}$ with respect to the p-norm.** The following results give an explicit expression for the superior (inferior) s.i.p. in $\mathbf{X}^2$ with respect to the p-norm.

Lemma 10. *The superior (inferior) s.i.p. in $\mathbf{X}^2$ with respect to the norm $\|(\cdot,\cdot)\|_p$ $(1<p<\infty)$, for any $(x,y),(u,v) \in \mathbf{X}^2$, are given by*

$$\langle (x,y),(u,v)\rangle_{p,s(i)} = \|(u,v)\|_p^{2-p}\left(\|u\|^{p-2}\langle x,u\rangle_{s(i)} + \|v\|^{p-2}\langle y,v\rangle_{s(i)}\right), \tag{8.1}$$

where $\langle\cdot,\cdot\rangle_{s(i)}$ are the superior (inferior) s.i.p. with respect to the norm $\|\cdot\|$ on $\mathbf{X}$.

Proof. We consider the following cases:
Case 1: If $(u,v) = (0,0)$, then

$$\langle (x,y),(u,v)\rangle_{p,s(i)} = \lim_{t\to 0^\pm} \frac{\|(u,v)+t(x,y)\|_p^2 - \|(u,v)\|_p^2}{2t} = \lim_{t\to 0^\pm} \frac{t^2\|(x,y)\|_p^2}{2t} = 0,$$

for any $(x,y) \in \mathbf{X}^2$, so (8.1) holds.
Case 2: Assume that $(u,v) \neq (0,0)$. We define the function $f : \mathbf{X}^2 \to \mathbb{R}$, where $f(x,y) = \|(x,y)\|_p^p$ $(1<p<\infty)$ for any $(x,y) \in \mathbf{X}^2$. We have

$$\begin{aligned}
(\nabla_\pm f(u,v))(x,y) &:= \lim_{t\to 0^\pm} \frac{\|(u,v)+t(x,y)\|_p^p - \|(u,v)\|_p^p}{t} \\
&= p\|(u,v)\|_p^{p-1} \lim_{t\to 0^\pm} \frac{\|(u,v)+t(x,y)\|_p - \|(u,v)\|_p}{t} \\
&= p\|(u,v)\|_p^{p-1}(\nabla_\pm \|(\cdot,\cdot)\|_p(u,v))(x,y) \\
&= p\|(u,v)\|_p^{p-2}\langle (x,y),(u,v)\rangle_{p,s(i)}. \qquad (8.2)
\end{aligned}$$

If $u, v \neq 0$, we have the following

$$\begin{aligned}(\nabla_{\pm} f(u,v))(x,y) &= \lim_{t\to 0^{\pm}} \frac{\|(u,v)+t(x,y)\|_p^p - \|(u,v)\|_p^p}{t} \\ &= \lim_{t\to 0^{\pm}} \frac{\|u+tx\|^p - \|u\|^p}{t} + \lim_{t\to 0^{\pm}} \frac{\|v+ty\|^p - \|v\|^p}{t} \\ &= p\left[\|u\|^{p-1}(\nabla_{\pm}\|\cdot\|(u))(x) + \|v\|^{p-1}(\nabla_{\pm}\|\cdot\|(v))(y)\right] \\ &= p\left(\|u\|^{p-2}\langle x,u\rangle_{s(i)} + \|v\|^{p-2}\langle y,v\rangle_{s(i)}\right).\end{aligned}$$

Thus, $\|(u,v)\|_p^{p-2}\langle (x,y),(u,v)\rangle_{p,s(i)} = \|u\|^{p-2}\langle x,u\rangle_{s(i)} + \|v\|^{p-2}\langle y,v\rangle_{s(i)}$, and

$$\langle (x,y),(u,v)\rangle_{p,s(i)} = \|(u,v)\|_p^{2-p}\left(\|u\|^{p-2}\langle x,u\rangle_{s(i)} + \|v\|^{p-2}\langle y,v\rangle_{s(i)}\right).$$

If $u = 0$ and $v \neq 0$, then (8.2) gives us

$$(\nabla_{\pm} f(u,v))(x,y) = p\|v\|^{p-2}\langle (x,y),(u,v)\rangle_{p,s(i)},$$

and therefore

$$\begin{aligned}(\nabla_{\pm} f(u,v))(x,y) &= \lim_{t\to 0^{\pm}} \frac{\|(u,v)+t(x,y)\|_p^p - \|(u,v)\|_p^p}{t} \\ &= \lim_{t\to 0^{\pm}} \frac{\|v+ty\|^p - \|v\|^p}{t} \\ &= p\|v\|^{p-1}(\nabla_{\pm}\|\cdot\|(v))(y) = p\|v\|^{p-2}\langle y,v\rangle_{s(i)}.\end{aligned}$$

Thus, we have the following

$$\langle (x,y),(u,v)\rangle_{p,s(i)} = \langle y,v\rangle_{s(i)} = \|(u,v)\|_p^{2-p}\left(\|u\|^{p-2}\langle x,u\rangle_{s(i)} + \|v\|^{p-2}\langle y,v\rangle_{s(i)}\right),$$

since $\langle x,u\rangle_{s(i)} = 0$ and $\|(u,v)\|_p = \|v\|$. Analogously, for $u \neq 0$ and $v = 0$, we have

$$\langle (x,y),(u,v)\rangle_{p,s(i)} = \langle x,u\rangle_{s(i)} = \|(u,v)\|_p^{2-p}\left(\|u\|^{p-2}\langle x,u\rangle_{s(i)} + \|v\|^{p-2}\langle y,v\rangle_{s(i)}\right). \qquad \square$$

Remark 9. Note that in l^p $(1 < p < \infty)$ spaces (see [24, p. 183] for references), the superior (inferior) s.i.p. of two vectors $x = (x_i)$ and $y = (y_i)$ are given by

$$\langle x,y\rangle_i = \langle x,y\rangle_s = \|y\|_{l^p}^{2-p}\sum_{i=1}^{\infty}|y_i|^{p-2}y_i x_i. \tag{8.3}$$

If $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, then $(\mathbf{X}^2, \|(\cdot,\cdot)\|_p) = l_2^p$, and the superior (inferior) s.i.p. (by Lemma 10) are given by

$$\langle (x,y),(u,v)\rangle_{p,s(i)} = |(u,v)|_{l^p}^{2-p}\left(|u|^{p-2}xu + |v|^{p-2}yv\right),$$

which recapture the definition of superior (inferior) s.i.p. given in (8.3), for l_2^p spaces.

Lemma 11. *The superior (inferior) s.i.p. in $\mathbf{X}^2$ with respect to the norm $\|(\cdot,\cdot)\|_1$ are given by*

$$\langle (x,y),(u,v)\rangle_{1,s(i)} = \begin{cases} \|(u,v)\|_1\left[(\nabla_{\pm}\|\cdot\|(u))(x) + (\nabla_{\pm}\|\cdot\|(v))(y)\right], & \text{if } u,v \neq 0; \\ \langle x,u\rangle_{s(i)} \pm \|u\|\|y\|, & \text{if } u \neq 0,\ v = 0; \\ \langle y,v\rangle_{s(i)} \pm \|v\|\|x\|, & \text{if } u = 0,\ v \neq 0; \\ 0, & \text{if } (u,v) = (0,0), \end{cases}$$

for any $(x,y),(u,v) \in \mathbf{X}^2$ (here, $\langle\cdot,\cdot\rangle_{s(i)}$ are the superior(inferior) s.i.p. with respect to the norm $\|\cdot\|$ on $\mathbf{X}$).

Proof. The proof for $(u,v)=(0,0)$ is trivial, so we consider the case where $(u,v)\neq(0,0)$. If $u,v\neq 0$, then

$$\begin{aligned}(\nabla_{\pm}\|(\cdot,\cdot)\|_1(u,v))(x,y) &= \lim_{t\to 0^{\pm}}\frac{\|(u,v)+t(x,y)\|_1-\|(u,v)\|_1}{t}\\ &= \lim_{t\to 0^{\pm}}\frac{\|u+tx\|-\|u\|}{t}+\lim_{t\to 0^{\pm}}\frac{\|v+ty\|-\|v\|}{t}\\ &= (\nabla_{\pm}\|\cdot\|(u))(x)+(\nabla_{\pm}\|\cdot\|(v))(y),\end{aligned}$$

which implies that $\langle (x,y),(u,v)\rangle_{1,s(i)} = \|(u,v)\|_1\left[(\nabla_{\pm}\|\cdot\|(u))(x)+(\nabla_{\pm}\|\cdot\|(v))(y)\right]$. Now, if $u\neq 0$ and $v=0$, we have

$$\begin{aligned}(\nabla_{\pm}\|(\cdot,\cdot)\|_1(u,0))(x,y) &= \lim_{t\to 0^{\pm}}\frac{\|(u,0)+t(x,y)\|_1-\|u\|}{t}\\ &= \lim_{t\to 0^{\pm}}\frac{\|u+tx\|-\|u\|}{t}+\lim_{t\to 0^{\pm}}\frac{|t|\|y\|}{t}\\ &= (\nabla_{\pm}\|\cdot\|(u))(x)\pm\|y\|,\end{aligned}$$

which implies that

$$\langle (x,y),(u,0)\rangle_{1,s(i)} = \|u\|\left[(\nabla_{\pm}\|\cdot\|(u))(x)\pm\|y\|\right] = \langle x,u\rangle_{s(i)}\pm\|u\|\|y\|,$$

and analogously for $u=0$ and $v\neq 0$, we have

$$\langle (x,y),(0,v)\rangle_{1,s(i)} = \|v\|\left[(\nabla_{\pm}\|\cdot\|(v))(y)\pm\|x\|\right] = \langle y,v\rangle_{s(i)}\pm\|v\|\|x\|.$$

□

Remark 10. Note that in l^1 space (see [24, p. 183] for references), the superior (inferior) s.i.p. of two vectors $x=(x_i)$ and $y=(y_i)$ are given by

$$\langle x,y\rangle_{s(i)} = \|y\|_{l^1}\left(\sum_{y_i\neq 0}\frac{y_i}{|y_i|}x_i \pm \sum_{y_i=0}|x_i|\right) = \|y\|_{l^1}\left(\sum_{y_i\neq 0}\operatorname{sgn}(y_i)x_i \pm \sum_{y_i=0}|x_i|\right). \tag{8.4}$$

If we take $(\mathbf{X},\|\cdot\|)=(\mathbb{R},|\cdot|)$, then $(\mathbf{X}^2,\|(\cdot,\cdot)\|_1)=l_2^1$, and the superior (inferior) s.i.p. (by Lemma 11) are given by

$$\langle (x,y),(u,v)\rangle_{1,s(i)} = \begin{cases} 0, & \text{if } (u,v)=(0,0);\\ \|(u,v)\|_{l^1}(x\ \operatorname{sgn}(u)+y\ \operatorname{sgn}(v)), & \text{if } u,v\neq 0;\\ |u|(x\ \operatorname{sgn}(u)\pm|y|), & \text{if } u\neq 0,\ v=0;\\ |v|(y\ \operatorname{sgn}(v)\pm|x|), & \text{if } u=0,\ v\neq 0,\end{cases}$$

which recapture the definition of superior (inferior) s.i.p. given in (8.4) for l_2^1 spaces.

Lemma 12. *The superior (inferior) s.i.p. in $\mathbf{X}^2$ with respect to the norm $\|(\cdot,\cdot)\|_\infty$, for any vector $(x,y),(u,v)\in\mathbf{X}^2$ with $\|u\|\neq\|v\|$, are given by*

$$\langle (x,y),(u,v)\rangle_{\infty,s(i)} = \begin{cases}\langle x,u\rangle_{s(i)}, & \text{if } \|u\|>\|v\|;\\ \langle y,v\rangle_{s(i)}, & \text{if } \|u\|<\|v\|,\end{cases}$$

where $\langle\cdot,\cdot\rangle_{s(i)}$ are the superior (inferior) s.i.p. with respect to the norm $\|\cdot\|$ on $\mathbf{X}$.

Proof. Without loss of generality, assume that $\|u\|>\|v\|$. Define $h(t)=\|u+tx\|-\|v+ty\|$ for $t\in\mathbb{R}$, then by our assumption, we have $h(0)=\|u\|-\|v\|>0$. We claim that there exists $\varepsilon>0$ such that $h(t)>0$ for all $t\in(-\varepsilon,\varepsilon)$. Suppose that the claim is false, then given $\varepsilon=\frac{1}{n}$

($n \in \mathbb{N}$), we can find $t_n \in (-\varepsilon, \varepsilon)$ such that $h(t_n) \le 0$. So, we have a sequence (t_n) with $t_n \to 0$ as $n \to \infty$. Then, by the continuity of h, we have

$$h(0) = \lim_{n\to\infty} h(t_n) \le 0,$$

which contradicts our assumption.

Thus, there exists an $\varepsilon > 0$ such that $\|u + tx\| > \|v + ty\|$ for all $t \in (-\varepsilon, \varepsilon)$, or equivalently, $\|(u, v) + t(x, y)\|_\infty = \|u + tx\|$ for all $t \in (-\varepsilon, \varepsilon)$. Therefore, for all $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$, we have

$$\frac{\|(u, v) + t(x, y)\|_\infty - \|(u, v)\|_\infty}{t} = \frac{\|u + tx\| - \|u\|}{t}.$$

By taking $t \to 0^\pm$ to obtain $(\nabla_\pm\|(\cdot,\cdot)\|_\infty(u, v))(x, y) = (\nabla_\pm\|\cdot\|(u))(x)$. It implies that

$$\langle (x, y), (u, v)\rangle_{\infty,s(i)} = \|(u, v)\|_\infty(\nabla_\pm\|\cdot\|(u))(x) = \|u\|(\nabla_\pm\|\cdot\|(u))(x) = \langle x, u\rangle_{s(i)}. \qquad \square$$

Remark 11. For the case where $\|u\| = \|v\|$, we have the following for any $(x, y) \in \mathbf{X}^2$:

(1) If $u, v = 0$, then $\langle (x, y), (u, v)\rangle_{\infty,s(i)} = 0$;

(2) If $\|u + tx\| \ge \|v + ty\|$ for $t \to 0^+$, then $\langle (x, y), (u, v)\rangle_{\infty,s} = \lim_{t\to 0^+} \frac{\|u + tx\|^2 - \|u\|^2}{2t} = \langle u, x\rangle_s$; similarly, if $\|u + tx\| \ge \|v + ty\|$ for $t \to 0^-$, then $\langle (x, y), (u, v)\rangle_{\infty,i} = \langle u, x\rangle_i$;

(3) If $\|u + tx\| \le \|v + ty\|$ for $t \to 0^+$, then $\langle (x, y), (u, v)\rangle_{\infty,s} = \lim_{t\to 0^+} \frac{\|v + ty\|^2 - \|v\|^2}{2t} = \langle v, y\rangle_s$; similarly, if $\|u + tx\| \le \|v + ty\|$ for $t \to 0^-$, then $\langle (x, y), (u, v)\rangle_{\infty,i} = \langle v, y\rangle_i$.

8.2. **Semi-inner products in $\mathbf{X}^2$ with respect to the p-HH-norm.** Let f be a continuous real-valued function defined on $D := \{(x, t) : x \in [0, 1], t \in \mathbb{R} \setminus \{0\}\}$. Then, the mapping $x \mapsto f(x, t)$ is continuous for any fixed $t \in \mathbb{R} \setminus \{0\}$ and therefore is Lebesgue integrable on $[0, 1]$.

Proposition 12. *Let f be defined as above and $\lim_{t\to 0^\pm} f(x, t) = g_\pm(x)$, where $g_\pm$ is a Lebesgue integrable function defined on $[0, 1]$. Then*

$$\lim_{t\to 0^\pm}\left(\int_0^1 f(x, t)dx\right) = \int_0^1 g_\pm(x)dx = \int_0^1 \left(\lim_{t\to 0^\pm} f(x, t)\right) dx.$$

Proof. We will prove the statement for the right-sided limit (the left-sided limit can be proven in a similar way). Given $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $|f(x, t) - g_+(x)| < \varepsilon$ whenever $0 < t < \delta_0$. We also have

$$\begin{aligned}\left|\int_0^1 f(x, t)dx - \int_0^1 g_+(x)dx\right| &= \left|\int_0^1 [f(x, t) - g_+(x)]\, dx\right| \\ &\le \int_0^1 |f(x, t) - g_+(x)|\, dx < \int_0^1 \varepsilon\ dx = \varepsilon,\end{aligned}$$

which shows that

$$\lim_{t\to 0^+}\left(\int_0^1 f(x, t)dx\right) = \int_0^1 g_+(x)dx = \int_0^1 \left(\lim_{t\to 0^+} f(x, t)\right) dx. \qquad \square$$

Proposition 13. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $D := \{(\sigma, t) : \sigma \in [0, 1], t \in \mathbb{R} \setminus \{0\}\}$, and consider the real-valued function F_p on D defined by*

$$F_p(\sigma, t) := \frac{\|(1 - \sigma)(u + tx) + \sigma(v + ty)\|^p - \|(1 - \sigma)u + \sigma v\|^p}{t},$$

for any $x, y, u, v \in \mathbf{X}$, where $u \neq 0$ or $v \neq 0$, and $1 \leq p < \infty$. Then, F_p is continuous on D. Furthermore, if $\lim_{t\to 0^{\pm}} F_p(\sigma, t) = G_{p,\pm}(\sigma)$, then $G_{p,\pm}$ is Lebesgue integrable, and

$$\int_0^1 G_{p,\pm}(\sigma)d\sigma = p\int_0^1 \|(1-\sigma)u + \sigma v\|^{p-2}\langle (1-\sigma)x + \sigma y, (1-\sigma)u + \sigma v\rangle_{s(i)} d\sigma. \tag{8.5}$$

Proof. The continuity can be easily verified. Note that for any $1 \leq p < \infty$, if u, v are linearly independent, then $\|(1-\sigma)u + \sigma v\| \neq 0$ for all $\sigma \in [0,1]$. Thus

$$\begin{aligned} G_{p,\pm}(\sigma) &= \lim_{t\to 0^{\pm}} F_p(\sigma, t) \\ &= p\|(1-\sigma)u + \sigma v\|^{p-1}\left[\nabla_{\pm}\|(\cdot,\cdot)\|_{p-HH}[(1-\sigma)u + \sigma v]\right][(1-\sigma)x + \sigma y] \\ &= p\|(1-\sigma)u + \sigma v\|^{p-2}\langle (1-\sigma)x + \sigma y, (1-\sigma)u + \sigma v\rangle_{s(i)}. \end{aligned}$$

By the Cauchy-Schwarz inequality and the convexity of the given norm, we have

$$\begin{aligned} \int_0^1 G_{p,\pm}(\sigma)d\sigma &\leq p\int_0^1 \|(1-\sigma)u + \sigma v\|^{p-1}\|(1-\sigma)x + \sigma y\| d\sigma \\ &\leq p\int_0^1 \left[(1-\sigma)\|u\|^{p-1} + \sigma\|v\|^{p-1}\right]\left[(1-\sigma)\|x\| + \sigma\|y\|\right] d\sigma < \infty, \end{aligned}$$

which shows that $G_{p,\pm}$ is Lebesgue integrable, and therefore (8.5) holds.

If u, v are linearly dependent, then there exists a unique $\sigma_0 \in [0,1]$ such that $(1-\sigma_0)u + \sigma_0 v = 0$. For $1 \leq p < \infty$, and $\sigma \neq \sigma_0$,

$$G_{p,\pm}(\sigma) = \lim_{t\to 0^{\pm}} F_p(\sigma, t) = p\|(1-\sigma)u + \sigma v\|^{p-2}\langle (1-\sigma)x + \sigma y, (1-\sigma)u + \sigma v\rangle_{s(i)}.$$

For $\sigma = \sigma_0$, we have

$$\begin{aligned} G_{p,\pm}(\sigma_0) = \lim_{t\to 0^{\pm}} F_p(\sigma_0, t) &= \lim_{t\to 0^{\pm}} \frac{|t|^p\|(1-\sigma_0)x + \sigma_0 y\|^p}{t} \\ &= \begin{cases} \pm\|(1-\sigma_0)x + \sigma_0 y\|, & p = 1; \\ 0, & p \neq 1. \end{cases} \end{aligned}$$

Note that, in this case, the integrability of $G_{p,\pm}$ is implied by the previous case (the case where u, v are linearly independent). Since

$$G_{p,\pm}(\sigma) = p\|(1-\sigma)u + \sigma v\|^{p-2}\langle (1-\sigma)x + \sigma y, (1-\sigma)u + \sigma v\rangle_{s(i)}$$

almost everywhere on $[0,1]$, then (8.5) holds. □

Lemma 13. *The superior (inferior) s.i.p. in $\mathbf{X}^2$ with respect to the norm $\|(\cdot,\cdot)\|_{p-HH}$ ($1 \leq p < \infty$) are given by*

$$\begin{aligned} &\langle (x,y),(u,v)\rangle_{p-HH,s(i)} \\ &\quad = \|(u,v)\|_{p-HH}^{2-p}\int_0^1 \|(1-\sigma)u + \sigma v\|^{p-2}\langle (1-\sigma)x + \sigma y, (1-\sigma)u + \sigma v\rangle_{s(i)} d\sigma, \end{aligned}$$

for any $(x,y),(u,v) \in \mathbf{X}^2$ (here, $\langle \cdot,\cdot\rangle_{s(i)}$ are the superior (inferior) s.i.p. with respect to the norm $\|\cdot\|$ on $\mathbf{X}$).

Proof. The proof for the case where $(u,v) = (0,0)$ is trivial. Assume that $(u,v) \neq (0,0)$ and define the function $g : \mathbf{X}^2 \to \mathbb{R}$, where $g(x,y) = \|(x,y)\|_{p-HH}^p$ ($1 \leq p < \infty$) for any $(x,y) \in \mathbf{X}^2$.

We have

$$\begin{aligned}
(\nabla_{\pm}g(u,v))(x,y) &:= \lim_{t\to 0^{\pm}} \frac{\|(u,v)+t(x,y)\|_{p-HH}^p - \|(u,v)\|_{p-HH}^p}{t} \\
&= p\|(u,v)\|_{p-HH}^{p-1} \lim_{t\to 0^{\pm}} \frac{\|(u,v)+t(x,y)\|_{p-HH} - \|(u,v)\|_p}{t} \\
&= p\|(u,v)\|_{p-HH}^{p-1}(\nabla_{\pm}\|(\cdot,\cdot)\|_{p-HH}(u,v))(x,y) \\
&= p\|(u,v)\|_{p-HH}^{p-2}\langle (x,y),(u,v)\rangle_{p-HH,s(i)}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\lim_{t\to 0^{\pm}} \frac{\|(u,v)+t(x,y)\|_{p-HH}^p - \|(u,v)\|_{p-HH}^p}{t} \\
&\quad = \lim_{t\to 0^{\pm}} \int_0^1 \frac{\|(1-\sigma)u+\sigma v + t((1-\sigma)x+\sigma y)\|^p - \|(1-\sigma)u+\sigma v\|^p}{t} d\sigma \\
&\quad = \lim_{t\to 0^{\pm}} \int_0^1 F_p(\sigma,t)d\sigma = \int_0^1 G_{p,\pm}(\sigma)d\sigma,
\end{aligned}$$

where F_p and $G_{p,\pm}$ are as defined in Proposition 13. Thus, we have the following identity

$$p\|(u,v)\|_{p-HH}^{p-2}\langle (x,y),(u,v)\rangle_{p-HH,s(i)} = \int_0^1 G_{p,\pm}(t),$$

that is,

$$\langle (x,y),(u,v)\rangle_{p-HH,s(i)} = \frac{1}{p}\|(u,v)\|_{p-HH}^{2-p}\int_0^1 G_{p,\pm}(t),$$

and the proof is completed by (8.5). □

Remark 12. Particularly for $p=2$, we have the following for any (x,y) and (u,v) in $\mathbf{X}^2$,

$$\langle (x,y),(u,v)\rangle_{HH,s(i)} = \int_0^1 \langle (1-\sigma)x+\sigma y, (1-\sigma)u+\sigma v\rangle_{s(i)} d\sigma.$$

9. Geometrical properties

We are interested in investigating whether the geometrical properties of $\mathbf{X}^2$, with respect to the p-norm and the p-HH-norm, are implied by those of $(\mathbf{X}, \|\cdot\|)$. The results can be stated in the following subsections.

9.1. Smoothness.

The space $L^p([0,1],\mathbf{B})$ is smooth (Fréchet smooth), when $\mathbf{B}$ is, by Lemma 2. Therefore, the smoothness (Fréchet smoothness) of $(\mathbf{B}^2, \|(\cdot,\cdot)\|_p)$ and $(\mathbf{B}^2, \|(\cdot,\cdot)\|_{p-HH})$ for $1<p<\infty$ are inherited from $L^p([0,1],\mathbf{B})$, by the embedding argument as described in Section 7. Here, we provide an alternative proof for the smoothness using the superior (inferior) s.i.p., and we do not require the space to be complete. We also prove that 1-HH-norm is a smooth norm via the superior (inferior) s.i.p.

Corollary 3. *The space* $(\mathbf{X}^2, \|(\cdot,\cdot)\|_p)$ $(1<p<\infty)$ *is a smooth normed space, whenever* $\mathbf{X}$ *is.*

Proof. Since $\mathbf{X}$ is smooth, we have $\langle x,y\rangle_i = \langle x,y\rangle_s$ for all $x,y\in\mathbf{X}$. Therefore

$$\begin{aligned}
\langle (x,y),(u,v)\rangle_{p,i} &= \frac{\|u\|^{p-2}\langle x,u\rangle_i + \|v\|^{p-2}\langle y,v\rangle_i}{\|(u,v)\|_p^{p-2}} \\
&= \frac{\|u\|^{p-2}\langle x,u\rangle_s + \|v\|^{p-2}\langle y,v\rangle_s}{\|(u,v)\|_p^{p-2}} = \langle (x,y),(u,v)\rangle_{p,s},
\end{aligned}$$

for all $(x,y),(u,v)\in \mathbf{X}^2$. □

Remark 13. Note that the space $(\mathbf{X}^2,\|(\cdot,\cdot)\|_1)$ is not always smooth, even if $\mathbf{X}$ is. For example, choose $\mathbf{X}=\mathbb{R}$, then take $(x,y)=(1,0)$ and $(u,v)=(0,1)$ in $(\mathbb{R}^2,\|\cdot\|_{l^1})$. We have

$$(\nabla_+\|(\cdot,\cdot)\|_1(1,0))(0,1)=1\neq -1=(\nabla_-\|(\cdot,\cdot)\|_1(1,0))(0,1).$$

The space $(\mathbf{X}^2,\|(\cdot,\cdot)\|_\infty)$ might be a non-smooth space, even when $\mathbf{X}$ is smooth. For example, let $\mathbf{X}=\mathbb{R}$, then take $(x,y)=(1,1)$ and $(u,v)=(-1,1)$ in $(\mathbb{R}^2,\|\cdot\|_{l^\infty})$. We have

$$(\nabla_+\|(\cdot,\cdot)\|_\infty(1,1))(-1,1)=1\neq -1=(\nabla_-\|(\cdot,\cdot)\|_\infty(1,1))(-1,1).$$

Corollary 4. *The space $(\mathbf{X}^2,\|(\cdot,\cdot)\|_{p-HH})$ $(1\le p<\infty)$ is a smooth normed space, if $\mathbf{X}$ is.*

Proof. The proof is trivial for $(u,v)=(0,0)$. Since $\mathbf{X}$ is smooth, $\langle x,y\rangle_i=\langle x,y\rangle_s$ for all $x,y\in\mathbf{X}$. It implies that for any $(x,y),(u,v)\in\mathbf{X}^2$ with nonzero (u,v), we have the following

$$\begin{aligned}\langle (x,y),(u,v)\rangle_{p-HH,s} &= \|(u,v)\|_{p-HH}^{2-p}\int_0^1 \|(1-\sigma)u+\sigma v\|^{p-2}\langle (1-\sigma)x+\sigma y,(1-\sigma)u+\sigma v\rangle_s d\sigma\\ &= \|(u,v)\|_{p-HH}^{2-p}\int_0^1 \|(1-\sigma)u+\sigma v\|^{p-2}\langle (1-\sigma)x+\sigma y,(1-\sigma)u+\sigma v\rangle_i d\sigma\\ &= \langle (x,y),(u,v)\rangle_{p-HH,i},\end{aligned}$$

for any $1\le p<\infty$. □

9.2. **Reflexivity.**

Corollary 5. *If $1\le p\le\infty$, then $(\mathbf{B}^2,\|(\cdot,\cdot)\|_p)$ and $(\mathbf{B}^2,\|(\cdot,\cdot)\|_{p-HH})$ are reflexive, provided that $\mathbf{B}$ is.*

Proof. For $1<p<\infty$, if $\mathbf{B}$ is reflexive, then so is $L^p([0,1],\mathbf{B})$ (Lemma 3). Since $(\mathbf{B}^2,\|(\cdot,\cdot)\|_p)$ is isomorphic to a closed subspace of $L^p([0,1],\mathbf{B})$, $(\mathbf{B}^2,\|(\cdot,\cdot)\|_p)$ is also reflexive. Since all the norms $\|(\cdot,\cdot)\|_p$ and $\|(\cdot,\cdot)\|_{p-HH}$ $(1\le p\le\infty)$ are equivalent, the reflexivity of the remaining cases follows by Lemma 1. □

Alternative proof for Corollary 5. Let $1<p<\infty$. Suppose that $(\mathbf{B}^2,\|(\cdot,\cdot)\|_p)$ is not a reflexive normed space. By Proposition 2, we are able to find a continuous linear functional F on $\mathbf{B}^2$ such that for any $(u,v)\in\mathbf{B}^2$, either one of the following holds:

(1) there exists $(x_0,y_0)\in\mathbf{B}^2$ such that $\langle (x_0,y_0),(u,v)\rangle_{p,i}>F(x_0,y_0)$ or $\langle (x_0,y_0),(u,v)\rangle_{p,s}<F(x_0,y_0)$;
(2) $\|F\|\neq\|(u,v)\|_p$.

Suppose that (1) holds (either (2) holds or does not hold). Define a continuous linear functional f on $\mathbf{B}$, by $f(x)=F(x,y_0)$. For any $u\in\mathbf{B}$ ($(u,0)\in\mathbf{B}^2$), there exists $x_0\in\mathbf{B}$ such that

$$\langle x_0,u\rangle_i=\langle (x_0,y_0),(u,0)\rangle_{p,i}>F(x_0,y_0)=f(x_0)$$

$$\text{or } \langle x_0,u\rangle_s=\langle (x_0,y_0),(u,0)\rangle_{p,s}<F(x_0,y_0)=f(x_0),$$

which contradicts the fact that $\mathbf{B}$ is reflexive.

Suppose that only (2) holds, i.e. there exists a continuous linear functional G on $\mathbf{B}^2$, such that for any $(u,v)\in\mathbf{B}^2$, we have

$$\langle (x,y),(u,v)\rangle_{p,i}\le G(x,y)\le\langle (x,y),(u,v)\rangle_{p,s},$$

for any $(x,y)\in\mathbf{B}^2$ and $\|G\|\neq\|(u,v)\|_p$. By Cauchy-Schwarz inequality, we always have $\|G\|\le\|(u,v)\|_p$. Thus, we conclude that $\|G\|<\|(u,v)\|_p$ for any $(u,v)\in\mathbf{B}^2$. Define a continuous linear functional g on $\mathbf{B}$, by $g(x) = G(x,0)$. Then, for any $u \in \mathbf{B}$ ($(u,0) \in \mathbf{B}^2$), we have

$$\langle x, u\rangle_i = \langle (x,0),(u,0)\rangle_{p,i} \le G(x,0) = g(x) \le \langle (x,0),(u,0)\rangle_{p,s} = \langle x,u\rangle_s$$

and

$$\|g\| = \sup_{\substack{x\in\mathbf{X}\\ \|x\|\neq 0}} \frac{|g(x)|}{\|x\|} = \sup_{\substack{(x,0)\in\mathbf{X}^2\\ \|(x,0)\|_p\neq 0}} \frac{|G(x,0)|}{\|(x,0)\|_p} \le \|G\| < \|(u,0)\|_p = \|u\|$$

which contradicts the fact that $\mathbf{B}$ is reflexive. The proof for the remaining cases follows by the norm equivalency and Lemma 1. □

9.3. **Strict convexity and uniform convexity.**

Corollary 6. *If* $(\mathbf{B}, \|\cdot\|)$ *is a strictly (uniformly) convex normed space, then so are* $(\mathbf{B}^2, \|(\cdot,\cdot)\|_p)$ *and* $(\mathbf{B}^2, \|(\cdot,\cdot)\|_{p-HH})$*, for any* $1 < p < \infty$.

Proof. The proof follows directly by Proposition 3, Lemma 4 and the fact that $\mathbf{B}^2$ together with $\|(\cdot,\cdot)\|_p$ and $\|(\cdot,\cdot)\|_{p-HH}$ are homeomorphic to a subspace of $L^p([0,1],\mathbf{B})$ (see Section 7). □

Remark 14. Note that in general, $(\mathbf{B}^2, \|(\cdot,\cdot)\|_1)$ is not strictly (uniformly) convex, even if $\mathbf{B}$ is. For example, take $(x,y) = (1,0)$ and $(u,v) = (0,1)$ in $(\mathbb{R}^2, \|\cdot\|_{l^1})$. Observe that $\|(x,y)\|_{l^1} = \|(u,v)\|_{l^1} = 1$, but $\|(x,y)+(u,v)\|_{l^1} = 2$, which shows that this space is not strictly convex (which also implies that it is not uniformly convex).

The space $(\mathbf{B}^2, \|(\cdot,\cdot)\|_\infty)$ is not strictly (uniformly) convex, even when $\mathbf{B}$ is. As an example, take $(x,y) = (1,1)$ and $(u,v) = (-1,1)$ in $(\mathbb{R}^2, \|\cdot\|_{l^\infty})$. Observe that $\|(x,y)\|_{l^\infty} = \|(u,v)\|_{l^\infty} = 1$, but $\|(x,y)+(u,v)\|_{l^\infty} = 2$, which shows that this space is not strictly (uniformly) convex.

The $(\mathbf{B}^2, \|(\cdot,\cdot)\|_{1-HH})$ is not always strictly (uniformly) convex, even if $\mathbf{B}$ is. For example, take $(\mathbf{B}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $(x,y) = (2,0)$ and $(u,v) = (0,2)$ in $\mathbb{R}^2$. Observe that $\|(x,y)\|_{1-HH} = \int_0^1 2(1-t)\,dt = 1$ and $\|(u,v)\|_{1-HH} = \int_0^1 2t dt = 1$, but $\|(x,y)+(u,v)\|_{l^1} = \int_0^1 2\,dt = 2$, which shows that this space is not strictly (uniformly) convex.

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