A Generalization of the Cauchy-Schwarz Inequality with Four Free Parameters and Applications

A GENERALIZATION OF THE CAUCHY-SCHWARZ INEQUALITY WITH FOUR FREE PARAMETERS AND APPLICATIONS

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Abstract. A generalization of the well-known Cauchy-Schwarz inequality with four free parameters is given for both discrete and continuous cases. Some particular cases of interest are also analyzed.

1. Introduction

Let $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ be two sequences of real numbers. It is well known that the discrete version of the Cauchy-Schwarz inequality [3] can be stated as:

$$\left(\sum_{k=1}^n a_k b_k\right)^2 \le \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2. \tag{1.1}$$

The equality case holds in (1.1) if and only if the sequences are proportional meaning that there exists a real number r so that $a_k = r b_k$ for each $k \in \{1, ..., n\}$.

To date, a large number of generalizations and refinements of (1.1) have been mentioned in the literature, see for example the survey paper [4], the book [7] and the numerous references therein.

In this paper, we present a further generalization of the Cauchy-Schwarz inequality in terms of four free parameters and study some particular cases of interest.

2. A Generalization of the Cauchy-Schwarz Inequality

The first result is incorporated in:

Theorem 1. *If $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are two sequences of real numbers and $p, q, r, s \in \mathbb{R}$ then*

$$\begin{aligned}\left[\sum_{k=1}^n a_k b_k + A_1 \sum_{k=1}^n a_k \sum_{k=1}^n b_k + B_1\left(\sum_{k=1}^n a_k\right)^2 + C_1\left(\sum_{k=1}^n b_k\right)^2\right]^2 \\ \le \left[\sum_{k=1}^n a_k^2 + A_2 \sum_{k=1}^n a_k \sum_{k=1}^n b_k + B_2\left(\sum_{k=1}^n a_k\right)^2 + C_2\left(\sum_{k=1}^n b_k\right)^2\right] \\ \times \left[\sum_{k=1}^n b_k^2 + A_3 \sum_{k=1}^n a_k \sum_{k=1}^n b_k + B_3\left(\sum_{k=1}^n a_k\right)^2 + C_3\left(\sum_{k=1}^n b_k\right)^2\right],\end{aligned} \tag{2.1}$$

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in which the coefficients involved are defined in the following matrix equation

$$M=\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}=\frac{1}{n}\begin{pmatrix} p+s+ps+qr & r(1+p) & q(1+s) \\ 2q(1+p) & p(p+2) & q^2 \\ 2r(1+s) & r^2 & s(s+2) \end{pmatrix}.$$

Moreover, the inequality (2.1) is equivalent to

$$\begin{aligned}(2.2)\quad &\Bigg[\sum_{k=1}^{n}a_kb_k+\frac{p+s+ps+qr}{n}\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k \\ &\quad +\frac{r(1+p)}{n}\left(\sum_{k=1}^{n}a_k\right)^2+\frac{q(1+s)}{n}\left(\sum_{k=1}^{n}b_k\right)^2\Bigg]^2 \\ &\leq\left[\sum_{k=1}^{n}a_k^2+\frac{1}{n}\sum_{k=1}^{n}(pa_k+qb_k)\sum_{k=1}^{n}((p+2)a_k+qb_k)\right] \\ &\quad\times\left[\sum_{k=1}^{n}b_k^2+\frac{1}{n}\sum_{k=1}^{n}(ra_k+sb_k)\sum_{k=1}^{n}(ra_k+(s+2)b_k)\right],\end{aligned}$$

and is a generalization of the Cauchy-Schwarz inequality for $A_i=B_i=C_i=0$, $i=1,2,3$. The equality holds if $a_k=b_k$ and $A_i=B_i=C_i$ for each $i=1,2,3$.

Proof. Let us define the positive quadratic polynomial $Q:\mathbb{R}\to\mathbb{R}$ as

$$\begin{aligned}(2.3)\quad Q(x;p,q,r,s)=\sum_{k=1}^{n}\Bigg[&\left(a_k+\frac{p}{n}\sum_{k=1}^{n}a_k+\frac{q}{n}\sum_{k=1}^{n}b_k\right)x \\ &+\left(b_k+\frac{r}{n}\sum_{k=1}^{n}a_k+\frac{s}{n}\sum_{k=1}^{n}b_k\right)\Bigg]^2,\end{aligned}$$

in which $p,q,r,s\in\mathbb{R}$ and $\{a_k\}_{k=1}^n$, $\{b_k\}_{k=1}^n$ are real numbers. Since a simple calculation reveals that

$$\begin{aligned}(2.4)\quad Q(x;p,q,r,s)&=\sum_{k=1}^{n}\left(a_k+\frac{p}{n}\sum_{k=1}^{n}a_k+\frac{q}{n}\sum_{k=1}^{n}b_k\right)^2x^2 \\ &+2\sum_{k=1}^{n}\left(a_k+\frac{p}{n}\sum_{k=1}^{n}a_k+\frac{q}{n}\sum_{k=1}^{n}b_k\right)\left(b_k+\frac{r}{n}\sum_{k=1}^{n}a_k+\frac{s}{n}\sum_{k=1}^{n}b_k\right)x \\ &+\sum_{k=1}^{n}\left(b_k+\frac{r}{n}\sum_{k=1}^{n}a_k+\frac{s}{n}\sum_{k=1}^{n}b_k\right)^2\geq 0\end{aligned}$$

for any $x \in \mathbb{R}$, the discriminant Δ of Q must be negative, i.e.

$$\begin{aligned}(2.5)\quad \frac{1}{4}\Delta = &\left[\sum_{k=1}^{n}\left(a_k + \frac{p}{n}\sum_{k=1}^{n} a_k + \frac{q}{n}\sum_{k=1}^{n} b_k\right)\left(b_k + \frac{r}{n}\sum_{k=1}^{n} a_k + \frac{s}{n}\sum_{k=1}^{n} b_k\right)\right]^2 \\ &- \left(\sum_{k=1}^{n} (a_k + \frac{p}{n}\sum_{k=1}^{n} a_k + \frac{q}{n}\sum_{k=1}^{n} b_k)^2\right) \\ &\times \left(\sum_{k=1}^{n} (b_k + \frac{r}{n}\sum_{k=1}^{n} a_k + \frac{s}{n}\sum_{k=1}^{n} b_k)^2\right) \leq 0.\end{aligned}$$

On the other hand, the elements of $\Delta/4$ can be simplified as follows:

$$\begin{aligned}(2.6a)\quad &\sum_{k=1}^{n}\left(a_k + \frac{p}{n}\sum_{k=1}^{n} a_k + \frac{q}{n}\sum_{k=1}^{n} b_k\right)\left(b_k + \frac{r}{n}\sum_{k=1}^{n} a_k + \frac{s}{n}\sum_{k=1}^{n} b_k\right) \\ &= \sum_{k=1}^{n} a_k b_k + \frac{p+s+ps+qr}{n}\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \\ &\quad + \frac{r(1+p)}{n}\left(\sum_{k=1}^{n} a_k\right)^2 + \frac{q(1+s)}{n}\left(\sum_{k=1}^{n} b_k\right)^2,\end{aligned}$$

$$\begin{aligned}(2.6b)\quad &\sum_{k=1}^{n}\left(a_k + \frac{p}{n}\sum_{k=1}^{n} a_k + \frac{q}{n}\sum_{k=1}^{n} b_k\right)^2 \\ &= \sum_{k=1}^{n} a_k^2 + \frac{2q(1+p)}{n}\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \\ &\quad + \frac{p(p+2)}{n}\left(\sum_{k=1}^{n} a_k\right)^2 + \frac{q^2}{n}\left(\sum_{k=1}^{n} b_k\right)^2,\end{aligned}$$

$$\begin{aligned}(2.6c)\quad &\sum_{k=1}^{n}\left(b_k + \frac{r}{n}\sum_{k=1}^{n} a_k + \frac{s}{n}\sum_{k=1}^{n} b_k\right)^2 \\ &= \sum_{k=1}^{n} b_k^2 + \frac{2r(1+s)}{n}\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \\ &\quad + \frac{r^2}{n}\left(\sum_{k=1}^{n} a_k\right)^2 + \frac{s(s+2)}{n}\left(\sum_{k=1}^{n} b_k\right)^2.\end{aligned}$$

So, by replacing the results (2.6) in inequality (2.5), the first part of Theorem 1 is proved.

To prove the second part (i.e. the equality condition) let us assume that $b_k = va_k$ and substitute it into (2.1) to get

$$\left[v\sum_{k=1}^{n} a_k^2 + (C_1v^2 + A_1v + B_1)\left(\sum_{k=1}^{n} a_k\right)^2\right]^2 = \left[\sum_{k=1}^{n} a_k^2 + (C_2v^2 + A_2v + B_2)\left(\sum_{k=1}^{n} a_k\right)^2\right] \times \left[v^2\sum_{k=1}^{n} a_k^2 + (C_3v^2 + A_3v + B_3)\left(\sum_{k=1}^{n} a_k\right)^2\right]. \tag{2.7}$$

After some computations, the above equality leads to the nonlinear system

$$\begin{cases} (C_1v^2 + A_1v + B_1)^2 = (C_2v^2 + A_2v + B_2)(C_3v^2 + A_3v + B_3), \\ 2v(C_1v^2 + A_1v + B_1) = v^2(C_2v^2 + A_2v + B_2) + (C_3v^2 + A_3v + B_3). \end{cases} \tag{2.8}$$

Obviously, one of the solutions of equation (2.8) is: $A_i = B_i = C_i$ for each $i = 1, 2, 3$ and $v = 1$. □

Remark 1. *We can observe that there exist various sub-cases of inequality (2.1). However, due to page limitations, we only consider a particular case of (2.1) and investigate its sub-cases. Naturally, other special cases can be separately studied. The details are left to the interested reader.*

3. The Particular Case $B_1 = C_1 = 0$

A total of four cases can occur for the inequality (2.1) when $B_1 = C_1 = 0$. They are, respectively:

(i) $(r, q) = (0, 0)$,
(ii) $(r, s) = (0, -1)$,
(iii) $(p, q) = (-1, 0)$,
(iv) $(p, s) = (-1, -1)$.

3.1. **Case $q = r = 0$ and $p, s \in R$ in (2.1).**

In this case $B_1 = C_1 = A_2 = C_2 = A_3 = B_3 = 0$ and the inequality (2.1) is reduced to

$$\left[\sum_{k=1}^{n} a_kb_k + \frac{p+s+ps}{n}\sum_{k=1}^{n} a_k\sum_{k=1}^{n} b_k\right]^2 \le \left[\sum_{k=1}^{n} a_k^2 + \frac{p(p+2)}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right]\left[\sum_{k=1}^{n} b_k^2 + \frac{s(s+2)}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right]. \tag{3.1}$$

This inequality has some interesting sub-cases as follows:

3.1.1. ***Sub-case 1.*** $p = s \in \mathbb{R} \setminus (-2, 0)$ ***(A generalization of the Wagner inequality).*** The following inequality for sequences of real numbers is known in the literature as the Wagner inequality [9] (see also [6]):

Let $\{a_k\}_{k=1}^n$ *and* $\{b_k\}_{k=1}^n$ *be two sequences of real numbers. If* $w \geq 0$ *then*

$$\left[\sum_{k=1}^n a_k b_k + w \sum_{k=1}^n a_k \sum_{k=1}^n b_k\right]^2 \leq \left[\sum_{k=1}^n a_k^2 + w \left(\sum_{k=1}^n a_k\right)^2\right] \left[\sum_{k=1}^n b_k^2 + w \left(\sum_{k=1}^n b_k\right)^2\right]. \tag{3.2}$$

To obtain (3.2) it is enough in (3.1) to assume that

$$\frac{p+s+ps}{n} = \frac{p(p+2)}{n} = \frac{s(s+2)}{n} \geq 0,$$

which holds for $p = s \in \mathbb{R} \setminus (-2, 0)$ and gives the Wagner inequality for $w = \frac{p(p+2)}{n} \geq 0$.

Note that in (3.1) if $p(p+2) \leq 0$ and $s(s+2) \leq 0$, then

$$\begin{aligned}
&\left[\sum_{k=1}^n a_k b_k + \frac{p+s+ps}{n} \sum_{k=1}^n a_k \sum_{k=1}^n b_k\right]^2 \\
&\leq \left[\sum_{k=1}^n a_k^2 + \frac{p(p+2)}{n} \left(\sum_{k=1}^n a_k\right)^2\right] \left[\sum_{k=1}^n b_k^2 + \frac{s(s+2)}{n} \left(\sum_{k=1}^n b_k\right)^2\right] \\
&\leq \left(\sum_{k=1}^n a_k^2\right)\left(\sum_{k=1}^n b_k^2\right) \Leftrightarrow p \in [-2, 0] \quad \text{and} \quad s \in [-2, 0].
\end{aligned} \tag{3.3}$$

3.1.2. ***Sub-case 2.*** $p = s \in [-2, 0]$ ***(A refinement for the Cauchy-Schwarz inequality)***. Suppose in (3.1) that $p = s \in [-2, 0]$ and $p(p+2) = u$. Consequently $u \in [-1, 0]$. By noting these assumptions we can obtain a refinement for inequality (1.1). For this purpose, first the following inequality should be considered, which is directly provable via some algebraic computations

$$\begin{aligned}
&\left[\sum_{k=1}^n a_k^2 + \frac{u}{n} \left(\sum_{k=1}^n a_k\right)^2\right] \left[\sum_{k=1}^n b_k^2 + \frac{u}{n} \left(\sum_{k=1}^n b_k\right)^2\right] \\
&\leq \left[\left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^n b_k^2\right)^{\frac{1}{2}} + \frac{u}{n} \sum_{k=1}^n a_k \sum_{k=1}^n b_k\right]^2,
\end{aligned} \tag{3.4}$$

because (3.4) eventually leads to

$$\frac{u}{n}\left[\left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{2}} \sum_{k=1}^n b_k - \left(\sum_{k=1}^n b_k^2\right)^{\frac{1}{2}} \sum_{k=1}^n a_k\right]^2 \leq 0 \quad \text{for} \quad u \in [-1, 0]. \tag{3.5}$$

Hence, by referring to inequalities (3.1) and (3.4), one can at last conclude:

Corollary 1. *Let $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ be two positive sequences of real numbers and $\alpha \in [0,1]$. Then*

$$\begin{aligned}(3.6)\qquad &\left[\sum_{k=1}^n a_k b_k - \frac{\alpha}{n}\sum_{k=1}^n a_k \sum_{k=1}^n b_k\right]^2 \\ &\le \left[\sum_{k=1}^n a_k^2 - \frac{\alpha}{n}\left(\sum_{k=1}^n a_k\right)^2\right]\left[\sum_{k=1}^n b_k^2 - \frac{\alpha}{n}\left(\sum_{k=1}^n b_k\right)^2\right] \\ &\le \left[\left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^n b_k^2\right)^{\frac{1}{2}} - \frac{\alpha}{n}\sum_{k=1}^n a_k \sum_{k=1}^n b_k\right]^2,\end{aligned}$$

which is equivalent to

$$\begin{aligned}(3.7)\qquad &\left(\sum_{k=1}^n a_k b_k\right)^2 \\ &\le \left(\frac{\alpha}{n}\sum_{k=1}^n a_k \sum_{k=1}^n b_k + \sqrt{\sum_{k=1}^n a_k^2 - \frac{\alpha}{n}\left(\sum_{k=1}^n a_k\right)^2}\sqrt{\sum_{k=1}^n b_k^2 - \frac{\alpha}{n}\left(\sum_{k=1}^n b_k\right)^2}\right)^2 \\ &\le \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2.\end{aligned}$$

The equality holds in (3.7) when $b_k = \lambda a_k$ (where λ is a constant).

For other refinements of the Cauchy-Schwarz inequality we refer the reader to [1] and [10].

3.1.3. ***Sub-case 3.*** It may be interesting to add that if $\frac{1}{p}+\frac{1}{s} = -1$ for $p, s \in \mathbb{R}\backslash\{0\}$, then (3.1) is reduced to

$$\begin{aligned}(3.8)\qquad \left(\sum_{k=1}^n a_k b_k\right)^2 \le &\left[\sum_{k=1}^n a_k^2 + \frac{p(p+2)}{n}\left(\sum_{k=1}^n a_k\right)^2\right] \\ &\times \left[\sum_{k=1}^n b_k^2 + \frac{s(s+2)}{n}\left(\sum_{k=1}^n b_k\right)^2\right],\end{aligned}$$

where $p = s = -2$ gives the Cauchy-Schwarz inequality.

3.2. **Case $r = 0$, $s = -1$ and $p, q \in \mathbb{R}$ in (2.1).** In this case $B_1 = C_1 = A_3 = B_3 = 0$ and the inequality (2.1) is reduced to:

$$\left[\sum_{k=1}^{n} a_k b_k - \frac{1}{n}\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k\right]^2 \leq \left[\sum_{k=1}^{n} b_k^2 - \frac{1}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right] \times \left[\sum_{k=1}^{n} a_k^2 - \frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2 + \frac{1}{n}\left(\sum_{k=1}^{n} (p+1)a_k + qb_k\right)^2\right]. \tag{3.9}$$

However, since

$$\sum_{k=1}^{n} a_k^2 - \frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2 \geq 0, \tag{3.10}$$

the best option for p, q in (3.9) is when $p = -1$ and $q = 0$. Furthermore, note that the third mentioned case, i.e. $p = -1$, $q = 0$ and $r, s \in \mathbb{R}$, gives the same result as in (3.9).

3.3. **Case $p = s = -1$ and $q, r \in \mathbb{R}$ in (2.1).** In this case $B_1 = C_1 = A_2 = A_3 = 0$ and the inequality (2.1) is reduced to

$$\left[\sum_{k=1}^{n} a_k b_k + \frac{qr-1}{n}\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k\right]^2 \leq \left[\sum_{k=1}^{n} a_k^2 - \frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2 + \frac{q^2}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right] \left[\sum_{k=1}^{n} b_k^2 - \frac{1}{n}\left(\sum_{k=1}^{n} b_k\right)^2 + \frac{r^2}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right]. \tag{3.11}$$

An interesting case in (3.11) is when $q = r = 1$, i.e.

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left[\sum_{k=1}^{n} a_k^2 + \frac{1}{n}\left\{\left(\sum_{k=1}^{n} b_k\right)^2 - \left(\sum_{k=1}^{n} a_k\right)^2\right\}\right] \left[\sum_{k=1}^{n} b_k^2 + \frac{1}{n}\left\{\left(\sum_{k=1}^{n} a_k\right)^2 - \left(\sum_{k=1}^{n} b_k\right)^2\right\}\right]. \tag{3.12}$$

4. A Generalization of the Cauchy-Bunyakovsky Inequality

In a similar manner, the integral version of the Cauchy-Schwarz inequality, which is known in the literature as the Cauchy-Bunyakovsky inequality [2] and has the form

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx, \tag{4.1}$$

can also be generalized as follows.

Theorem 2. *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions on $[a, b]$ and $p, q, r, s \in \mathbb{R}$. Then, the following inequality holds*

$$\begin{aligned}(4.2)\qquad &\left[\int_a^b f(x)g(x)dx + A_1^* \int_a^b f(x)dx \int_a^b g(x)dx\right.\\
&\qquad\left. + B_1^*\left(\int_a^b f(x)dx\right)^2 + C_1^*\left(\int_a^b g(x)dx\right)^2\right]^2\\
&\le \left[\int_a^b f^2(x)dx + A_2^* \int_a^b f(x)dx \int_a^b g(x)dx\right.\\
&\qquad\left. + B_2^*\left(\int_a^b f(x)dx\right)^2 + C_2^*\left(\int_a^b g(x)dx\right)^2\right]\\
&\qquad\times\left[\int_a^b g^2(x)dx + A_3^* \int_a^b f(x)dx \int_a^b g(x)dx\right.\\
&\qquad\qquad\left. + B_3^*\left(\int_a^b f(x)dx\right)^2 + C_3^*\left(\int_a^b g(x)dx\right)^2\right],\end{aligned}$$

in which

$$M^* = \begin{pmatrix} A_1^* & B_1^* & C_1^* \\ A_2^* & B_2^* & C_2^* \\ A_3^* & B_3^* & C_3^* \end{pmatrix} = \frac{1}{b-a}\begin{pmatrix} p+s+ps+qr & r(1+p) & q(1+s) \\ 2q(1+p) & p(p+2) & q^2 \\ 2r(1+s) & r^2 & s(s+2) \end{pmatrix}.$$

Moreover, the inequality (4.2) is equivalent to

$$\begin{aligned}(4.3)\quad &\left[\int_a^b f(x)g(x)dx + \frac{p+s+ps+qr}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx\right.\\
&\qquad\left. + \frac{r(1+p)}{b-a}\left(\int_a^b f(x)dx\right)^2 + \frac{q(1+s)}{b-a}\left(\int_a^b g(x)dx\right)^2\right]^2\\
&\le \left[\int_a^b f^2(x)dx + \frac{1}{b-a}\int_a^b (pf(x)+qg(x))dx \int_a^b ((p+2)f(x)+qg(x))dx\right]\\
&\times\left[\int_a^b g^2(x)dx + \frac{1}{b-a}\int_a^b (rf(x)+sg(x))dx \int_a^b (rf(x)+(s+2)g(x))dx\right],\end{aligned}$$

and is a generalization of the Cauchy-Bunyakovsky inequality for $A_i^* = B_i^* = C_i^* = 0$, $i = 1, 2, 3$. The equality holds if $f(x) = g(x)$ and $A_i^* = B_i^* = C_i^*$ for each $i = 1, 2, 3$.

Although the proof is similar to the proof of Theorem 1, by defining the positive quadratic polynomial

$$(4.4)\quad R(x;p,q,r,s)=\int_a^b\left[\left(f(t)+\frac{p}{b-a}\int_a^b f(x)dx+\frac{q}{b-a}\int_a^b g(x)dx\right)x+\left(g(t)+\frac{r}{b-a}\int_a^b f(x)dx+\frac{s}{b-a}\int_a^b g(x)dx\right)\right]^2 dt\geq 0,$$

we should however note that the following relations are to be used in the proof:

$$(4.5)\quad \int_a^b\left(f(x)+\frac{p}{b-a}\int_a^b f(x)dx+\frac{q}{b-a}\int_a^b g(x)dx\right)\times\left(g(x)+\frac{r}{b-a}\int_a^b f(x)dx+\frac{s}{b-a}\int_a^b g(x)dx\right)dx$$
$$=\int_a^b f(x)g(x)dx+\frac{p+s+ps+qr}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx+\frac{r(1+p)}{b-a}\left(\int_a^b f(x)dx\right)^2+\frac{q(1+s)}{b-a}\left(\int_a^b g(x)dx\right)^2,$$

and

$$\int_a^b\left(f(x)+\frac{p}{b-a}\int_a^b f(x)dx+\frac{q}{b-a}\int_a^b g(x)dx\right)^2 dx$$
$$=\int_a^b f^2(x)dx+\frac{2q(1+p)}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx+\frac{p(p+2)}{b-a}\left(\int_a^b f(x)dx\right)^2+\frac{q^2}{b-a}\left(\int_a^b g(x)dx\right)^2,$$

and

$$\int_a^b\left(g(x)+\frac{r}{b-a}\int_a^b f(x)dx+\frac{s}{b-a}\int_a^b g(x)dx\right)^2 dx$$
$$=\int_a^b g^2(x)dx+\frac{2r(1+s)}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx+\frac{r^2}{b-a}\left(\int_a^b f(x)dx\right)^2+\frac{s(s+2)}{b-a}\left(\int_a^b g(x)dx\right)^2,$$

respectively. Moreover, we note that all the mentioned sub-cases for inequality (2.1) could similarly be considered for the continuous case (4.2).

For the sake of completeness, we can state, for instance, the following result:

Corollary 2 (A refinement of the Cauchy- Bunyakovsky inequality)**.**

Let $f, g : [a, b] \to \mathbb{R}$ be two positive integrable functions on $[a, b]$ and $\alpha \in [0, 1]$. Then

$$\left(\int_a^b f(x)g(x)dx\right)^2 \tag{4.6}$$
$$\le \left(\frac{\alpha}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx + \sqrt{\int_a^b f^2(x)dx - \frac{\alpha}{b-a}\left(\int_a^b f(x)dx\right)^2}\right.$$
$$\left.\times\sqrt{\int_a^b g^2(x)dx - \frac{\alpha}{b-a}\left(\int_a^b g(x)dx\right)^2}\right)^2$$
$$\le \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

5. A Unified Approach for the Classification of (2.1) and (4.2)

As we observed in the previous sections, there were respectively two special matrices M and M^* for inequalities (2.1) and (4.2) having 9 elements. Hence, each sub-case of these two inequalities can be characterized by M or M^* directly. For instance, the discrete inequality

$$\left[\sum_{k=1}^n a_k b_k - \frac{s+2}{n}\sum_{k=1}^n a_k \sum_{k=1}^n b_k - \frac{r}{n}\left(\sum_{k=1}^n a_k\right)^2\right]^2 \tag{5.1}$$
$$\le \sum_{k=1}^n a_k^2 \left[\sum_{k=1}^n b_k^2 + \frac{1}{n}\sum_{k=1}^n (ra_k + sb_k)\sum_{k=1}^n (ra_k + (s+2)b_k)\right],$$

which is a generalization of (1.1) for $r = 0$ and $s = -2$, has the characteristic matrix

$$M(\text{Ineq. } (5.1)) = \frac{1}{n}\begin{pmatrix} -s-2 & -r & 0 \\ 0 & 0 & 0 \\ 2r(1+s) & r^2 & s(s+2) \end{pmatrix}, \tag{5.2}$$

while the continuous inequality

$$\left[\int_a^b f(x)g(x)dx + \frac{p}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx + \frac{q}{b-a}\left(\int_a^b g(x)dx\right)^2\right]^2 \tag{5.3}$$
$$\le \left[\int_a^b f^2(x)dx + \frac{1}{b-a}\int_a^b (pf(x) + qg(x))dx\right.$$
$$\left.\times \int_a^b ((p+2)f(x) + qg(x))dx\right]\left(\int_a^b g^2(x)dx\right),$$

corresponds to the matrix

$$M^*(\text{Ineq. } (5.3)) = \frac{1}{b-a}\begin{pmatrix} p & 0 & q \\ 2q(1+p) & p(p+2) & q^2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.4}$$

Similarly, for inequalities (3.2), (3.9), (3.11) and (3.12) we have

$$M(\text{Ineq. (3.2)}) = \frac{p(p+2)}{n} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.5}$$

$$M(\text{Ineq. (3.9)}) = \frac{1}{n} \begin{pmatrix} -1 & 0 & 0 \\ 2q(1+p) & p(p+2) & q^2 \\ 0 & 0 & -1 \end{pmatrix},$$

$$M(\text{Ineq. (3.11)}) = \frac{1}{n} \begin{pmatrix} qr-1 & 0 & 0 \\ 0 & -1 & q^2 \\ 0 & r^2 & -1 \end{pmatrix},$$

$$M(\text{Ineq. (3.12)}) = \frac{1}{n} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Finally we mention that the usual Cauchy-Schwarz and Cauchy-Bunyakovsky inequalities correspond to respectively $M = 0$ and $M^* = 0$, which can be obtained for $p = q = r = s = 0$.

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