Generalizations of an Integral Inequality

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GENERALIZATIONS OF AN INTEGRAL INEQUALITY

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Abstract. In this short paper, an integral inequality posed in the 11th International Mathematical Competition for University Students, Skopje, Macedonia, 25–26 July 2004 is generalized.

1. introduction

The Problem 2 of the 11th International Mathematical Competition for University Students, Skopje, Macedonia, 25–26 July 2004 (see [1]) reads as follows.

Proposition 1 ([1]). *Let $f,g:[a,b]\to[0,\infty)$ be two continuous and non-decreasing functions such that*

$$\int_a^x \sqrt{f(t)}\,\mathrm{d}t \le \int_a^x \sqrt{g(t)}\,\mathrm{d}t \tag{1}$$

for $x\in[a,b]$ and

$$\int_a^b \sqrt{f(t)}\,\mathrm{d}t = \int_a^b \sqrt{g(t)}\,\mathrm{d}t. \tag{2}$$

Then

$$\int_a^b \sqrt{1+f(t)}\,\mathrm{d}t \ge \int_a^b \sqrt{1+g(t)}\,\mathrm{d}t. \tag{3}$$

It is clear that, considering (2), inequality (1) can be rewritten as

$$\int_x^b \sqrt{f(t)}\,\mathrm{d}t \ge \int_x^b \sqrt{g(t)}\,\mathrm{d}t. \tag{4}$$

If replacing $f(x)$ by $\sqrt{f(x)}$ and $g(x)$ by $\sqrt{g(x)}$, then Proposition 1 can be simplified into the following Proposition 2.

Proposition 2. *Let $f,g:[a,b]\to[0,\infty)$ be two continuous and non-decreasing functions such that*

$$\int_x^b f(t)\,\mathrm{d}t \ge \int_x^b g(t)\,\mathrm{d}t \tag{5}$$

for $x\in[a,b]$ and

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b g(t)\,\mathrm{d}t. \tag{6}$$

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Then

$$\int_a^b \sqrt{1+f^2(t)}\,\mathrm{d}t \geq \int_a^b \sqrt{1+g^2(t)}\,\mathrm{d}t. \tag{7}$$

If denoting

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \quad \text{and} \quad G(x) = \int_a^x g(t)\,\mathrm{d}t, \tag{8}$$

then $F(x) \leq G(x)$ for $x \in [a,b]$ and $G(x)$ is a convex function on $[a,b]$. On the other hand, since $F(a) = G(a)$ and $F(b) = G(b)$, then inequality (7) is valid apparently, because the length of the curve $y = F(x)$ is not less than that of the curve $y = G(x)$. This explains the geometric meaning of Proposition 2 and gives a solution to Proposition 1.

The main aim of this paper is to generalize Proposition 1 and Proposition 2 above.

Our main results are included in a couple of theorems below.

Theorem 1. *Let $f : [a,b] \to [0,\infty)$ be a continuous function and $g : [a,b] \to [0,\infty)$ a continuous and non-decreasing function satisfying* (5) *and* (6)*. Then*

$$\int_a^b h(f(t))\,\mathrm{d}t \geq \int_a^b h(g(t))\,\mathrm{d}t \tag{9}$$

validates for every convex function h on $[0,\infty)$.

Theorem 2. *Let $f : [a,b] \to [0,\infty)$ be a continuous function and $g : [a,b] \to [0,\infty)$ a continuous and non-increasing function satisfying* (6) *and the reverse of* (5)*. Then inequality* (9) *holds true for every convex function h on $[0,\infty)$.*

Remark 1. It is easy to see that the function $\sqrt{1+x^2}$ is convex on $[a,b]$. Hence, inequality (7) can be deduced easily from (9).

2. Proofs of Theorem 1 and Theorem 2

In order to prove our theorems, the well known second mean value theorem for integrals will be availble.

Lemma 1 ([2, p. 35])**.** *Let $f(x)$ be bounded and monotonic and $g(x)$ integrable on $[a,b]$. Then there exists some $\xi \in [a,b]$ such that*

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(a)\int_a^\xi g(x)\,\mathrm{d}x + f(b)\int_\xi^b g(x)\,\mathrm{d}x. \tag{10}$$

Proof of Theorem 1. Denote

$$\phi(x) = -\int_x^b f(t)\,\mathrm{d}t \quad \text{and} \quad \varphi(x) = -\int_x^b g(t)\,\mathrm{d}t. \tag{11}$$

Since h is convex, then

$$h(t) \geq h(s) + (t-s)h'(s)$$

for $a \leq s, t \leq b$, therefore,

$$h(\phi'(t)) \geq h(\varphi'(t)) + [\phi'(t) - \varphi'(t)]h'(\varphi'(t)) \tag{12}$$

which gives

$$\int_a^b h(\phi'(t))\,\mathrm{d}t \geq \int_a^b h(\varphi'(t))\,\mathrm{d}t + \int_a^b [\phi'(t) - \varphi'(t)]h'(\varphi'(t))\,\mathrm{d}t. \tag{13}$$

Now it is sufficient to prove that

$$\int_a^b [\phi'(t) - \varphi'(t)]h'(\varphi'(t))\,\mathrm{d}t \ge 0. \tag{14}$$

Since $h(t)$ is convex, the function $h'(t)$ is non-decreasing; Since $g(t)$ is non-decreasing, the function $\varphi'(t)$ is also non-decreasing. Thus, the composite function $h'(\varphi'(t))$ is non-decreasing with respect to t. Using Lemma 1 yields

$$\begin{aligned}
&\int_a^b [\phi'(t) - \varphi'(t)]h'(\varphi'(t))\,\mathrm{d}t \\
&\quad = h'(\varphi'(a))\int_a^\xi [\phi'(t) - \varphi'(t)]\,\mathrm{d}t + h'(\varphi'(b))\int_\xi^b [\phi'(t) - \varphi'(t)]\,\mathrm{d}t \\
&\quad = h'(g(a))[\phi(\xi) - \phi(a) - \varphi(\xi) + \varphi(a)] + h'(g(b))[\phi(b) - \phi(\xi) - \varphi(b) + \varphi(\xi)] \\
&\quad = [\phi(\xi) - \varphi(\xi)][h'(g(a)) - h'(g(b))] \\
&\quad \ge 0,
\end{aligned}$$

where $\xi \in [a, b]$, since $\phi(\xi) \le \varphi(\xi)$ and $h'(g(a)) \le h'(g(b))$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Denote

$$\psi(x) = \int_a^x f(t)\,\mathrm{d}t \quad \text{and} \quad \theta(x) = \int_a^x g(t)\,\mathrm{d}t \tag{15}$$

for $a \le x \le b$. Along with the proof of Theorem 1, it is obtained that, in order to show Theorem 2, it suffices to prove

$$\int_a^b [\psi'(t) - \theta'(t)][-h'(\theta'(t))]\,\mathrm{d}t \le 0. \tag{16}$$

Since h is convex, then $-h'$ is non-increasing; Since g is non-increasing, then θ' is also non-increasing. Consequently, the composite function $-h'(\theta'(t))$ is non-increasing with respect to t. Utilizing Lemma 1 leads to

$$\begin{aligned}
&\int_a^b [\psi'(t) - \theta'(t)]h'(\theta'(t))\,\mathrm{d}t \\
&\qquad\qquad = h'(\theta'(a))\int_a^\xi [\psi'(t) - \theta'(t)]\,\mathrm{d}t + h'(\theta'(b))\int_\xi^b [\psi'(t) - \theta'(t)]\,\mathrm{d}t \\
&\qquad\qquad = [\psi(\xi) - \theta(\xi)][h'(g(a)) - h'(g(b))] \\
&\qquad\qquad \ge 0,
\end{aligned}$$

where $\xi \in [a, b]$, since $\psi(\xi) \ge \theta(\xi)$ and $h'(g(a)) \ge h'(g(b)$. The proof of Theorem 2 is complete. □

References

[1] *The 11th International Mathematical Competition for University Students, Skopje, Macedonia, 25–26 July 2004, Solutions for problems on Day 2*, Available online at `http://www.imc-math.org.uk/index.php?year=2004` or `http://www.imc-math.org.uk/imc2004/day2_solutions.pdf`.

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