New Approximate Schemes for Generalized General Set-Valued Mixed Quasi Variational Inequalities

NEW APPROXIMATE SCHEMES FOR GENERALIZED GENERAL SET-VALUED MIXED QUASI VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we suggest and consider a class of new three-step approximation schemes for generalized general set-valued mixed quasi variational inequalities. We also consider and analyze a new class of extragradient-type methods for solving generalized set-valued variational inequalities. The proposed methods include several new and known methods as special cases. Our results present a significant improvement of previously known methods for solving variational inequalities and related optimization problems.

1. Introduction

In recent years, variational inequalities have been generalized and extended in many different directions using novel and innovative techniques to study wider classes of unrelated problems in mechanics, physics, optimization and control, nonlinear programming, economics, regional, structural, transportation, elasticity, and applied sciences, etc, see [1]-[9] and the reference therein. An important and useful generalization of variational inequalities is called generalized general set-valued mixed quasi variational inequality involving the nonlinear bifunction, which is introduced and studied by Chao Feng Shi, San Yang Liu and Jun Li Lian [1]. Chao Feng Shi [1] prove the existence of the solution of the auxiliary problem for the generalized general set-valued mixed quasi variational inequalities, and suggest a predictor-corrector method for solving the generalized general set-valued mixed quasi variational inequalities by using the auxiliary principle techniques. In this paper, we suggest and analyze a new class of three-step approximation schemes for solving generalized general mixed quasi variational inequalities and related problems. These new methods include the Mann and Ishikawa iterative schemes and modified forward-backward splitting methods of Noor [10] as special cases. Inspired and motivated by the recent research [11-15], we suggest and analyze a modified extragradient methods for solving generalized general set-valued variational inequalities. Our results represent an important improvement and refinement of the previously known results in this field.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $< \cdot, \cdot >$and $\| \cdot \|$, respectively. Let K be a nonempty closed convex set in H. Let $\phi(\cdot, \cdot) : H \times H \to H, g : H \to H$, consider the problem of finding $u \in H, w \in Tu, y \in$

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Vu such that

$$< N(w,y), g(v)-g(u) > +\phi(g(v),g(u)) - \phi(g(u),g(u)) \geq 0, \forall g(v) \in H. \quad (2.1)$$

If $\phi \equiv 0, g: H \to K$, then variational inequality (2.1) reduces to the following:

$$< N(w,y), g(v)-g(u) > \geq 0, \forall g(v) \in K, \quad (2.2)$$

which is called the generalized general set-valued variational inequalities. An inequality of type (2.1) is called a generalized set-valued mixed quasi variational inequality, which was introduced and studied by Chao Feng Shi et al. [1] in 2003. It turned out that many problems arising in various branches of pure and applied sciences can be studied via generalized general set-valued mixed quasi variational inequalities (see [1]).

We need the following concepts.

Definition 2.1 $N(\cdot,\cdot): H \times H \to H$ is said to be

(1) g-strongly monotone with respect to the set-valued mapping T, if there exists a constant $\alpha > 0$ such that

$$< N(w_1,\cdot) - N(w_2,\cdot), g(u_1) - g(u_2) > \geq \alpha \|g(u_1) - g(u_2)\|^2,$$

where $w_1 \in Tu_1, w_2 \in Tu_2$;

(2) g-monotone with respect to the set-valued mapping V, if

$$< N(w_1,\cdot) - N(w_2,\cdot), g(u_1) - g(u_2) > \geq 0,$$

where $w_1 \in Tu_1, w_2 \in Tu_2$;

(3) g-pseudomonotone with respect to the set-valued mapping T and V, if $< N(w,y), g(v) - g(u) > \geq 0$, implies

$$< N(\bar{w},\bar{y}), g(v) - g(u) > \geq 0,$$

where $w \in Tu, y \in Vu, \bar{w} \in Tu, \bar{y} \in Vu$;

(4) Lipschitz continuous with respect to the set-valued mapping T, if there exists a constant $\beta > 0$ such that

$$\|N(w_1,\cdot) - N(w_2,\cdot)\| \leq \beta \|w_1 - w_2\|.$$

Definition 2.2 $T: H \to CB(H)$ is said to be M-Lipschitz continuous, if there exists a constant $\mu > 0$ such that

$$D(Tu,Tv) \leq \mu \|u - v\|,$$

where $D(\cdot,\cdot)$ is a Hausdorff metric.

3. Three -step approximation schemes

In this section, we suggest and analyze some new approximation schemes for solving generalized general set-valued mixed quasi variational inequality (2.1).

Lemma 3.1 The function $u \in H, w \in Tu, y \in Vu$ is a solution of the variational inequality (2.1) if and only if $u \in H$ satisfies the relation

$$g(u) = J_{\phi(g(u))}[g(u) - \rho N(w,y)], \quad (3.1)$$

where $J_{\phi(g(u))} = [I + \rho\partial\phi(g(u))]^{-1}$ is the resolvent operator, $\phi(u) = \phi(\cdot,u)$, $\rho > 0$ is a constant and g is inverse.

Proof.

$$g(u) = [I + \rho\partial\phi(g(u))]^{-1}[g(u) - \rho N(w,y)]$$
$$\iff -N(w,y) \in \partial\phi(g(u))$$

$$\Longleftrightarrow < N(w,y), g(v)-g(u) > +\phi(g(u),g(v)) - \phi(g(u),g(u)) \geq 0$$

$\Longleftrightarrow u \in H, w \in Tu, y \in Vu$ is a solution of the variational inequality (2.1).

Lemma 3.1 implies that the generalized general set-valued mixed quasi variational inequality (2.1) is equivalent to the fixed point problem. This alternative equivalent formulation is very useful from the numerical and theoretical points of view. The relation (3.1) can be written as

$$F(u) = u - g(u) + J_{\phi(g(u))}[g(u) - \rho N(w,y)]. \tag{3.2}$$

We now study the conditions under which the generalized general set-valued mixed quasi variational inequality (2.1) has a unique solution and this is the main motivation of our next results.

Assumption 3.1

$$\|J_{\phi(g(u))}(\cdot) - J_{\phi(g(v))}(\cdot)\| \leq c\|g(u) - g(v)\|.$$

Theorem 3.1 Let N, T, V, g satisfy the following conditions.
(1) $N(\cdot,\cdot) : H \times H \to H$ is strongly monotone with respect to set-valued mapping T and a constant $\alpha > 0$;
(2) $N(\cdot,\cdot) : H \times H \to H$ is monotone with respect to set-valued mapping V;
(3) $N(\cdot,\cdot) : H \times H \to H$ is Lipschitz continuous with respect to the first argument and a constant $\beta > 0$, and Lipschitz continuous with respect to the second argument and a constant $\gamma > 0$;
(4) $T : H \to CB(H)$ is M-Lipschitz continuous with a constant $\mu > 0$;
(5) $V : H \to CB(H)$ is M-Lipschitz continuous with a constant $\xi > 0$;
(6) $g : H \to H$ is Lipschitz continuous with a constant $\delta > 0$ and strong monotone with a constant $\sigma > 0$. If Assumption 3.1 holds and

$$|\rho - \frac{\alpha}{(\mu\beta+\gamma\xi)^2}| < \frac{\sqrt{\alpha^2 - (\mu\beta+\gamma\xi)^2 k(2-k)}}{\mu\beta+\gamma\xi}, \alpha > (\mu\beta+\gamma\xi)\sqrt{k(2-k)}, k + c\delta < 1, \tag{3.3}$$

where

$$k = 2\sqrt{1 - 2\sigma + \delta^2}, \tag{3.4}$$

then there exists a unique solution $u \in H, g(u) \in K, w \in Tu, y \in Vu$ of the generalized general set-valued mixed quasi variational inequality (2.1).

Proof. From lemma 3.1, it follows that problem (3.2) and (2.1) are equivalent. Thus it is enough to show that the map $F(u)$ has a fixed point. For all $u, v \in H, w \in Tu, y \in Vu, w' \in Tv, y' \in Vv$,

$$\begin{aligned} &\|F(u) - F(v)\| \\ &= \|u - v - (g(u) - g(v)) + J_{\phi(g(u))}[g(u) - \rho N(w,y)] - J_{\phi(g(v))}[g(v) - \rho N(w',y')]\| \\ &\leq \|u - v - (g(u) - g(v))\| + \|J_{\phi(g(u))}[g(u) - \rho N(w,y)] - J_{\phi(g(v))}[g(v) - \rho N(w',y')]\| \\ &\leq \|u - v - (g(u) - g(v))\| + \|g(u) - \rho N(w,y)] - (g(v) - \rho N(w',y'))\| + c\|g(u) - g(v)\| \\ &\leq 2\|u - v - (g(u) - g(v))\| + \|u - v - \rho(N(w,y) - N(w',y'))\| + c\|g(u) - g(v)\|, \end{aligned} \tag{3.5}$$

where we have used the fact that the operator $J_{\phi(g(u))}$ is nonexpansive and Assumption 3.1.

From conditions (1)-(5), we have

$$\begin{aligned} &\|u - v - \rho(N(w,y) - N(w',y'))\|^2 \\ &\leq \|u - v\|^2 - 2\rho < N(w,y) - N(w',y'), u - v > +\rho^2\|N(w,y) - N(w',y')\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2(\mu\beta+\gamma\xi)^2)\|u - v\|^2. \end{aligned} \tag{3.6}$$

In a similar way, from conditions (6), we have

$$\|u - v - (g(u) - g(v))\|^2 \leq (1 - 2\sigma + \delta^2)\|u - v\|^2. \tag{3.7}$$

From (3.5), (3.6)and (3.7), we have

$$\begin{aligned} \|F(u) - F(v)\| &\leq (2\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\rho\alpha + \rho^2(\mu\beta + \gamma\xi)^2})\|u - v\| \\ &= (k + t(\rho) + c\delta)\|u - v\| \\ &= \theta\|u - v\|, \end{aligned} \tag{3.8}$$

where

$$t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2(\mu\beta + \gamma\xi)^2} \tag{3.9}$$

and

$$\theta = k + t(\rho) + c\delta. \tag{3.10}$$

From (3.3), if follows that $\theta < 1$, which implies that the map has a fixed point, which is a unique solution of (2.1). The proof is completed.

Using the auxiliary principle technique, see Chaofeng Shi, Sanyang Liu and Junli Lian [1], we can suggest the predictor-corrector type algorithm for solving the generalized general set-valued mixed quasi variational inequalities (2.1).

We now suggest another three-step approximation scheme for solving the generalized general set-valued mixed quasi variational inequality (2.1).

Algorithm 3.1 For a given $u_0 \in H, w_0 \in Tu_0, y_0 \in Vu_0$, compute the approximate solution by the iterative schemes.

$$p_n = (1 - \gamma_n)u_n + \gamma_n\{u_n - g(u_n) + J_{\phi(g(u_n))}[g(u_n) - \rho N(w_n, y_n)]\}, \tag{3.11}$$

$$q_n = (1 - \beta_n)u_n + \beta_n\{p_n - g(p_n) + J_{\phi(g(p_n))}[g(p_n) - \rho N(w'_n, y'_n)]\}, \tag{3.12}$$

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{q_n - g(q_n) + J_{\phi(g(q_n))}[g(q_n) - \rho N(\xi_n, \eta_n)]\}, \tag{3.13}$$

$$\begin{gathered} w_n \in Tu_n, \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n+1})D(Tu_n, Tu_{n+1}), \\ y_n \in Vu_n, \|y_n - y_{n+1}\| \leq (1 + \frac{1}{n+1})D(Vu_n, Vu_{n+1}), \\ w'_n \in Tp_n, \|w'_n - w'_{n+1}\| \leq (1 + \frac{1}{n+1})D(Tp_n, Tp_{n+1}), \\ y'_n \in Vu_n, \|y'_n - y'_{n+1}\| \leq (1 + \frac{1}{n+1})D(Vp_n, Vp_{n+1}), \\ \xi_n \in Tq_n, \|\xi_n - \xi_{n+1}\| \leq (1 + \frac{1}{n+1})D(Tq_n, Tq_{n+1}), \\ \eta_n \in Vq_n, \|\eta_n - \eta_{n+1}\| \leq (1 + \frac{1}{n+1})D(Vq_n, Vq_{n+1}), \\ n = 0, 1, 2, \cdots, \end{gathered}$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$; for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges.

Now we suggest a perturbed iterative scheme for solving the generalized general set-valued mixed quasi variational inequality (2.1).

Algorithm 3.2 For a given $u_0 \in H, w_0 \in Tu_0, y_0 \in Vu_0$, compute the approximate solution u_n by the iterative schemes.

$$\begin{aligned} p_n &= (1 - \gamma_n)u_n + \gamma_n\{u_n - g(u_n) + J_{\phi(g(u_n))}[g(u_n) - \rho N(w_n, y_n)]\} + \gamma_n h_n, \\ q_n &= (1 - \beta_n)u_n + \beta_n\{p_n - g(p_n) + J_{\phi(g(p_n))}[g(p_n) - \rho N(w'_n, y'_n)]\} + \beta_n f_n, \\ u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n\{q_n - g(q_n) + J_{\phi(g(q_n))}[g(q_n) - \rho N(\xi_n, \eta_n)]\} + \alpha_n e_n, \end{aligned}$$

$$w_n \in Tu_n, \|w_n - w_{n+1}\| \leq (1+\frac{1}{n+1})D(Tu_n, Tu_{n+1}),$$
$$y_n \in Vu_n, \|y_n - y_{n+1}\| \leq (1+\frac{1}{n+1})D(Vu_n, Vu_{n+1}),$$
$$w'_n \in Tp_n, \|w'_n - w'_{n+1}\| \leq (1+\frac{1}{n+1})D(Tp_n, Tp_{n+1}),$$
$$y'_n \in Vu_n, \|y'_n - y'_{n+1}\| \leq (1+\frac{1}{n+1})D(Vp_n, Vp_{n+1}),$$
$$\xi_n \in Tq_n, \|\xi_n - \xi_{n+1}\| \leq (1+\frac{1}{n+1})D(Tq_n, Tq_{n+1}),$$
$$\eta_n \in Vq_n, \|\eta_n - \eta_{n+1}\| \leq (1+\frac{1}{n+1})D(Vq_n, Vq_{n+1}),$$
$$n = 0, 1, 2, \cdots,$$

where $\{e_n\}, \{f_n\}$ and $\{h_n\}$ are sequences of element of H introduced to take into account possible inexact computations and $J_{\phi_n(\cdot)}$ is the corresponding perturbed resolvent operator, and the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges.

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative schemes and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving the generalized general set-valued quasi variational inequality (2.1).

In brief, for a suitable and appropriate choice of the operators N, T, V, g and the space H, one can obtain a number of new and previously known iterative schemes for solving variational inequalities and related problems.

We now study the convergence criteria of Algorithm 3.1. In a similar way, one can analyze the convergence criteria of Algorithm 3.2.

Theorem 3.2 Let the operator N, T, V, g satisfy all the assumptions of theorem 3.1, if the condition (3.3) and assumption 3.1 holds, then the approximate solution $\{u_n\}, \{w_n\}, \{y_n\}$ obtained from algorithm 3.1 converge to the exact solution $u \in H, w \in Tu, y \in Vu$ of the generalized general set-valued mixed quasi variational inequality (2.1) strongly in H, respectively.

Proof. From Theorem 3.1, we see that there exists a solution $u \in H, w \in Tu, y \in Vu$ of the generalized general set-valued mixed quasi variational inequality (2.1). Then, using lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n\{u - g(u) + J_{\phi(g(u))}[g(u) - \rho N(w, y)]\} \tag{3.14}$$
$$= (1-\beta_n)u + \beta_n\{u - g(u) + J_{\phi(g(u))}[g(u) - \rho N(w, y)]\} \tag{3.15}$$
$$= (1-\gamma_n)u + \gamma_n\{u - g(u) + J_{\phi(g(u))}[g(u) - \rho N(w, y)]\} \tag{3.16}$$

From (3.13) and (3.14), we have

$$\begin{aligned} \|u_{n+1} - u\| &= \|(1-\alpha_n)(u_n - u) + \alpha_n(q_n - u - g(q_n) - g(u)))+ \\ &\alpha_n\{J_{\phi(g(q_n))}[g(q_n) - \rho N(\xi_n, \eta_n)] - J_{\phi(g(u))}[g(u) - \rho N(w, y)]\}\| \\ &\leq (1-\alpha_n)\|u_n - u\| + 2\alpha_n\|q_n - u - g(q_n) - g(u)))\|+ \\ &\alpha_n\|g(q_n) - g(u) - \rho(N(\xi_n, \eta_n) - N(w, y))\| + c\delta\|q_n - u\| \\ &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n(k + t(\rho) + c\delta)\|q_n - u\|, \\ &= (1-\alpha_n)\|u_n - u\| + \alpha_n\theta\|q_n - u\|, Using (3.10) \end{aligned} \tag{3.17}$$

In a similar way, from (3.4),(3.9),(3.10),(3.12) and (3.15), we have

$$\|q_n - u\| \leq (1-\beta_n)\|u_n - u\| + 2\beta_n\|p_n - u - (g(p_n) - g(u))\|$$

$$+\beta_n\|g(p_n)-g(u)-\rho(N(\xi_n,\eta_n)-N(w,y))\|$$
$$\leq(1-\beta_n)\|u_n-u\|+\beta_n(k+t(\rho)+c\delta)\|p_n-u\|,$$
$$=(1-\beta_n)\|u_n-u\|+\beta_n\theta\|p_n-u\|, \tag{3.18}$$

and from (3.10),(3.11) and (3.16), we obtain

$$\|p_n-u\|=(1-\gamma_n)\|u_n-u\|+\gamma_n\theta\|u_n-u\|,$$
$$\leq(1-(1-\theta)\gamma_n)\|u_n-u\|$$
$$\leq\|u_n-u\|. \tag{3.19}$$

From (3.18) and (3.19), we obtain

$$\|q_n-u\|\leq(1-\beta_n)\|u_n-u\|+\beta_n\theta\|u_n-u\|=(1-(1-\theta)\beta_n)\|u_n-u\|\leq\|u_n-u\|. \tag{3.20}$$

Combining (3.17) and (3.20), we have

$$\|u_{n+1}-u\|\leq(1-\alpha_n)\|u_n-u\|+\alpha_n\theta\|u_n-u\|=[1-(1-\theta)\alpha_n]\|u_n-u\|$$
$$\leq\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|u_0-u\|.$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\prod_{i=0}^{\infty}[1-(1-\theta)\alpha_i]=0$, consequently the sequence $\{u_n\}$ converges strongly to u. From (3.18) and (3.19), it follows that the sequences $\{p_n\}$ and $\{q_n\}$ also converge to u strongly in H. Since $\{w_n\}$ is a Cauchy sequence, there exists $w\in H$ such that $\{w_n\}$ strongly converges to w in H. Also, we have there exists $y\in H$ such that $\{y_n\}$ strongly converges to y in H. Since $\{w_n\}$ is a Cauchy sequence, there exists $w\in H$ such that $\{w_n\}$ strongly converges to w in H. Also, we have there exists $y\in H$ such that $\{y_n\}$ strongly converges to y in H.
Since

$$d(w,Tu)\leq d(w,w_n)+D(Tu_n,Tu)\to 0, w\in Tu.$$

Also, we have $y\in Vu$.

4. New extragradient-type methods for generalized set-valued variational inequalities.

Lemma 4.1 For a given $z\in H, u\in H$ satisfies the inequality $<u-z,v-u>\leq 0$, for all $v\in K$, if and only if

$$u=P_k[z], \tag{4.1}$$

where P_k is the projection of H onto K. Also, the projection operator P_k is nonexpansive and satisfies the inequality

$$\|P_K[z]-u\|^2\leq\|z-u\|^2-\|z-P_k[z]\|^2.$$

In this section, we use the projection technique to suggest and analyze extragradient-type methods for solving the generalized general set-valued variational inequalities (2.2). For this purpose, we need the following result, which can be proved by invoking lemma 4.1.

Lemma 4.2 The functions $u\in H, g(u)\in K, w\in Tu, y\in Vu$ is a solution of (2.2) if and only if

$$g(u)=P_k[g(u)-\rho N(w,y)], \tag{4.2}$$

where $\rho>0$ is a constant and g is onto K.

Lemma 4.2 implies that problem (2.2) and (4.2) are equivalent. This alternative formulation is very important from the numerical analysis point of view. We invoke this fixed-point formulation to suggest the following the extragradient-type method for solving generalized set-valued variational inequality (2.2).

Let

$$R(u) = g(u) - P_k[g(u) - \rho N(w, y)], \tag{4.3}$$

Algorithm 4.1 For a given $u_0 \in H, w_0 \in Tu_0, y_0 \in Vu_0$, compute the approximate solution u_{n+1} by the iterative schemes.

Predictor step

$$g(p_n) = g(u_n) - \eta_n R(u_n), \tag{4.4}$$

where $\eta_n = \alpha_{m_k}$, and m_k is the smallest nonnegative integer m such that

$$\rho_n \eta_n < N(w_n, y_n) - N(w'_n, y'_n), R(u_n) >\leq \sigma \|R(u_n)\|^2, \sigma \in (0, 1) \tag{4.5}$$

$$w_n \in Tu_n, y_n \in Vu_n, w'_n \in Tg^{-1}(g(u_n) - \alpha^{m_k} R(u_n)), y'_n \in Vg^{-1}(g(u_n) - \alpha^{m_k} R(u_n)), \tag{4.6}$$

Corrector step

$$g(u_{n+1}) = P_K[g(u_n) - \alpha_n N(w'_n, y'_n)], w'_n \in Tg^{-1}(g(u_n) - \eta_n R(u_n)), \tag{4.7}$$

$$y'_n \in Vg^{-1}(g(u_n) - \eta_n R(u_n)), \tag{4.8}$$

where

$$\alpha_n = \frac{(\eta_n - \sigma)\|R(u_n)\|^2}{\|N(w'_n, y'_n)\|^2}. \tag{4.9}$$

For the convergence analysis of Algorithm 4.1 , we need the following results.

Lemma 4.3 Let $\bar{u} \in H, \bar{w} \in Tu, \bar{y} \in Vu$ be a solution of (2.2), if the operator $N(\cdot, \cdot)$ is g-pseudomonotone with respect to set-valued operator T and V, then

$$< g(u) - g(\bar{u}), N(w', y') >\geq (\eta - \sigma)\|R(u)\|^2, \forall u \in H, \tag{4.10}$$

$$w' \in Tg^{-1}(g(u) - \eta R(u)),$$

$$y' \in Vg^{-1}(g(u) - \eta R(u)).$$

Proof. Let $\bar{u} \in H, g(\bar{u}) \in K, \bar{w} \in T\bar{u}, \bar{y} \in V\bar{u}$ be a solution of (2.2), then

$$< N(\bar{w}, \bar{y}), g(v) - (\bar{u}) >\geq 0, \forall g(v) \in K.$$

This implies

$$< N(\tilde{w}, \tilde{y}), g(v) - g(\bar{u}) \geq 0,$$

where

$$\tilde{w} \in Tv, \tilde{y} \in Vv, \tag{4.11}$$

since $N(\cdot, \cdot)$ is g-pseudomonotone with respect to set-valued mapping T and V. Now taking $g(v) = g(u) - \eta R(u)$ in (4.11), we obtain $< N(w', y'), g(u) - \eta R(u) - g(\bar{u}) >\geq 0$, from which we have

$$\begin{aligned} &< g(u) - g(\bar{u}), \rho N(w', y') > \\ &> \eta\rho < R(u), N(w', y') > \\ &\geq -\eta\rho < R(u), N(w, y) - N(w', y') > + \eta\rho < N(w, y), R(u) > \\ &\geq -\sigma\|R(u)\|^2 + \rho\eta < N(w, y), R(u) > . \end{aligned} \tag{4.12}$$

Taking $z = g(u) - \rho N(w, y), u = P_K[g(u) - \rho N(w, y)], v = g(u)$ in (4.1), we obtain

$$< P_K[g(u) - \rho N(w, y)] - g(u) + \rho N(w, y), g(u) - P_k[g(u) - \rho N(w, y)] >\geq 0,$$

from which it follows that

$$< \rho N(w, y), R(u) >\geq \|R(u)\|^2. \tag{4.13}$$

Combining (4.12) and (4.13) , we have

$$< g(u) - g(\bar{u}), \rho N(w', y') >\geq (\eta - \sigma)\|R(u)\|^2,$$

the required results.

Lemma 4.4 Let $\bar{u} \in H, \bar{w} \in T\bar{u}, \bar{y} \in V\bar{u}$ be a solution of (2.2) and let u_{n+1} be the approximate solution from algorithm 4.1, then

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2. \tag{4.12}$$

where

$$w_n \in Tg^{-1}(g(u_n) - \eta_n R(u_n)), y_n \in Vg^{-1}(g(u_n) - \eta_n R(u_n)).$$

Proof. From (4.7)-(4.10), we have

$$\|g(u_{n+1} - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{[u]}) - \alpha_n N(w_n', y_n')\|^2$$
$$\leq \|g(u_n) - g(\bar{u})\|^2 - 2\alpha_n < g(u_n) - g(\bar{u}), N(w_n', y_n') > +\alpha_n^2\|N(w_n', y_n')\|^2$$
$$\leq \|g(u_n) - g(\bar{u})\|^2 - 2\alpha_n(\eta_n - \sigma)\|R(u_n)\|^2 + \alpha_n^2\|N(w_n', y_n')\|^2$$
$$\leq \|g(u_n) - g(\bar{u})\|^2 - \frac{(\eta_n - \sigma)^2\|R(u_n)\|^4}{\|N(w_n', y_n')\|^2},$$

the required results.

Theorem 4.1 Let $u_{n+1} \in H, w_{n+1} \in Tu_{n+1}, y_{n+1} \in Vu_{n+1}$, be the approximate solution obtained form algorithm 4.1 and $\bar{u} \in H, \bar{w} \in T\bar{u}, \bar{y} \in V\bar{u}$ be the solution of (2.2). If H is a finite dimensional space and g is injective, then

$$\lim_{n\to\infty} u_n = \bar{u}.$$

Proof. Let $u^* \in H, w^* \in Tu^*, y^* \in Vu^*$ be a solution of (2.2). Then, from (4.14), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \frac{(\eta_n - \sigma)^2\|R(u_n)\|^4}{\|N(w_n', y_n')\|^2} \leq \|g(u_0) - g(u^*)\|^2,$$

which implies that either

$$\lim_{n\to\infty} R(u_n) = 0 \tag{4.15}$$

or

$$\lim_{n\to\infty} \eta_n = 0. \tag{4.16}$$

Assume that (4.15) holds. Let $\bar{u} \in H$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\bar{u}$.
Since R is continuous, if follows that

$$R(\bar{u}) = \lim_{j\to\infty} R(u_{n_j}) = 0,$$

which implies that $\bar{u}$ is a solution of (2.2) by imoking lemma 4.2 and

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2.$$

Thus the sequence $\{u_n\}$ has exactly one cluster point and consequently

$$\lim_{n\to\infty} g(u_n) = g(\bar{u}).$$

Since g is injective, if follows that

$$\lim_{n\to\infty} u_n = \bar{u} \in H$$

Assume that (4.16) holds, that is, $\lim_{n\to\infty}\eta_n = 0$. If (4.5) does not hold, then by a choice of η_n,we obtain

$$\sigma\|R(u_n)\|^2 \leq \rho_n\eta_n < N(w_n, y_n) - N(w'_n, y'_n), R(u_n) > . \tag{4.17}$$

Let $\bar{u}$ be a cluster point of $\{u_n\}$ and let $\{u_{n_j}\}$ be the corresponding subsequence of $\{u_n\}$ converging to $\bar{u}$. Taking the limit in (4.17), we have

$$\sigma\|R(\bar{u})\|^2 \leq 0,$$

which implies that $R(\bar{u})$. Repeating the above argument, we conclude that

$$\lim_{n\to\infty} u_n = \bar{u}.$$

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