A Logarithmically Completely Monotonic Function Involving the Gamma Function

This is the Published version of the following publication


The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository https://vuir.vu.edu.au/17631/
A LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION INVOLVING THE GAMMA FUNCTION

MIAO-QING AN

Abstract. In this paper, we present some logarithmically completely monotonic functions. Furthermore, we obtain a function similar to an open problem given by FENG QI in 2004.

1. Introduction and Main Results
A function \( f \) is said to be complete monotonic in an interval \( I \) if \( f \) has derivatives of all orders in \( I \) which alternate successively in sign, that is
\[
(-1)^n f^{(n)}(x) \geq 0 \quad (1)
\]
for \( x \in I \) and \( n \geq 0 \). If inequality (1) is strict for all \( x \in I \) and for all \( n \geq 0 \), then \( f \) is said to be strictly complete monotonic.

Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory \([5]\), probability theory \([7, 9, 11]\), physics \([8]\), numerical and asymptotic analysis \([10, 13]\), and combinatorics \([3]\). A detailed collection of the most important properties of completely monotonic functions can be found in \([12]\), and in an abstract in \([6]\).

A function \( f \) is said to be logarithmically completely monotonic on an interval \( I \) if its logarithm \( \ln f \) satisfies
\[
(-1)^k \ln f^{(k)}(x) \geq 0 \quad (2)
\]
for \( k \in \mathbb{N} \) on \( I \). If inequality (2) is strict for all \( x \in I \) and for all \( k \geq 1 \), then \( f \) is said to be strictly logarithmically completely monotonic. A (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic \([4]\). But not conversely, since a convex function may not be logarithmically convex (see Remark 1.16 at page 7 in \([14]\)).

The classical gamma function
\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \, dt \quad (\text{Re} \ z > 0) \quad (3)
\]
is one of the most important functions in analysis and its applications. The history and development of this function are described in detail \([1]\). The psi or digamma function \( \psi(x) = \frac{\Gamma(x)}{\Gamma(x)} \), the logarithmic derivative of the gamma function, and the polygamma functions can be expressed \([2]\) for \( x > 0 \) and \( k \in \mathbb{N} \) as
\[
\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, dt, \quad (4)
\]
\[
\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} \, dt, \quad (5)
\]
where \( \gamma = 0.57721566490153286 \ldots \) is the Euler-Mascheroni constant.

2000 Mathematics Subject Classification. 33B15; 26D07.
Key words and phrases. Gamma function, logarithmically complete monotonicity.
In [16] it is proved that the function \( \frac{\Gamma(x+1)^{\frac{1}{x}}}{x^e} (1 + \frac{1}{x})^x \) is strictly logarithmically completely monotonic on \((0, \infty)\). In this paper we establish a similar function and give its logarithmically complete monotonicity.

Let \( a, b, c, \) and \( d \) be real numbers, and define function

\[
F(x) = \frac{\Gamma(x+1)^{\frac{1}{x}}}{x^e} (1 + \frac{a}{x})^{x+b}, \quad x > 0
\]

Theorem 1. The function \( f(x) = \frac{\Gamma(x+1)^{\frac{1}{x}}}{x^e} \) is strictly logarithmically completely monotonic on \((0, \infty)\) where \( c \geq 0 \) and \( d < 0 \).

Theorem 2. The function \( g(x) = (1 + \frac{a}{x})^{x+b} \) is strictly logarithmically completely monotonic on \((0, \infty)\) where \( a > 0 \).

Theorem 3. The function \( F(x) \) defined by (6) is strictly logarithmically completely monotonic on \((0, \infty)\) where \( a > 0, c \geq 0, d < 0 \).

2. Proofs of Theorems

Proof of Theorem 1.

Considering \( f(x) \), taking logarithm and differentiation yields

\[
(\log f(x))' = d^{x(x+1) - \log \Gamma(x+1)} - \frac{c}{x}
\]

and

\[
(\log f(x))^{(n)} = d^{x(x+1)} + (-1)^n(n-1)! \frac{c}{x^n}
\]

where \( n \geq 2, \psi^{(-1)}(x+1) = \log \Gamma(x+1), \psi^{(0)}(x+1) = \psi(x+1), \) and

\[
g_n(x) = \sum_{k=0}^{n} \frac{(-1)^{(n-k)!}x^k \psi^{(k-1)}(x+1)}{k!}
\]

\[
g_n'(x) = x^n \psi^{(n)}(x+1) \begin{cases} > 0, & \text{if } n \text{ is odd}, \\ < 0, & \text{if } n \text{ is even}. \end{cases}
\]

Let \( h(x) = (-1)^n x^{n+1} (\log f(x))^{(n)} \), we have

\[
h'(x) = (-1)^n dx^n \psi^{(n)}(x+1) + (-1)^{2n} c(n-1)!
\]

It is easy to know \( h'(x) > 0 \) where \( c \geq 0 \) and \( d < 0 \). Thus the function \( (-1)^n x^{n+1} (\log f(x))^{(n)} \) is increasing in \((0, \infty)\). Since

\[
\lim_{x \to 0} \left\{ (-1)^n x^{n+1} (\log f(x))^{(n)} \right\} = 0
\]

we have \( (-1)^n x^{n+1} (\log f(x))^{(n)} \), then \( (-1)^n (\log f(x))^{(n)} > 0 \) for \( n \geq 2 \) in \((0, \infty)\). Since \( (\log f(x))^{(n)} > 0 \), the function \( (\log f(x))^{(n)} \) increases. It is easy to see

\[
\lim_{x \to \infty} (\log f(x))^{(n)} = 0
\]

so \( (\log f(x))^{(n)} \) is decreasing in \((0, \infty)\). The Proof of Theorem 1 is complete.
Proof of Theorem 2.

Taking the logarithm of \( g(x) \) and differentiating yields

\[
\log g(x) = (x + b)(\log(x + a) - \log x)
\]  
(13)

\[
[\log g(x)]' = \log(x + a) - \log x + \frac{b - a}{x + a} - \frac{b}{x}
\]  
(14)

and for \( n \geq 2 \),

\[
[\log g(x)]^{(n)} = (-1)^{n-2} \frac{(n - 2)!}{(x + a)^{n-1}} + (-1)^{n-1} \frac{(n - 1)!}{x^{n-1}}
\]

\[
(-1)^{n-1} \frac{(b - a)(n - 1)!}{x^n} + (-1)^n \frac{b(n - 1)!}{x^n}
\]

It is also known that (see [15], p.884)

\[
\frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1}e^{-xt}dt, (n \in \mathbb{N}; x \in \mathbb{R}^+).
\]  
(15)

Hence,

\[
(-1)^n[\log f(x)]^{(n)} = \int_0^\infty t^{n-2}e^{-(x+a)t}dt - \int_0^\infty t^{n-2}e^{-xt}dt
\]

\[
-(b - a) \int_0^\infty t^{n-1}e^{-xt}dt + b \int_0^\infty t^{n-1}e^{-xt}dt
\]

\[
= \int_0^\infty [1 - e^{at} - (b - a)te^{at} + bte^{at}]t^{n-2}e^{-(x+a)t}dt
\]

\[
\triangleq \int_0^\infty \phi(t)t^{n-2}e^{-(x+a)t}dt
\]

\[
\phi(0) = 0
\]

\[
\phi'(t) = a^2te^{at}
\]

So we have \( \phi'(t) > 0 \) for \( a > 0 \). Then we obtain that \( \phi(t) \) is strictly increasing in \((0, \infty)\). As a result of \( \phi(0) = 0 \), we obtain \( \phi(t) > 0 \) in \((0, \infty)\). This means that \((-1)^n(\log g(x))^{(n)} > 0\) for \( n \geq 2 \) in \((0, \infty)\). Since \( [\log g(x)]'' > 0 \), the function \([\log g(x)]'\) is increasing.

It is not difficult to know that

\[
\lim_{x \to \infty} [\log g(x)]' = 0,
\]  
(16)

so \([\log g(x)]' < 0\) and \(\log g(x)\) is decreasing in \((0, \infty)\). In conclusion, the function \(\log g(x)\) is strictly completely monotonic in \((0, \infty)\). The Proof of Theorem 2 is complete.

Proof of Theorem 3.
It is easy to know that the product of (strictly) logarithmically completely monotonic functions is also (strictly) logarithmically completely monotonic functions. Write
\[ F(x) = f(x)g(x). \]  
(17)

Clearly, the function \( F(x) \) is strictly logarithmically completely monotonic on \((0, \infty)\). The Proof of Theorem 3 is complete.

REFERENCES


(miao-qing an) College of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan 454003, China

E-mail address: anxv1013@126.com, anxv1013@163.com