# *A Logarithmically Completely Monotonic Function Involving the Gamma Function*

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# A LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION INVOLVING THE GAMMA FUNCTION

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Abstract. In this paper ,we present some logarithmically completely monotonic functions . Furthermore ,we obtain a function similar to an open problem given by FENG QI in 2004.

## 1. Introduction and Main resluts

A function $f$ is said to be complete monotonic in an interval $I$ if $f$ has derivatives of all orders in $I$ which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1}$$

for $x \in I$ and $n \geq 0$.If inequality (1) is strict for all $x \in I$ and for all $n \geq 0$,then $f$ is said to be strictly complete monotonic.

Completely monotonic functions have remarkable applications in different branches . For instance ,they play a role in potential theory [5], probability theory [7, 9, 11] , physics[8] , numerical and asymptotic analysis [10, 13], and combinatorics [3]. A detailed collection of the most important properties of completely monotonic functions can be found in [12], and in an abstract in [6].

A function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0 \tag{2}$$

for $k \in \mathbb{N}$ on $I$. If inequality (2) is strict for all $x \in I$ and for all $k \geq 1$, then $f$ is said to be strictly logarithmically completely monotonic. A (strictly) logarthmically completely monotonic function is also (strictly) completely monotonic[4]. But not conversely , since a convex function may not be logarithmically convex (see Remark. 1.16 at page 7 in [14]).

The classical gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}t \quad (\operatorname{Re} z > 0) \tag{3}$$

is one of the most important functions in analysis and its applications.The history and development of this function are described in detail [1]. The psi or digamma function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed[2] for $x > 0$ and $k \in \mathbb{N}$ as

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \,\mathrm{d}t, \tag{4}$$

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} \,\mathrm{d}t, \tag{5}$$

where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant.

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In [16] it is proved that the function $\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{x}(1+\frac{1}{x})^x$ is strictly logarthmically completely monotonic on $(0,\infty)$.In this paper we establish a similar function and give its logarithmically complete monotonicity.

Let a,b,c, and d be real numbers , and define function

$$F(x)=\frac{[\Gamma(x+1)]^{\frac{d}{x}}}{x^c}(1+\frac{a}{x})^{x+b}, x>0 \tag{6}$$

**Theorem 1.** *The function* $f(x)=\frac{[\Gamma(x+1)]^{\frac{d}{x}}}{x^c}$ *is strictly logarithmically comletely monotonic on* $(0,\infty)$ *where* $c\geq 0$ *and* $d<0$.

**Theorem 2.** *The function* $g(x)=(1+\frac{a}{x})^{x+b}$ *is strictly logarithmically comletely monotonic on* $(0,\infty)$ *where* $a>0$.

**Theorem 3.** *The function* $F(x)$ *defined by* (6) *is strictly logarithmically comletely monotonic on* $(0,\infty)$ *where* $a>0$, $c\geq 0$, $d<0$.

## 2. Proofs of Theorems

Proof of Theorem 1.

Considering $f(x)$ ,taking logarithm and differentiation yields

$$\begin{aligned}(\log f(x))^{'} &= d\tfrac{x\psi(x+1)-\log\Gamma(x+1)}{x^2}-\tfrac{c}{x}\\ &= d\tfrac{\psi(x+1)}{x}-d\tfrac{\log\Gamma(x+1)}{x^2}-\tfrac{c}{x}\end{aligned}$$

and

$$(\log f(x))^{(n)}=\frac{dg_n(x)}{x^{n+1}}+(-1)^n(n-1)!\frac{c}{x^n} \tag{7}$$

where $n\geq 2$, $\psi^{(-1)}(x+1)=\log\Gamma(x+1)$, $\psi^{(0)}(x+1)=\psi(x+1)$, and

$$g_n(x)=\sum_{k=0}^{n}\frac{(-1)^{(n-k)}n!x^k\psi^{(k-1)}(x+1)}{k!} \tag{8}$$

$$g_n^{'}(x)=x^n\psi^{(n)}(x+1)\begin{cases}>0, & \text{if n is odd,}\\ <0, & \text{if n is even.}\end{cases} \tag{9}$$

Let $h(x)=(-1)^n x^{n+1}(\log f(x))^{(n)}$, we have

$$h^{'}(x)=(-1)^n dx^n\psi^{(n)}(x+1)+(-1)^{2n}c(n-1)! \tag{10}$$

It is easy to know $h^{'}(x)>0$ where $c\geq 0$ and $d<0$. Thus the function $(-1)^n x^{n+1}[\log f(x)]^{(n)}$ is increasing in $(0,\infty)$.Since

$$\lim_{x\to 0}\left\{(-1)^n x^{n+1}[\log f(x)]^{(n)}\right\}=0 \tag{11}$$

we have $(-1)^n x^{n+1}[\log f(x)]^{(n)}>0$ , then $(-1)^n[\log f(x)]^{(n)}>0$ for $n\geq 2$ in $(0,\infty)$. Since $[\log f(x)]^{''}>0$, the function $[\log f(x)]^{'}$ increases . It is easy to see

$$\lim_{x\to\infty}[\log f(x)]^{'}=0 \tag{12}$$

so $[\log f(x)]^{'}<0$ and $\log f(x)$ is decreasing in $(0,\infty)$. The Proof of Theorem 1 is complete.

Proof of Theorem 2.

Taking the logarithm of $g(x)$ and differentiating yields

$$\log g(x) = (x+b)(\log(x+a) - \log x) \tag{13}$$

$$[\log g(x)]^{'} = \log(x+a) - \log x + \frac{b-a}{x+a} - \frac{b}{x} \tag{14}$$

and for $n \geq 2$,

$$\begin{aligned} [\log g(x)]^{(n)} &= (-1)^{n-2}\frac{(n-2)!}{(x+a)^{n-1}} + (-1)^{n-1}\frac{(n-1)!}{x^{n-1}} \\ &\quad (-1)^{n-1}\frac{(b-a)(n-1)!}{x^n} + (-1)^n\frac{b(n-1)!}{x^n} \end{aligned}$$

It is also known that (see[15],p.884])

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-xt}dt, (n \in \mathbb{N}; x \in \mathbb{R}^+). \tag{15}$$

Hence,

$$\begin{aligned} (-1)^n[\log f(x)]^{(n)} &= \int_0^\infty t^{n-2}e^{-(x+a)t}dt - \int_0^\infty t^{n-2}e^{-xt}d \\ &\quad -(b-a)\int_0^\infty t^{n-1}e^{-xt}dt + b\int_0^\infty t^{n-1}e^{-xt}dt \\ &= \int_0^\infty [1 - e^{at} - (b-a)te^{at} + bte^{at}]t^{n-2}e^{-(x+a)t}dt \\ &\triangleq \int_0^\infty \phi(t)t^{n-2}e^{-(x+a)t}dt \end{aligned}$$

$$\begin{aligned} \phi(0) &= 0 \\ \phi^{'}(t) &= a^2te^{at} \end{aligned}$$

So we have $\phi^{'}(t) > 0$ for $a > 0$ . Then we obtain that $\phi(t)$ is strictly increasing in $(0, \infty)$. As a result of $\phi(0) = 0$ , we obtain $\phi(t) > 0$ in $(0, \infty)$. This means that $(-1)^n(\log g(x))^{(n)} > 0$ for $n \geq 2$ in $(0, \infty)$. Since $[\log g(x)]^{''} > 0$, the function $[\log g(x)]^{'}$ is increasing .

It is not difficult to know that

$$\lim_{x\to\infty}[\log g(x)]^{'} = 0, \tag{16}$$

so $[\log g(x)]^{'} < 0$ and $\log g(x)$ is decreasing in $(0, \infty)$. In conclusion, the function $\log g(x)$ is strictly completely monotonic in $(0, \infty)$. The Proof of Theorem 2 is complete.

Proof of Theorem 3.

It is easy to know that the product of (strictly) logarithmically completely monotonic functions is also (strictly) logarithmically completely monotonic functions. Write

$$F(x) = f(x)g(x). \tag{17}$$

Clearly, the function $F(x)$ is strictly logarithmically completely monotonic on $(0, \infty)$. The Proof of Theorem 3 is complete.

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