# *Ostrowski's Inequality for Vector-Valued Functions and Applications*

# OSTROWSKI'S INEQUALITY FOR VECTOR-VALUED FUNCTIONS AND APPLICATIONS

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ABSTRACT. Some Ostrowski type inequalities for vector-valued functions are obtained. Applications for operatorial inequalities and numerical approximation for the solutions of certain differential equations in Banach spaces are also given.

## 1. INTRODUCTION

The concepts of Riemann and Lebesgue integrability are well known for a scalar-valued function $F : [a, b] \to \mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers and $-\infty < a < b < \infty$. It is known, for example, that if $F$ is an absolutely continuous function, then it is differentiable almost everywhere and its derivative function $f := F'$ is a Lebesgue integrable function. Moreover, in this case, the following fundamental formula of calculus, holds:

$$F(t) = F(a) + (L) \int_a^t f(s)\, ds, \quad \text{for all } t \in [a, b], \tag{1.1}$$

where $(L) \int_a^t f(s)\, ds$ is Lebesgue's integral. If we replace $\mathbb{K}$ with a real or complex linear space $X$, that is, if $F$ is a vector-valued function, then the above result will not hold. More precisely, if $X$ is a Banach space, then the concept of Lebesgue integrability can be replaced with the concept of Bochner integrability (see for example [3], [11], [2]). However, there exist $X-$valued functions defined on $[a, b]$ which are absolutely continuous, and the set of points $t \in [a, b]$ for which $f$ is not differentiable with respect to $t$, is of non-null Lebesgue measure.

A Banach space $X$ with the property that every absolutely continuous $X-$valued function is almost everywhere differentiable is said to be a *Radon-Nikodym* space [5, pp. 217–219] or [11, 2]. For example, every reflexive Banach space (in particular, every Hilbert space) is a Radon-Nikodym space, but the space $L_\infty[0, 1]$ of all $\mathbb{K}-$valued, essentially bounded functions defined on the interval $[0, 1]$, endowed with the norm

$$\|g\|_\infty := ess \sup_{t\in[0,1]} |g(t)|$$

is a Banach space which is not a Radon-Nikodym space.

However, if $f : [a, b] \to X$ (where $X$ is an arbitrary Banach space) is a Bochner integrable function on $[a, b]$, then the function

$$t \mapsto F(t) := (B) \int_a^t f(s)\, ds : [a, b] \to X$$

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is differentiable almost everywhere on $[a,b]$, i.e., $F' = f$ a.e. and (1.1) holds. It should be noted that the integral is being considered in the Bochner sense.

A function $f : [a,b] \to X$ is *measurable* if there exists a sequence of simple functions $(f_n)$ (with $f_n : [a,b] \to X$) which converges punctually a.e. at $f$ on $[a,b]$.

It is well-known that a measurable function $f : [a,b] \to X$ is Bochner integrable if and only if its norm, i.e., the function $t \longmapsto \|f\|(t) := \|f(t)\| : [a,b] \to \mathbb{R}_+$ is Lebesgue integrable on $[a,b]$, (see for example [10]).

It is known that if $f$ is a scalar-valued and Riemann integrable function on $[a,b]$, then its primitive function, that is, the function $t \mapsto F(t) := (R)\int_a^t f(s)\,ds :$ $[a,b] \to \mathbb{K}$ is differentiable almost everywhere and (1.1) holds a.e. on $[a,b]$. Such a result, however, is not valid for vector-valued functions. For example, the function $f : [0,1] \to L_\infty[0,1]$ given by $f(t) = 1_{[0,t]}(\cdot)$, $t \in [0,1]$ (where $1_{[0,t]}$ is the characteristic function of the interval $[0,t]$) is a Riemann integrable vector valued function and its Riemann integral is given by

$$F(t) := (R)\int_0^t f(s)\,ds = (t - \cdot)\,1_{[0,t]}(\cdot), \quad t \in [0,1]. \tag{1.2}$$

The function $F : [0,1] \to L_\infty[0,1]$, defined in (1.2) is absolutely continuous (in fact, it is even Lipschitz continuous on $[0,1]$) but nowhere differentiable because

$$\frac{F(t+h) - F(t)}{h}(\cdot) = 1_{[0,t]}(\cdot) + \frac{1}{h}(t + h - \cdot)\,1_{[t,t+h]}(\cdot)$$

does not converge in $L_\infty[0,1]$ as $h \to 0$ for any $0 \le t \le 1$.

Another example can be found in [11, p. 172].

In Section 2, we will use the integration by parts formula. This holds under the following general conditions:

Let $-\infty < a < b < \infty$ and $f, g$ be two mappings defined on $[a,b]$ such that $f$ is $\mathbb{C}$-valued and $g$ is $X$-valued, where $X$ is a real or complex Banach space. If $f, g$ are differentiable on $[a,b]$ and their derivatives are Bochner integrable on $[a,b]$, then

$$(B)\int_a^b f'g = f(b)\,g(b) - f(a)\,g(b) - (B)\int_a^b fg'.$$

Using this in Section 2, we obtain some Ostrowski type inequalities for vector-valued functions and show that the mid-point inequality is the best possible inequality in the class. In Section 3, a quadrature formula of the Riemann type for the Bochner integral and the error bounds are considered. Section 4 is devoted to operator inequalities that can be obtained via Ostrowski type inequalities for vector-valued functions for which, in the last section, a numerical approximation for the mild solution of inhomogeneous vector-valued differential equations is given. In the last section, two numerical examples are considered.

For some results on the Ostrowski inequality for real-valued functions, see [1], [4], [8] and [9], and the references therein.

## 2. Ostrowski's Inequality for the Bochner Integral

The following theorem concerning a version of Ostrowski's inequality for vector-valued functions holds.

**Theorem 1.** *Let $(X; \|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f : [a,b] \to X$ an absolutely continuous function on $[a,b]$ with the property that*

$f' \in L_\infty([a,b];X)$, *i.e.*,

$$|||f'|||_{[a,b],\infty} := ess \sup_{t\in[a,b]} \|f'(t)\| < \infty.$$

*Then we have the inequalities:*

$$
\begin{aligned}
&\left\| f(s) - \frac{1}{b-a}(B)\int_a^b f(t)\,dt \right\| \qquad (2.1)\\
\leq\ & \frac{1}{b-a}\left[\int_a^s (t-a)\|f'(t)\|\,dt + \int_s^b (b-t)\|f'(t)\|\,dt\right]\\
\leq\ & \frac{1}{2(b-a)}\left[(s-a)^2 |||f'|||_{[a,s],\infty} + (b-s)^2 |||f'|||_{[s,b],\infty}\right]\\
\leq\ & \left[\frac{1}{4} + \left(\frac{s-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)|||f'|||_{[a,b],\infty}\\
\leq\ & \frac{1}{2}(b-a)|||f'|||_{[a,b],\infty}
\end{aligned}
$$

*for any* $s \in [a,b]$, *where* $(B)\int_a^b f(t)\,dt$ *is the Bochner integral of* $f$.

*Proof.* Using the integration by parts formula, we may write that

$$(B)\int_a^s (t-a) f'(t)\,dt = (s-a) f(s) - (B)\int_a^s f(t)\,dt$$

and

$$(B)\int_s^b (b-t) f'(t)\,dt = (b-s) f(s) - (B)\int_s^b f(t)\,dt,$$

for any $s \in [a,b]$; from which we get the identity:

$$
\begin{aligned}
&(b-a) f(s) - (B)\int_a^b f(t)\,dt \qquad (2.2)\\
=\ & (B)\int_a^s (t-a) f'(t)\,dt + (B)\int_s^b (b-t) f'(t)\,dt.
\end{aligned}
$$

Taking the norm on $X$, we obtain

$$
\begin{aligned}
\left\|(b-a) f(s) - (B)\int_a^b f(t)\,dt\right\| &= \left\|(B)\int_a^s (t-a) f'(t)\,dt + (B)\int_s^b (b-t) f'(t)\,dt\right\|\\
&\leq \left\|(B)\int_a^s (t-a) f'(t)\,dt\right\| + \left\|(B)\int_s^b (b-t) f'(t)\,dt\right\|\\
&\leq \int_a^s (t-a)\|f'(t)\|\,dt + \int_s^b (b-t)\|f'(t)\|\,dt\\
&= : B(s),
\end{aligned}
$$

which proves the first inequality in (2.1).

We also have

$$\int_a^s (t-a)\|f'(t)\|\,dt \leq |||f'|||_{[a,s],\infty}\int_a^s (t-a)\,dt = |||f'|||_{[a,s],\infty}\cdot\frac{(s-a)^2}{2}$$

and

$$\int_s^b (b-t)\|f'(t)\|\,dt \le |||f'|||_{[s,b],\infty} \int_s^b (b-t)\,dt = |||f'|||_{[s,b],\infty} \cdot \frac{(b-s)^2}{2}$$

from whence, by addition, we get the second part of (2.1).

Since

$$\max\left\{|||f'|||_{[a,s],\infty}, |||f'|||_{[s,b],\infty}\right\} \le |||f'|||_{[a,b],\infty}$$

and, by the parallelogram identity for real numbers, we have,

$$\frac{1}{2}\left[(s-a)^2+(b-s)^2\right] = \frac{1}{4}(b-a)^2 + \left(s-\frac{a+b}{2}\right)^2$$

then the last part of (2.1) is also proved. ∎

**Remark 1.** *We observe that for the scalar function $B:[a,b]\to\mathbb{R}$, we have*

$$B'(s) = (s-a)\|f'(s)\| - (b-s)\|f'(s)\| = 2\left(s-\frac{a+b}{2}\right)\|f'(s)\|$$

*for any $s\in[a,b]$, showing that $B$ is monotonic nonincreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nondecreasing on $\left[\frac{a+b}{2},b\right]$ and*

$$\begin{aligned}\inf_{s\in[a,b]} B(s) &= B\left(\frac{a+b}{2}\right) \\ &= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(t-a)\|f'(t)\|\,dt + \int_{\frac{a+b}{2}}^b (b-t)\|f'(t)\|\,dt\right].\end{aligned} \tag{2.3}$$

Consequently, the best inequalities we can obtain from (2.1) are embodied in the following corollary.

**Corollary 1.** *With the assumptions of Theorem 1, we have the inequality:*

$$\begin{aligned}&\left\|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}(B)\int_a^b f(t)\,dt\right\| \\ \le\ & \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(t-a)\|f'(t)\|\,dt + \int_{\frac{a+b}{2}}^b (b-t)\|f'(t)\|\,dt\right] \\ \le\ & \frac{b-a}{2}\left[|||f'|||_{\left[a,\frac{a+b}{2}\right],\infty} + |||f'|||_{\left[\frac{a+b}{2},b\right],\infty}\right] \\ \le\ & \frac{1}{4}(b-a)\,|||f'|||_{[a,b],\infty}\,.\end{aligned} \tag{2.4}$$

Bounds involving the $p-$norms, $p\in[1,\infty)$, of the derivative $f'$, are embodied in the following theorem.

**Theorem 2.** *Let $(X,\|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f:[a,b]\to X$ be an absolutely continuous function on $[a,b]$ with the property that $f'\in L_p([a,b];X)$, $p\in[1,\infty)$, i.e.,*

$$|||f'|||_{[a,b],p} := \left(\int_a^b \|f'(t)\|^p\,dt\right)^{\frac{1}{p}} < \infty. \tag{2.5}$$

*Then we have the inequalities*

$$\left\| f(s) - \frac{1}{b-a}(B)\int_a^b f(t)\,dt \right\| \tag{2.6}$$
$$\leq \frac{1}{b-a}\left[\int_a^s (t-a)\|f'(t)\|\,dt + \int_s^b (b-t)\|f'(t)\|\,dt\right]$$
$$\leq \begin{cases} \dfrac{1}{b-a}\left[(s-a)\,|||f'|||_{[a,s],1} + (b-s)\,|||f'|||_{[s,b],1}\right] \\ \qquad\qquad \text{if } f' \in L_1([a,b];X); \\ \dfrac{1}{(b-a)(q+1)^{\frac{1}{q}}}\left[(s-a)^{\frac{1}{q}+1}\,|||f'|||_{[a,s],p} + (b-s)^{\frac{1}{q}+1}\,|||f'|||_{[s,b],p}\right] \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ and } f' \in L_p([a,b];X) \end{cases}$$
$$\leq \begin{cases} \left[\dfrac{1}{2} + \left|\dfrac{s-\frac{a+b}{2}}{b-a}\right|\right]|||f'|||_{[a,b],1} \qquad \text{if } f' \in L_1([a,b];X); \\ \dfrac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\dfrac{s-a}{b-a}\right)^{q+1} + \left(\dfrac{b-s}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\,|||f'|||_{[a,b],p} \\ \qquad\qquad \text{if } f' \in L_p([a,b];X). \end{cases}$$

*Proof.* We have

$$\int_a^s (t-a)\|f'(t)\|\,dt \leq (s-a)\int_a^s \|f'(t)\|\,dt = (s-a)\,|||f'|||_{[a,s],1}$$

and

$$\int_s^b (b-t)\|f'(t)\|\,dt \leq (b-s)\int_s^b \|f'(t)\|\,dt = (b-s)\,|||f'|||_{[s,b],1}$$

and the first part of the second inequality in (2.6) is proved.

Using Hölder's integral inequality for scalar functions we have (for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$) that

$$\begin{aligned} \int_a^s (t-a)\|f'(t)\|\,dt &\leq \left(\int_a^s |t-a|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^s \|f'(t)\|^p\,dt\right)^{\frac{1}{p}} \\ &= \frac{(s-a)^{\frac{1}{q}+1}}{(q+1)^{\frac{1}{q}}}\,|||f'|||_{[a,s],p} \end{aligned}$$

and

$$\begin{aligned} \int_s^b (b-t)\|f'(t)\|\,dt &\leq \left(\int_s^b |b-t|^q\,dt\right)^{\frac{1}{q}}\left(\int_s^b \|f'(t)\|^p\,dt\right)^{\frac{1}{p}} \\ &= \frac{(b-s)^{\frac{1}{q}+1}}{(q+1)^{\frac{1}{q}}}\,|||f'|||_{[s,b],p}, \end{aligned}$$

giving the second part of the second inequality.

Since

$$\begin{aligned}
& (s-a)\,|||f'|||_{[a,s],1} + (b-s)\,|||f'|||_{[s,b],1} \\
\leq\ & \max\{s-a,b-s\}\left[|||f'|||_{[a,s],1} + |||f'|||_{[s,b],1}\right] \\
=\ & \left[\frac{1}{2}(b-a) + \left|s-\frac{a+b}{2}\right|\right] |||f'|||_{[a,b],1}\,,
\end{aligned}$$

the first part of the third inequality in (2.6) is proved.

For the last part, we note that for any $\alpha$, $\beta$, $\gamma$, $\delta > 0$ and $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$ we have:

$$(\alpha^q+\beta^q)^{\frac{1}{q}}(\gamma^p+\delta^p)^{\frac{1}{p}} \geq \alpha\gamma+\beta\delta,$$

and then:

$$\begin{aligned}
& (s-a)^{1+\frac{1}{q}}\,|||f'|||_{[a,s],p} + (b-s)^{1+\frac{1}{q}}\,|||f'|||_{[s,b],p} \\
\leq\ & \left[(s-a)^{q\left(1+\frac{1}{q}\right)} + (b-s)^{q\left(1+\frac{1}{q}\right)}\right]^{\frac{1}{q}} \left[|||f'|||_{[a,s],p}^{p} + |||f'|||_{[s,b],p}^{p}\right]^{\frac{1}{p}} \\
=\ & \left[(s-a)^{1+q} + (b-s)^{1+q}\right]^{\frac{1}{q}} \left[\int_a^s \|f'(s)\|^p\,ds + \int_s^b |||f'(s)|||^p\,ds\right]^{\frac{1}{p}} \\
=\ & \left[(s-a)^{1+q} + (b-s)^{1+q}\right]^{\frac{1}{q}} |||f'|||_{[a,b],p}\,.
\end{aligned}$$

The theorem is completely proved. ∎

**Remark 2.** *The above theorem both generalises and extends for vector-valued functions the results in* [6] *and* [7]*.*

The best inequalities we can obtain from (2.6) in the sense of providing the tightest bound are embodied in the following corollary concerning the mid-point rule.

**Corollary 2.** *With the assumptions in Theorem 3, we have*

$$\begin{aligned}
& \left\|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}(B)\int_a^b f(t)\,dt\right\| \\
\leq\ & \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} (t-a)\,\|f'(t)\|\,dt + \int_{\frac{a+b}{2}}^b (b-t)\,\|f'(t)\|\,dt\right]
\end{aligned} \tag{2.7}$$

$$\leq \begin{cases} \frac{1}{2}\left|\left|\left|f'\right|\right|\right|_{[a,b],1} & \text{if } f' \in L_1([a,b];X); \\ \frac{(b-a)^{\frac{1}{q}}}{2^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left[\left|\left|\left|f'\right|\right|\right|_{\left[a,\frac{a+b}{2}\right],p} + \left|\left|\left|f'\right|\right|\right|_{\left[\frac{a+b}{2},b\right],p}\right] \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ and } f' \in L_p([a,b];X) \end{cases}$$

$$\leq \begin{cases} \frac{1}{2}\left|\left|\left|f'\right|\right|\right|_{[a,b],1} & \text{if } f' \in L_1([a,b];X); \\ \frac{1}{2(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\left|\left|\left|f'\right|\right|\right|_{[a,b],p} \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ and } f' \in L_p([a,b];X). \end{cases}$$

## 3. A Quadrature Formula of the Riemann Type

Now, let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partitioning of the interval $[a,b]$ and define $h_i = x_{i+1} - x_i$, $\nu(h) := \max\{h_i | i = 0,\ldots,n-1\}$. Consider the mapping $f : [a,b] \to X$, where $X$ is a Banach space with the Radon-Nikodym property. Define the Riemann sum by:

$$A_n(f, I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} h_i f(\xi_i), \tag{3.1}$$

where $\boldsymbol{\xi} = (\xi_0,\ldots,\xi_{n-1})$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,\ldots,n-1)$ are intermediate (arbitrarily chosen) points.

The following theorem holds.

**Theorem 3.** *Let $f$ be as in Theorem 1. Then we have:*

$$(B)\int_a^b f(t)\,dt = A_n(f, I_n, \boldsymbol{\xi}) + R_n(f, I_n, \boldsymbol{\xi}), \tag{3.2}$$

*where $A_n(f, I_n, \boldsymbol{\xi})$ is the Riemann quadrature given by (3.1) and the remainder $R_n(f, I_n, \boldsymbol{\xi})$ in (3.2) satisfies the bound*

$$\begin{aligned} &\left\|R_n(f, I_n, \boldsymbol{\xi})\right\| \\ \leq\ & \sum_{i=0}^{n-1}\left[\int_{x_i}^{\xi_i}(t-x_i)\left\|f'(t)\right\|dt + \int_{\xi_i}^{x_{i+1}}(x_{i+1}-t)\left\|f'(t)\right\|dt\right] \\ \leq\ & \frac{1}{2}\sum_{i=0}^{n-1}\left[(\xi_i-x_i)^2\left|\left|\left|f'\right|\right|\right|_{[x_i,\xi_i],\infty} + (x_{i+1}-\xi_i)^2\left|\left|\left|f'\right|\right|\right|_{[\xi_i,x_{i+1}],\infty}\right] \\ \leq\ & \sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2\right]\left|\left|\left|f'\right|\right|\right|_{[x_i,x_{i+1}],\infty} \\ \leq\ & \frac{1}{2}\sum_{i=0}^{n-1}h_i^2\left|\left|\left|f'\right|\right|\right|_{[x_i,x_{i+1}],\infty} \\ \leq\ & \frac{1}{2}\left|\left|\left|f'\right|\right|\right|_{[a,b],\infty}\sum_{i=0}^{n-1}h_i^2 \leq \frac{1}{2}(b-a)\,\nu(h)\left|\left|\left|f'\right|\right|\right|_{[a,b],\infty}. \end{aligned} \tag{3.3}$$

*Proof.* Apply the inequality (2.1) on the interval $[x_i, x_{i+1}]$ to obtain

$$\begin{aligned} &\left\| h_i f(\xi_i) - \int_{x_i}^{x_{i+1}} f(t)\,dt \right\| \\ &\leq \int_{x_i}^{\xi_i} (t - x_i) \|f'(t)\|\,dt + \int_{\xi_i}^{x_{i+1}} (x_{i+1} - t) \|f'(t)\|\,dt \\ &\leq \frac{1}{2}\left[ (\xi_i - x_i)^2 \||f'\||_{[x_i,\xi_i],\infty} + (x_{i+1} - \xi_i)^2 \||f'\||_{[\xi_i,x_{i+1}],\infty} \right] \\ &\leq \left[ \frac{1}{4} + \left( \frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right)^2 \right] h_i^2 \||f'\||_{[x_i,x_{i+1}],\infty} \\ &\leq \frac{1}{2} h_i^2 \||f'\||_{[x_i,x_{i+1}],\infty} \end{aligned} \tag{3.4}$$

for any $i = 0, \ldots, n-1$.

Summing over $i$ from 0 to $n-1$ and using the generalised triangle inequality for norms, we obtain (3.3). ∎

If we consider the *midpoint quadrature rule* given by

$$M_n(f, I_n) := \sum_{i=0}^{n-1} h_i f\left( \frac{x_i + x_{i+1}}{2} \right) \tag{3.5}$$

then we may state the following corollary.

**Corollary 3.** *With the assumptions in Theorem 1, we have*

$$(B)\int_a^b f(t)\,dt = M_n(f, I_n) + W_n(f, I_n) \tag{3.6}$$

*where $M_n(f, I_n)$ is the vector-valued midpoint quadrature rule given in (3.5) and the remainder $W_n(f, I_n)$ satisfies the estimate:*

$$\begin{aligned} &\|W_n(f, I_n)\| \\ &\leq \sum_{i=0}^{n-1} \left[ \int_{x_i}^{\frac{x_i + x_{i+1}}{2}} (t - x_i) \|f'(t)\|\,dt + \int_{\frac{x_i + x_{i+1}}{2}}^{x_{i+1}} (x_{i+1} - t) \|f'(t)\|\,dt \right] \\ &\leq \frac{1}{8} \sum_{i=0}^{n-1} h_i^2 \left[ \||f'\||_{\left[x_i, \frac{x_i + x_{i+1}}{2}\right],\infty} + \||f'\||_{\left[\frac{x_i + x_{i+1}}{2}, x_{i+1}\right],\infty} \right] \\ &\leq \frac{1}{4} \sum_{i=0}^{n-1} h_i^2 \||f'\||_{[x_i,x_{i+1}],\infty} \leq \frac{1}{4} \||f'\||_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^2 \\ &\leq \frac{1}{4} (b - a) \||f'\||_{[a,b],\infty}\, \nu(h). \end{aligned} \tag{3.7}$$

**Remark 3.** *It is obvious that $\|W_n(f, I_n)\| \to 0$ as $\nu(h) \to 0$, showing that $M_n(f, I_n)$ is an approximation for the Bochner integral $(B)\int_a^b f(t)\,dt$ with order one accuracy.*

**Remark 4.** *Similar bounds for the remainder $R_n(f, I_n, \boldsymbol{\xi})$ and $W_n(f, I_n)$ may be obtained in terms of the $p-$norms $(p \in [1, \infty))$, but we omit the details.*

## 4. Applications for the Operator Inequality

Let $X$ be an arbitrary Banach space and $\mathcal{L}(X)$ the Banach space of all bounded linear operators on $X$. We recall that if $A \in \mathcal{L}(X)$ then its operatorial norm is defined by

$$\|A\| = \sup\{\|Ax\| : x \in X,\ \|x\| \le 1\}.$$

We recall also that the series $\left(\sum_{n\ge 0}\frac{(tA)^n}{n!}\right)$ converges absolutely and locally uniformly for $t \in \mathbb{R}$. If we denote by $e^{tA}$ its sum, then

$$\left\|e^{tA}\right\| \le e^{t\|A\|}, \quad \text{for all } t \ge 0. \tag{4.1}$$

Another definition of $e^{tA}$ is given in the next section.

**Proposition 1.** *Let $X$ be a Banach space, $A \in \mathcal{L}(X)$ and $0 \le a < b < \infty$. Then for each $s \in [a,b]$, we have:*

$$\begin{aligned} &\left\|e^{sA} - \frac{1}{b-a}\int_a^b e^{tA}dt\right\| \\ &\le \frac{1}{b-a}\left[(2s-a-b)\,e^{s\|A\|} + \frac{1}{\|A\|}\left(e^{a\|A\|} + e^{b\|A\|} - 2e^{s\|A\|}\right)\right]. \end{aligned} \tag{4.2}$$

*Proof.* We apply Theorem 1 with $X$ replaced by $\mathcal{L}(X)$ and $f(t) = e^{tA}$. Note that in this case the function $f$ is continuously differentiable, so that it is not necessary that $X$ be a Radon-Nikodym space. We have, by (4.1), that

$$\begin{aligned} \int_a^s (t-a)\|f'(t)\|\,dt &\le \|A\|\int_a^s (t-a)\,e^{t\|A\|}dt \\ &= (s-a)\,e^{s\|A\|} - \frac{1}{\|A\|}\left(e^{a\|A\|} - e^{s\|A\|}\right), \end{aligned}$$

and

$$\begin{aligned} \int_s^b (b-t)\|f'(t)\|\,dt &\le \|A\|\int_s^b (b-t)\,e^{t\|A\|}dt \\ &= -(b-s)\,e^{s\|A\|} + \frac{1}{\|A\|}\left(e^{b\|A\|} - e^{s\|A\|}\right). \end{aligned}$$

On adding the two above inequalities, we obtain the desired inequality (4.2). ∎

**Corollary 4.** *With the assumptions in Proposition 1, we have the following inequality*

$$\left\|e^{\frac{a+b}{2}A} - \frac{1}{b-a}\int_a^b e^{tA}dt\right\| \le \frac{1}{(b-a)\|A\|}\left(e^{\frac{a}{2}\|A\|} - e^{\frac{b}{2}\|A\|}\right)^2. \tag{4.3}$$

Let $GL(X)$ be the subset of $\mathcal{L}(X)$ consisting of all invertible operators. It is known that $GL(X)$ is an open set in $\mathcal{L}(X)$.

Using (4.3), we may state the following result as well.

**Corollary 5.** *Let $A \in GL(X)$. Then the following inequality holds:*

$$\begin{aligned} \left\|Ae^{\frac{a+b}{2}A} - \frac{1}{b-a}\left(e^{bA} - e^{aA}\right)\right\| &\le \|A\|\left\|e^{\frac{a+b}{2}A} - \frac{1}{b-a}A^{-1}\left(e^{bA} - e^{aA}\right)\right\| \\ &\le \frac{1}{b-a}\left(e^{\frac{a}{2}\|A\|} - e^{\frac{b}{2}\|A\|}\right)^2. \end{aligned}$$

*Proof.* The first inequality is obvious. For the second inequality we remark that

$$\int_a^b e^{tA}dt = A^{-1}\left(e^{bA} - e^{aA}\right)$$

and apply Corollary 4. ∎

**Remark 5.** *As a consequence of Corollary 5, we can obtain the well-known inequality for real numbers* $e^y \geq 1 + y$ *for each* $y \in \mathbb{R}$. *Indeed, if* $A = x \in (0, \infty)$, *then*

$$\left| xe^{\frac{a+b}{2}x} - \frac{1}{b-a}\left(e^{bx} - e^{ax}\right) \right| \leq \frac{1}{b-a}\left(e^{\frac{a}{2}x} - e^{\frac{b}{2}x}\right)^2.$$

*which is equivalent to*

$$e^{\frac{a-b}{2}x} \geq 1 + \frac{a-b}{2}x \quad \textit{and} \quad e^{\frac{b-a}{2}x} \geq 1 + \frac{b-a}{2}x.$$

Another example of an operatorial inequality is embodied in the following proposition.

**Proposition 2.** *Let* $X$ *be a Banach space,* $A \in \mathcal{L}(X)$ *and* $0 \leq a < b < \infty$. *Then for each* $s \in [a, b]$, *we have:*

$$\left\| \sin(sA) - \frac{1}{b-a}\int_a^b \sin(tA)\,dt \right\| \leq \left[\frac{1}{4} + \left(\frac{s - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|A\|. \tag{4.4}$$

*Proof.* We apply the first inequality from Theorem 1 for

$$f(t) = \sin(tA) := \sum_{n=0}^{\infty} (-1)^n \frac{(tA)^{2n+1}}{(2n+1)!}.$$

We have

$$\left\|(\sin(tA))'\right\| = \|A\cos(tA)\| \leq \|A\|.$$

Then

$$\int_a^s (t-a)\|f'(t)\|\,dt \leq \|A\| \cdot \frac{(s-a)^2}{2}$$

and

$$\int_s^b (b-t)\|f'(t)\|\,dt \leq \|A\| \cdot \frac{(s-b)^2}{2}.$$

On adding the above inequalities, we obtain the desired result (4.4). Here, $\cos(tA) = \sum_{n=0}^{\infty} (-1)^n \frac{(tA)^{2n}}{(2n)!}$. ∎

**Corollary 6.** *With the assumptions as in Proposition 2, we have the following inequality:*

$$\left\| \sin\left(\frac{a+b}{2} \cdot A\right) - \frac{1}{b-a}\int_a^b \sin(tA)\,dt \right\| \leq \frac{(b-a)^2}{4} \cdot \|A\|.$$

*If in addition $A \in GL(X)$, then*

$$\begin{aligned} & \left\| A \sin\left(\frac{a+b}{2} \cdot A\right) + \frac{1}{b-a}\left[\cos(bA) - \cos(aA)\right] \right\| \\ \leq \quad & \|A\| \cdot \left\| \sin\left(\frac{a+b}{2} \cdot A\right) + \frac{1}{b-a} A^{-1}\left[\cos(bA) - \cos(aA)\right] \right\| \\ \leq \quad & \frac{(b-a)^2}{4} \cdot \|A\|^2 . \end{aligned}$$

**Remark 6.** *In particular, for $A = x \in \mathbb{R}\backslash\{0\}$, it follows that*

$$\left| \sin\left(\frac{a+b}{2} \cdot x\right) \left[1 - \frac{\sin\frac{(b-a)x}{2}}{\frac{(b-a)x}{2}}\right] \right| \leq \frac{(b-a)^2}{4} |x| . \tag{4.5}$$

The similar result for $\cos(tA)$ will be summarised next.

**Proposition 3.** *With the above notations, we have:*

(i) $$\left\| \cos(sA) - \frac{1}{b-a} \int_a^b \cos(tA)\, dt \right\| \leq \left[\frac{1}{4} + \left(\frac{s - \frac{a+b}{2}}{b-a}\right)^2\right] (b-a) \|A\| .$$

(ii) $$\left\| \cos\left(\frac{a+b}{2} \cdot A\right) - \frac{1}{b-a} \int_a^b \cos(tA)\, dt \right\| \leq \frac{(b-a)^2}{4} \|A\| .$$

*If, in addition $A \in GL(X)$, then*

(iii) $$\begin{aligned} & \left\| A \cos\left(\frac{a+b}{2} \cdot A\right) - \frac{1}{b-a}\left[\sin(bA) - \sin(aA)\right] \right\| \\ & \leq \|A\| \left\| \cos\left(\frac{a+b}{2} \cdot A\right) - \frac{1}{b-a} \cdot A^{-1}\left[\sin(bA) - \sin(aA)\right] \right\| \\ & \leq \frac{(b-a)^2}{4} \|A\|^2 . \end{aligned}$$

**Remark 7.** *In particular, for $A = x \in \mathbb{R}\backslash\{0\}$, it follows that*

$$\left| \cos\left(\frac{a+b}{2} \cdot x\right) \cdot \left[1 - \frac{\sin\frac{(b-a)}{2} \cdot x}{\frac{(b-a)}{2} \cdot x}\right] \right| \leq \frac{(b-a)^2}{4} |x| . \tag{4.6}$$

**Remark 8.** *Taking the square of both sides of the inequalities (4.5) and (4.6) and then adding them, we obtain*

$$\left| 1 - \frac{\sin\frac{(b-a)}{2} \cdot x}{\frac{(b-a)}{2} \cdot x} \right| \leq \frac{\sqrt{2}}{4} (b-a)^2 |x| , \quad \textit{for all } x \in \mathbb{R}^* .$$

*In particular, if $b - a = 2$, then*

$$|\sin x - x| \leq \sqrt{2} x^2 , \quad \textit{for all } x \in \mathbb{R},$$

*which is an interesting scalar inequality.*

Another type of example is considered in the following.

A densely defined linear operator $A$ on a Banach space $X$ is said to be *sectorial* [13] if $(0, \infty) \subset \rho(A)$ and there exists $M = M_A > 0$ such that

$$\|R(t, A)\| \leq \frac{M}{1+t}, \text{ for all } t > 0, \tag{4.7}$$

where $R(t, A) := (tI - A)^{-1}$ is the resolvent operator of $A$.

**Proposition 4.** *Let $A$ be a sectorial operator on a Banach space $X$. Then for $0 \leq a \leq s \leq b < \infty$, we have:*

(*i*) $\left\|R^2(s,A) - R(a,A)R(b,A)\right\| \leq \frac{M^3}{(b-a)(s+1)^2} \cdot \left[\frac{(s-a)^2}{a+1} + \frac{(b-s)^2}{b+1}\right]$;

*and*

(*ii*) $\left\|R^2\left(\frac{a+b}{2},A\right) - R(a,A)R(b,A)\right\| \leq \frac{M^3(b-a)}{(a+1)(b+1)(a+b+2)}$.

*Proof.* By the resolvent identity

$$R(t,A) - R(s,A) = (s-t)R(t,A)R(s,A),$$

it follows that

$$\frac{d}{dt}[R(t,A)] = -R^2(t,A).$$

We apply Theorem 1 in Section 2 for $f(t) = R^2(t,A)$ giving, from (4.7)

$$\left\|\frac{d}{dt}\left[R^2(t,A)\right]\right\| = \left\|-2R^3(t,A)\right\| \leq \frac{2M^3}{(t+1)^3}.$$

Further,

$$\begin{aligned}
& \frac{1}{b-a}\left[\int_a^s (t-a)\|f'(t)\|\,dt + \int_s^b (b-t)\|f'(t)\|\,dt\right] \\
\leq\; & \frac{2M^3}{b-a}\left[\int_a^s \frac{(t-a)}{(1+t)^3}dt + \int_s^b \frac{(b-t)}{(1+t)^3}dt\right] \\
\leq\; & \frac{2M^3}{b-a}\left[\frac{(s-a)^2}{2(a+1)(s+1)^2} + \frac{(b-s)^2}{2(b+1)(s+1)^2}\right] \\
\leq\; & \frac{M^3}{(b-a)(s+1)^2}\left[\frac{(s-a)^2}{a+1} + \frac{(b-s)^2}{b+1}\right].
\end{aligned}$$

Statement $(i)$ is thus proved. Taking $s = \frac{a+b}{2}$ gives $(ii)$. ∎

**Remark 9.** *If $A = x \in (-\infty, 0)$, then we can choose $M_x = \sup_{t>0}\left[\frac{t+1}{t-x}\right] = -\frac{1}{x}$ and from $(i)$ we obtain the interesting inequality:*

$$(a-x)(b-x)(a+b-2x)^2 \geq (-x)^3(b-a)(a+1)(b+1)(a+b+2),$$

*for all $x \leq 0$ and all $0 \leq a < b < \infty$.*

## 5. Applications for Vector-Valued Differential Equations

Many problems of mathematical physics can be modelled using the following abstract Cauchy problem

$$(ACP_x) \qquad \begin{cases} \dot{u}(t) = Au(t)\,, & t \geq 0 \\ u(0) = x\,, \end{cases}$$

where $A$ is a linear, usually unbounded, operator with domain $D(A)$ on a Banach space $X$. For every particular mathematical physics problem, $X$ is a suitable Banach space of functions and $A$ is a partial differential operator. By the *classical solution* for $(ACP_x)$, we mean a continuous differentiable function $u_x : [0,\infty) \to D(A)$ which satisfies $(ACP_x)$. A continuous function $u : [0,\infty) \to X$ is said to be a *mild* *solution* for $(ACP_x)$ if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in D(A)$ such that for each $n$ the problem $(ACP_x)$ has a classical solution $u_{x_n}(\cdot)$ with $\lim\limits_{n\to\infty} u_{x_n}(t) = u(t)$ locally uniform on $[0,\infty)$. We say that the abstract Cauchy problem associated with a linear operator $A$ is *well-posed* if for each initial value $x \in D(A)$ the problem $(ACP_{x_n})$ has a unique classical solution. An example of an operator $A$ for which the associated abstract Cauchy problem is well-posed is presented in the following.

Let $X$ be a Banach space and $\mathcal{L}(X)$ the space of all bounded linear operators. We denote by $\|\cdot\|$ the norms of vectors and operators. A family $\mathbf{T} = \{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a semigroup of operators if the following conditions hold:

$(S_1)$ $T(0) = I$, $I$ is the identity operator on $X$;
$(S_2)$ $T(t+s) = T(t)\circ T(s)$ for all $t, s \geq 0$.

A semigroup $\mathbf{T}$ is said to be *uniformly continuous* if the mapping $t \longmapsto T(t) : [0,\infty) \to \mathcal{L}(X)$ is continuous at $t_0 = 0$ (or equivalently, is continuous on $\mathbb{R}_+$) in the operatorial norm in $\mathcal{L}(X)$.

A semigroup $\mathbf{T}$ is said to be *strongly continuous* (or $C_0-$semigroup) if the mapping $t \longmapsto T(t)x : [0,\infty) \to X$ is continuous at $t_0 = 0$ (or equivalently on $\mathbb{R}_+$) for all $x \in X$. It is well known [12] that if $\mathbf{T}$ is a uniformly continuous semigroup, then there exists an operator $A \in \mathcal{L}(X)$ such that

$$T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}; \quad t \geq 0.$$

In this case, the problem $(ACP_x)$ associated with $A$ has a unique classical (or mild) solution and it is given by

$$u_x(t) = u(t) = e^{tA}x, \ \ t \geq 0.$$

If $\mathbf{T}$ is a $C_0-$semigroup, then its generator $A$ with its domain $D(A)$ are given by

$$D(A) = \left\{ x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \ \text{ exists in } X \right\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t}, \quad x \in D(A).$$

It is easy to see that the function $t \mapsto T(t)x$ is differentiable on $\mathbb{R}_+$ for all $x \in D(A)$. It is well-known ([13], [12]) that the generator $A$ is a closed and densely defined operator (i.e., $D(A)$ is dense in $X$). In this case, the abstract Cauchy problem associated with $A$ is well-posed. The classical solution is given by $u_x(t) = T(t)x$ for $x \in D(A)$ and the mild solution is given by $u(t) = T(t)x$ for $x \in X$. The converse result is also true.

For example, if $A$ is a linear operator with domain $D(A)$, the abstract Cauchy problem associated with $A$ is well-posed and the resolvent set of $A$ $(\rho(A))$ is nonempty, then $A$ is the generator for a strongly continuous semigroup $\mathbf{T}$ ( [13], [12]). Every $C_0-$semigroup $\mathbf{T}$ has a growth bound. That is, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t}, \ \ \text{for all } t \geq 0. \tag{5.1}$$

Let $f:\mathbb{R}_+\to X$ be a locally Bochner integrable function. We consider the inhomogeneous abstract Cauchy problem

$$(A,f,0,x)\qquad \begin{cases} \dot{u}(t)=Au(t)+f(t)\ , & t\geq 0 \\ u(0)=x\ , \end{cases}$$

where $A$ is the generator of a strongly continuous semigroup $\mathbf{T}$ and $x\in X$. The function $T(t-\cdot)f(\cdot)$ is measurable, because if $\{f_n\}$ is a sequence of simple functions, then $g_n(\cdot):=T(t-\cdot)f_n(\cdot)$ are measurable for each $n\in\mathbb{N}$ (we used the strong continuity of $\mathbf{T}$), and $g_n(s)\to T(t-s)f(s)$ as $n\to\infty$, a.e. on $[0,t]$. Moreover, the function $T(t-\cdot)f(\cdot)$ is Bochner integrable on $[0,t]$, because $\|T(t-\cdot)f(\cdot)\|\leq Me^{\omega t}\|f(\cdot)\|$ and the function $f$ is Bochner integrable on $[0,t]$.

The mild solution of the problem $(A,f,0,x)$ can be represented by

$$u(t)=x+(B)\int_0^t T(t-s)f(s)\,ds,\ \ t\geq 0,\ x\in X,$$

We may state the following theorem in approximating the mild solutions of the inhomogeneous system $(A,f,0,x)$.

**Theorem 4.** *Let $0=\lambda_0<\lambda_1<\dots<\lambda_{n-1}<\lambda_n=1$ and $\mu_i\in[\lambda_i,\lambda_{i+1}]$ $\left(i=\overline{0,n-1}\right)$. If either*

(i) *$\mathbf{T}$ is a uniformly continuous semigroup and $f$ is a differentiable continuous $X-$valued function ($X$ is an arbitrary Banach space)*
*or*

(ii) *$\mathbf{T}$ is a strongly continuous semigroup, $f$ is differentiable continuous and $f(t)\in D(A)$ for all $t\geq 0$, and $Af(\cdot)$ is a locally bounded function on $[0,\infty)$*

*hold, then the mild solution $u(\cdot)$ of $(A,f,0,x)$ can be represented as*

$$u(t)=x+S_n(\boldsymbol{\lambda},\boldsymbol{\mu},t)+Q_n(\boldsymbol{\lambda},\boldsymbol{\mu},t)\,,\ \ t\geq 0, \tag{5.2}$$

*where*

$$S_n(\boldsymbol{\lambda},\boldsymbol{\mu},t):=t\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\,T\left[(1-\mu_i)\,t\right]f(\mu_i t) \tag{5.3}$$

*and the remainder $Q_n(\boldsymbol{\lambda},\boldsymbol{\mu},t)$ satisfies, in the first case, the estimates*

$$\begin{aligned} &\|Q_n(\boldsymbol{\lambda},\boldsymbol{\mu},t)\| \\ \leq\ & t^2e^{\|A\|t}\left[\|A\|\,|||f|||_{[0,t],\infty}+|||f'|||_{[0,t],\infty}\right] \\ &\times\sum_{i=0}^{n-1}\left[\frac{1}{4}(\lambda_{i+1}-\lambda_i)^2+\left(\mu_i-\frac{\lambda_i+\lambda_{i+1}}{2}\right)^2\right] \\ \leq\ & \frac{1}{2}t^2e^{\|A\|t}\left[\|A\|\,|||f|||_{[0,t],\infty}+|||f'|||_{[0,t],\infty}\right]\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)^2 \\ \leq\ & \frac{1}{2}\nu(\boldsymbol{\lambda})\,t^3e^{\|A\|t}\left[\|A\|\,|||f|||_{[0,t],\infty}+|||f'|||_{[0,t],\infty}\right], \end{aligned} \tag{5.4}$$

*where* $\nu(\boldsymbol{\lambda}) := \max_{i=\overline{0,n-1}} (\lambda_{i+1} - \lambda_i)$*, and, in the second case, the estimates*

$$\begin{aligned} (5.5) \qquad & \|Q_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t)\| \\ \leq\ & Mt^2 e^{\omega t}\left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right] \\ & \times \sum_{i=0}^{n-1}\left[\frac{1}{4}(\lambda_{i+1}-\lambda_i)^2 + \left(\mu_i - \frac{\lambda_i+\lambda_{i+1}}{2}\right)^2\right] \\ \leq\ & \frac{1}{2}t^2 M e^{\omega t}\left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right]\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)^2 \\ \leq\ & \frac{1}{2}\nu(\boldsymbol{\lambda})\, t^3 e^{\omega t}\left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right], \end{aligned}$$

*for each* $t \in [0,\infty)$*, where* $\omega$ *is a positive number such that the estimate (5.1) holds.*

*Proof.* For a fixed $t > 0$, consider the function $g(s) := T(t-s) f(s)$, $s \in [0,t]$. Then $g$ is differentiable on $(0,t)$ and

$$\frac{dg(s)}{ds} = \frac{d}{ds}\left[T(t-s) f(s)\right] = -AT(t-s) f(s) + T(t-s) f'(s),$$

for each $s \in (0,t)$.

We have, in the first case, that

$$\begin{aligned} \left|\left|\left|\frac{dg}{ds}\right|\right|\right|_{[0,t],\infty} &\leq |||AT(t-\cdot) f(\cdot)|||_{[0,t],\infty} + |||T(t-\cdot) f'(\cdot)|||_{[0,t],\infty} \\ &\leq \|A\| e^{\|A\|t} |||f|||_{[0,t],\infty} + e^{\|A\|t} |||f'|||_{[0,t],\infty} \\ &= e^{\|A\|t}\left[\|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right], \end{aligned}$$

for any $t \in [0,\infty)$.

In the second case, we have in a similar manner, that

$$\left|\left|\left|\frac{dg}{ds}\right|\right|\right|_{[0,t],\infty} \leq Me^{\omega t}\left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'(\cdot)|||_{[0,t],\infty}\right],$$

for each $t \in [0,\infty)$.

Now, consider the partitioning of the interval $[0,t]$ given by $x_i := \lambda_i t$ $\left(i = \overline{0,n-1}\right)$ where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$ and the intermediate points $\xi_i = \mu_i t$ $\left(i = \overline{0,n-1}\right)$ where $\mu_i \in [\lambda_i, \lambda_{i+1}]$ $\left(i = \overline{0,n-1}\right)$. If we apply Theorem 3 for $a = 0$, $b = t$, $x_i$, $\xi_i$ $\left(i = \overline{0,n-1}\right)$ and $g$ as defined above, then we deduce the representation (5.2) and the remainder $Q_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t)$ satisfies either the estimate (5.4) or the estimate (5.5). ∎

If we define the quadrature formula

$$(5.6) \qquad M_n(\boldsymbol{\lambda}, t) := t\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\, T\left[\left(1 - \frac{\lambda_i+\lambda_{i+1}}{2}\right)t\right] f\left(\frac{\lambda_i+\lambda_{i+1}}{2}\cdot t\right),$$

then we may state the following corollary.

**Corollary 7.** *Let* $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$*. If either (i) or (ii) in Theorem 4 hold, then the mild solution* $u(\cdot)$ *of* $(A, f, 0, x)$ *can be represented as*

$$(5.7) \qquad u(t) = x + M_n(\boldsymbol{\lambda}, t) + L_n(\boldsymbol{\lambda}, t),$$

*where $M_n(\boldsymbol{\lambda}, t)$ is as given in (5.6) and the remainder $L_n(\boldsymbol{\lambda}, t)$ satisfies, in the first case, the estimates*

$$\begin{aligned} \|L_n(\boldsymbol{\lambda}, t)\| &\leq \frac{1}{4} t^2 e^{\|A\| t} \left[ \|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty} \right] \sum_{i=0}^{n-1} h_i^2 \\ &\leq \frac{1}{4} t^3 \nu(h) e^{\|A\| t} \left[ \|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty} \right], \end{aligned} \tag{5.8}$$

*where $h_i := \lambda_{i+1} - \lambda_i > 0$ $\left(i = \overline{0, n-1}\right)$, and, in the second case, the estimates:*

$$\begin{aligned} \|L_n(\boldsymbol{\lambda}, t)\| &\leq \frac{1}{4} M t^2 e^{\omega t} \left[ |||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty} \right] \sum_{i=0}^{n-1} h_i^2 \\ &\leq \frac{1}{4} M \nu(h) t^3 e^{\omega t} \left[ |||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty} \right] \end{aligned} \tag{5.9}$$

*for each $t \in (0, \infty)$.*

**Remark 10.** *In practical applications, it is easier to consider a uniform partitioning of $[0, t]$ given by*

$$E_n : x_i = \left(\frac{i}{n}\right) \cdot t, \quad i = \overline{0, n},$$

*and then (5.6) becomes*

$$M_n(t) := \frac{t}{n} \sum_{i=0}^{n-1} T\left[\left(\frac{2n - 2i - 1}{2n}\right) t\right] f\left[\left(\frac{2i + 1}{2n}\right) t\right]. \tag{5.10}$$

*In this case, we have the representation of $u(\cdot)$ given by*

$$u(t) = x + M_n(t) + V_n(t), \tag{5.11}$$

*where the approximation $M_n(\cdot)$ is as defined above in (5.10) and the remainder $V_n(\cdot)$ satisfies the error bounds*

$$\|V_n(t)\| \leq \frac{1}{4n} t^3 e^{\|A\| t} \left[ \|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty} \right] \tag{5.12}$$

*in the first case, and*

$$\|V_n(t)\| \leq \frac{1}{4n} M t^3 e^{\omega t} \left[ |||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty} \right] \tag{5.13}$$

*in the second case, for each $t \in [0, \infty)$.*

## 6. Numerical Examples

Let $X = \mathbb{R}^2$, $x = (\xi, \eta) \in \mathbb{R}^2$, $\|x\|_2 = \sqrt{\xi^2 + \eta^2}$. We consider the linear, 2-dimensional, inhomogeneous differential systems

$$\begin{cases} \dot{u}_1(t) = \quad u_1(t) \qquad\qquad + \sin t & \\ & t \geq 0 \\ \dot{u}_2(t) = \qquad\qquad -u_2(t) \quad + \cos t & \\ u_1(0) = \quad u_2(0) = 0 & \end{cases}.$$

If we let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $f(t) = (\sin t, \cos t)$, $x_0 = (0,0)$ and identify $(\xi, \eta)$ by $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, then the above system is the Cauchy problem $(A, f, 0, x_0)$. We have: $e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$,

$$\begin{aligned} u(t) &= \int_0^t e^{(t-s)A} f(s)\, ds \\ &= \left( \int_0^t e^{(t-s)} \sin s ds, \int_0^t e^{-(t-s)} \cos s ds \right) \\ &= \left( \frac{1}{2} \left( e^t - \sin t - \cos t \right), \frac{1}{2} \left( \sin t + \cos t - e^{-t} \right) \right). \end{aligned} \tag{6.1}$$

Now, if we consider

$$\tilde{M}_n(t) := \frac{t}{n} \sum_{i=0}^{n-1} \left[ e^{\left[ \left( \frac{2n-2i-1}{2n} \right) t \right]} \sin \left[ \left( \frac{2i+1}{2n} \right) t \right], e^{-\left[ \left( \frac{2n-2i-1}{2n} \right) t \right]} \cos \left[ \left( \frac{2i+1}{2n} \right) t \right] \right]$$

then, by (5.11), the exact solution $u(\cdot)$ given in (6.1) may be represented by

$$u(t) = \tilde{M}_n(t) + \tilde{V}_n(t) \text{ for any } t \geq 0. \tag{6.2}$$

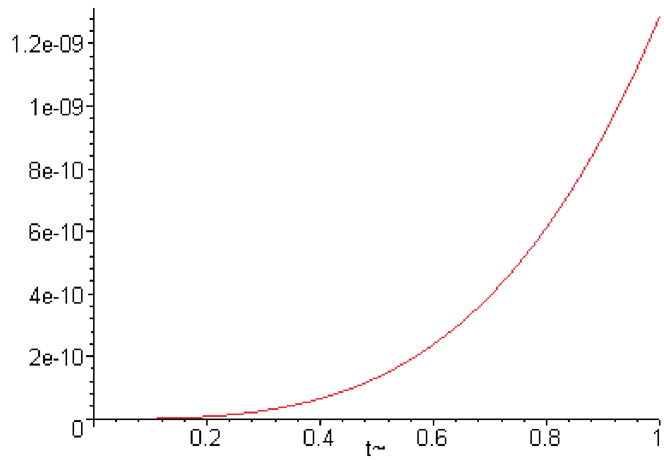
and, by (5.12), we know that

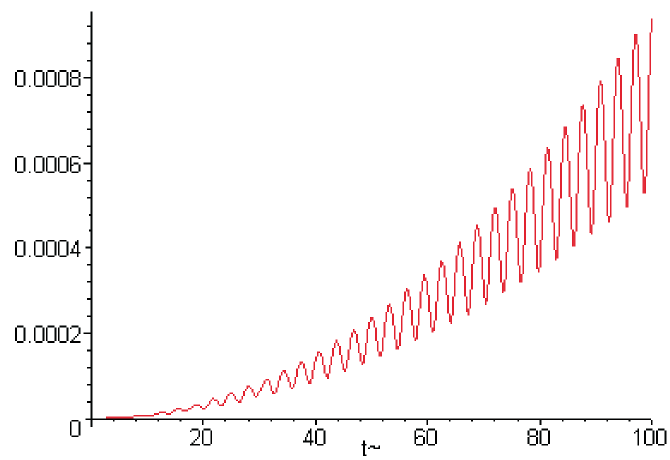
$$\lim_{n \to \infty} \left\| \tilde{V}_n(t) \right\|_2 = 0 \text{ for each } t \geq 0. \tag{6.3}$$

We have

$$\begin{aligned} B_n(t) :&= \left\| \tilde{V}_n(t) \right\|_2 \\ &= \left\{ \left[ \frac{1}{2} \left( e^t - \sin t - \cos t \right) - \frac{t}{n} \sum_{i=0}^{n-1} e^{\left[ \left( \frac{2n-2i-1}{2n} \right) t \right]} \sin \left[ \left( \frac{2i+1}{2n} \right) t \right] \right]^2 \right. \\ &\quad \left. + \left[ \frac{1}{2} \left( \sin t + \cos t - e^{-t} \right) - \frac{t}{n} \sum_{i=0}^{n-1} e^{-\left[ \left( \frac{2n-2i-1}{2n} \right) t \right]} \cos \left[ \left( \frac{2i+1}{2n} \right) t \right] \right]^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

If we implement $B_n(\cdot)$ for $n = 10^6$ and $t \in [0, 1]$, then the plot of the error in approximating the exact value of $u(\cdot)$ by its approximation $\tilde{M}_n(\cdot)$ on the interval $[0, 1]$ is embodied in Figure 1.




Let us now consider another system

$$\left\{\begin{array}{llll}\dot{u}_1(t)= & -u_1(t) & & +\sin t\\ \dot{u}_2(t)= & & -2u_2(t) & +\cos t\\ u_1(0)= & u_2(0)=0. & & \end{array}\right. . \tag{6.4}$$

The solution of this system is given by

$$u(t)=\left(\frac{1}{2}\left(e^{-t}+\sin t-\cos t\right),\frac{1}{5}\left(-2e^{-2t}+\sin t+2\cos t\right)\right). \tag{6.5}$$

Now, if we consider

$$\tilde{M}_n(t):=\frac{t}{n}\sum_{i=0}^{n-1}\left[e^{-\left[\left(\frac{2n-2i-1}{2n}\right)t\right]}\sin\left[\left(\frac{2i+1}{2n}\right)t\right],e^{-2\left[\left(\frac{2n-2i-1}{2n}\right)t\right]}\cos\left[\left(\frac{2i+1}{2n}\right)t\right]\right]$$

then by (5.11) the exact solution of the system (6.4), given in (6.5) may be represented as in (6.2), and by (5.13), we know that

$$\lim_{n\to\infty}\left\|\tilde{V}_n(t)\right\|_2 = 0$$

for any $t$ on $[0,\infty)$. We have

$$\begin{aligned}
B_n(t) \quad : & = \left\|\tilde{V}_n(t)\right\|_2 \\
& = \left\{\left[\frac{1}{2}\left(e^t - \sin t - \cos t\right) - \frac{t}{n}\sum_{i=0}^{n-1} e^{-\left[\left(\frac{2n-2i-1}{2n}\right)t\right]}\sin\left[\left(\frac{2i+1}{2n}\right)t\right]\right]^2\right. \\
& \quad \left. + \left[\frac{1}{5}\left(-2e^{-2t} + \sin t + 2\cos t\right) - \frac{t}{n}\sum_{i=0}^{n-1} e^{-2\left[\left(\frac{2n-2i-1}{2n}\right)t\right]}\cos\left[\left(\frac{2i+1}{2n}\right)t\right]\right]^2\right\}^{\frac{1}{2}}.
\end{aligned}$$

If we implement $B_n(\cdot)$ for $n = 10^3$, then the plot of the error in approximating the exact value $u(\cdot)$ by its approximation $\tilde{M}_n(\cdot)$ on the interval $[0, 100]$ is embodied in Figure 2.

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