On Weighted Ostrowski Type Inequalities for Operators and Vector-Valued Functions

ON WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR OPERATORS AND VECTOR-VALUED FUNCTIONS

N.S. BARNETT, C. BUŞE, P. CERONE, AND S.S. DRAGOMIR

Abstract. Some weighted Ostrowski type integral inequalities for operators and vector-valued functions in Banach spaces are given. Applications for linear operators in Banach spaces and differential equations are also provided.

1. Introduction

In [12], Pečarić and Savić obtained the following Ostrowski type inequality for weighted integrals (see also [7, Theorem 3]):

Theorem 1. *Let $w:[a,b]\to[0,\infty)$ be a weight function on $[a,b]$. Suppose that $f:[a,b]\to\mathbb{R}$ satisfies*

$$|f(t)-f(s)|\leq N|t-s|^{\alpha},\ \text{for all } t,s\in[a,b], \tag{1.1}$$

where $N>0$ and $0<\alpha\leq 1$ are some constants. Then for any $x\in[a,b]$

$$\left|f(x)-\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(t)\,dt}\right|\leq N\cdot\frac{\int_a^b|t-x|^{\alpha}w(t)\,dt}{\int_a^b w(t)\,dt}. \tag{1.2}$$

Further, if for some constants c and λ

$$0<c\leq w(t)\leq\lambda c,\ \text{ for all }\ t\in[a,b],$$

then for any $x\in[a,b]$, we have

$$\left|f(x)-\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(t)\,dt}\right|\leq N\cdot\frac{\lambda L(x)J(x)}{L(x)-J(x)+\lambda J(x)}, \tag{1.3}$$

where

$$L(x):=\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}$$

and

$$J(x):=\frac{(x-a)^{1+\alpha}+(b-x)^{1+\alpha}}{(1+\alpha)(b-a)}.$$

The inequality (1.2) was rediscovered in [4] where further applications for different weights and in Numerical Analysis were given.

For other results in connection to weighted Ostrowski inequalities, see [3], [8] and [10].

Date: May 17, 2001.
1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 46B99.
Key words and phrases. Ostrowski's Inequality, Vector-Valued Functions, Bochner Integral.

In the present paper we extend the weighted Ostrowski's inequality for vector-valued functions and Bochner integrals and apply the obtained results for operatorial inequalities and linear differential equations in Banach spaces. Some numerical experiments are also conducted.

2. Weighted Inequalities

Let X be a Banach space and $-\infty < a < b < \infty$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms of vectors or operators acting on X will be denoted by $\|\cdot\|$.

A function $f : [a,b] \to X$ is called *measurable* if there exists a sequence of simple functions $f_n : [a,b] \to X$ which converges punctually almost everywhere on $[a,b]$ at f. We recall also that a measurable function $f : [a,b] \to X$ is *Bochner integrable* if and only if its norm function (i.e. the function $t \mapsto \|f(t)\| : [a,b] \to \mathbb{R}_+$) is Lebesgue integrable on $[a,b]$.

The following theorem holds.

Theorem 2. *Assume that $B : [a,b] \to \mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e.,*

$$\|B(t) - B(s)\| \leq H |t-s|^{\alpha} \quad \text{for all } \ t,s \in [a,b], \tag{2.1}$$

where $H > 0$ and $\alpha \in (0,1]$.

If $f : [a,b] \to X$ is Bochner integrable on $[a,b]$, then we have the inequality:

$$\left\| B(t) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \tag{2.2}$$

$$\leq H \int_a^b |t-s|^{\alpha} \|f(s)\|\,ds$$

$$\leq H \times \begin{cases} \dfrac{(b-t)^{\alpha+1} + (t-a)^{\alpha+1}}{\alpha+1} |||f|||_{[a,b],\infty} & \text{if} \quad f \in L_{\infty}([a,b];X); \\[2ex] \left[\dfrac{(b-t)^{q\alpha+1} + (t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}} |||f|||_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f \in L_p([a,b];X); \\ \left[\dfrac{1}{2}(b-a) + \left| t - \dfrac{a+b}{2} \right|\right]^{\alpha} |||f|||_{[a,b],1} & \end{cases}$$

for any $t \in [a,b]$.

Proof. Firstly, we prove that the $X-$valued function $s \mapsto B(s) f(s)$ is Bochner integrable on $[a,b]$. Indeed, let (f_n) be a sequence of $X-$valued, simple functions which converge almost everywhere on $[a,b]$ at the function f. The maps $s \mapsto B(s) f_n(s)$ are measurable (because they are continuous with the exception of a finite number of points s in $[a,b]$). Then

$$\|B(s) f_n(s) - B(s) f(s)\| \leq \|B(s)\| \|f_n(s) - f(s)\| \to 0 \ \text{ a.e. on } \ [a,b]$$

when $n \to \infty$ so that the function $s \mapsto B(s) f(s) : [a,b] \to X$ is measurable. Now, using the estimate

$$\|B(s) f(s)\| \leq \sup_{\xi \in [a,b]} \|B(\xi)\| \cdot \|f(s)\|, \ \text{ for all } s \in [a,b],$$

it is easy to see that the function $s \mapsto B(s) f(s)$ is Bochner integrable on $[a,b]$.

We have successively

$$\begin{aligned}
&\left\| B(t) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \\
= &\left\| \int_a^b (B(t) - B(s)) f(s)\,ds \right\| \le \int_a^b \|(B(t) - B(s)) f(s)\|\,ds \\
\le &\int_a^b \|(B(t) - B(s))\| \, \|f(s)\|\,ds \le H \int_a^b |t-s|^{\alpha} \|f(s)\|\,ds =: M(t)
\end{aligned}$$

for any $t \in [a,b]$, proving the first inequality in (2.2).

Now, observe that

$$\begin{aligned}
M(t) &\le H \, |||f|||_{[a,b],\infty} \int_a^b |t-s|^{\alpha}\,ds \\
&= H \, |||f|||_{[a,b],\infty} \cdot \frac{(b-t)^{\alpha+1} + (t-a)^{\alpha+1}}{\alpha+1}
\end{aligned}$$

and the first part of the second inequality is proved.

Using Hölder's integral inequality, we may state that

$$\begin{aligned}
M(t) &\le H \left(\int_a^b |t-s|^{q\alpha}\,ds \right)^{\frac{1}{q}} \left(\int_a^b \|f(s)\|^p\,ds \right)^{\frac{1}{p}} \\
&= H \left[\frac{(b-t)^{q\alpha+1} + (t-a)^{q\alpha+1}}{q\alpha+1} \right]^{\frac{1}{q}} |||f|||_{[a,b],p} \, ,
\end{aligned}$$

proving the second part of the second inequality.

Finally, we observe that

$$\begin{aligned}
M(t) &\le H \sup_{s \in [a,b]} |t-s|^{\alpha} \int_a^b \|f(s)\|\,ds \\
&= H \max\left\{ (b-t)^{\alpha}, (t-a)^{\alpha} \right\} |||f|||_{[a,b],1} \\
&= H \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right]^{\alpha} |||f|||_{[a,b],1}
\end{aligned}$$

and the theorem is proved. ∎

The following corollary holds.

Corollary 1. *Assume that $B : [a,b] \to \mathcal{L}(X)$ is Lipschitzian with the constant $L > 0$. Then we have the inequality*

$$\begin{aligned}
&\left\| B(t) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \qquad (2.3) \\
\le \; & L \int_a^b |t-s| \, \|f(s)\|\,ds
\end{aligned}$$

$$\leq L\times\begin{cases}\left[\frac{1}{4}\left(b-a\right)^{2}+\left(t-\frac{a+b}{2}\right)^{2}\right]\left|\left|\left|f\right|\right|\right|_{[a,b],\infty} & \text{if} \quad f\in L_{\infty}\left(\left[a,b\right];X\right);\\ \left[\frac{\left(b-t\right)^{q+1}+\left(t-a\right)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left|\left|\left|f\right|\right|\right|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ & \text{and} \quad f\in L_{p}\left(\left[a,b\right];X\right);\\ \left[\frac{1}{2}\left(b-a\right)+\left|t-\frac{a+b}{2}\right|\right]\left|\left|\left|f\right|\right|\right|_{[a,b],1} & \end{cases}$$

for any $t\in[a,b]$.

Remark 1. *If we choose* $t=\frac{a+b}{2}$ *in (2.2) and (2.3), then we get the following midpoint inequalities:*

$$\left\|B\left(\frac{a+b}{2}\right)\int_{a}^{b}f\left(s\right)ds-\int_{a}^{b}B\left(s\right)f\left(s\right)ds\right\| \tag{2.4}$$

$$\leq H\int_{a}^{b}\left|s-\frac{a+b}{2}\right|^{\alpha}\left\|f\left(s\right)\right\|ds$$

$$\leq H\times\begin{cases}\frac{1}{2^{\alpha}\left(\alpha+1\right)}\left(b-a\right)^{\alpha+1}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty} & \text{if} \quad f\in L_{\infty}\left(\left[a,b\right];X\right);\\ \frac{1}{2^{\alpha}\left(q\alpha+1\right)^{\frac{1}{q}}}\left(b-a\right)^{\alpha+\frac{1}{q}}\left|\left|\left|f\right|\right|\right|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ & \text{and} \quad f\in L_{p}\left(\left[a,b\right];X\right);\\ \frac{1}{2^{\alpha}}\left(b-a\right)^{\alpha}\left|\left|\left|f\right|\right|\right|_{[a,b],1} & \end{cases}$$

and

$$\left\|B\left(\frac{a+b}{2}\right)\int_{a}^{b}f\left(s\right)ds-\int_{a}^{b}B\left(s\right)f\left(s\right)ds\right\| \tag{2.5}$$

$$\leq L\int_{a}^{b}\left|s-\frac{a+b}{2}\right|\left\|f\left(s\right)\right\|ds$$

$$\leq L\times\begin{cases}\frac{1}{4}\left(b-a\right)^{2}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty} & \text{if} \quad f\in L_{\infty}\left(\left[a,b\right];X\right);\\ \frac{1}{2\left(q+1\right)^{\frac{1}{q}}}\left(b-a\right)^{1+\frac{1}{q}}\left|\left|\left|f\right|\right|\right|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ & \text{and} \quad f\in L_{p}\left(\left[a,b\right];X\right);\\ \frac{1}{2}\left(b-a\right)\left|\left|\left|f\right|\right|\right|_{[a,b],1} & \end{cases}$$

respectively.

Remark 2. *Consider the function $\Psi_\alpha : [a,b] \to \mathbb{R}$, $\Psi_\alpha(t) := \int_a^b |t-s|^\alpha \|f(s)\| \, ds$, $\alpha \in (0,1)$. If f is continuous on $[a,b]$, then Ψ_α is differentiable and*

$$\begin{aligned} \frac{d\Psi_\alpha(t)}{dt} &= \frac{d}{dt}\left[\int_a^t (t-s)^\alpha \|f(s)\| \, ds + \int_t^b (s-t)^\alpha \|f(s)\| \, ds\right] \\ &= \alpha\left[\int_a^t \frac{\|f(s)\|}{(t-s)^{1-\alpha}} ds - \int_t^b \frac{\|f(s)\|}{(s-t)^{1-\alpha}} ds\right]. \end{aligned}$$

If $t_0 \in (a,b)$ is such that

$$\int_a^{t_0} \frac{\|f(s)\|}{(t_0-s)^{1-\alpha}} ds = \int_{t_0}^b \frac{\|f(s)\|}{(s-t_0)^{1-\alpha}} ds$$

and $\Psi_s'(\cdot)$ is negative on (a,t_0) and positive on (t_0,b), then the best inequality we can get in the first part of (2.2) is the following one

$$\left\|B(t_0)\int_a^b f(s)\, ds - \int_a^b B(s) f(s)\, ds\right\| \leq H \int_a^b |t_0 - s|^\alpha \|f(s)\| \, ds. \tag{2.6}$$

If $\alpha = 1$, then, for

$$\Psi(t) := \int_a^b |t-s| \|f(s)\| \, ds,$$

we have

$$\begin{aligned} \frac{d\Psi(t)}{dt} &= \int_a^t \|f(s)\| \, ds - \int_t^b \|f(s)\| \, ds, \quad t \in (a,b), \\ \frac{d^2\Psi(t)}{dt^2} &= 2\|f(t)\| \geq 0, \quad t \in (a,b), \end{aligned}$$

which shows that Ψ is convex on (a,b).

If $t_m \in (a,b)$ is such that

$$\int_a^{t_m} \|f(s)\| \, ds = \int_{t_m}^b \|f(s)\| \, ds,$$

then the best inequality we can get from the first part of (2.3) is

$$\left\|B(t_m)\int_a^b f(s)\, ds - \int_a^b B(s) f(s)\, ds\right\| \leq L \int_a^b sgn(s-t_m)\, s \|f(s)\| \, ds. \tag{2.7}$$

Indeed, as

$$\begin{aligned} &\inf_{t\in[a,b]} \int_a^b |t-s| \|f(s)\| \, ds \\ &= \int_a^b |t_m - s| \|f(s)\| \, ds = \int_a^{t_m} (t_m - s) \|f(s)\| \, ds + \int_{t_m}^b (s-t_m) \|f(s)\| \, ds \\ &= t_m \left(\int_a^{t_m} \|f(s)\| \, ds - \int_{t_m}^b \|f(s)\| \, ds\right) + \int_{t_m}^b s\|f(s)\| \, ds - \int_a^{t_m} s\|f(s)\| \, ds \\ &= \int_{t_m}^b s\|f(s)\| \, ds - \int_a^{t_m} s\|f(s)\| \, ds = \int_a^b sgn(s-t_m)\, s\|f(s)\| \, ds, \end{aligned}$$

then the best inequality we can get from the first part of (2.3) is obtained for $t = t_m \in (a,b)$.

We recall that a function $F:[a,b]\to\mathcal{L}(X)$ is said to be *strongly continuous* if for all $x\in X$, the maps $s\mapsto F(s)x:[a,b]\to X$ are continuous on $[a,b]$. In this case the function $s\mapsto\|B(s)\|:[a,b]\to\mathbb{R}_+$ is (Lebesgue) measurable and bounded ([6]). The linear operator $L=\int_a^b F(s)\,ds$ (defined by $Lx:=\int_a^b F(s)\,xds$ for all $x\in X$) is bounded, because

$$\|Lx\|\le\left(\int_a^b\|F(s)\|\,ds\right)\cdot\|x\|\quad\text{for all}\quad x\in X.$$

In a similar manner to Theorem 2, we may prove the following result as well.

Theorem 3. *Assume that $f:[a,b]\to X$ is Hölder continuous, i.e.,*

$$\|f(t)-f(s)\|\le K|t-s|^{\beta}\quad\textit{for all}\quad t,s\in[a,b],\tag{2.8}$$

where $K>0$ and $\beta\in(0,1]$.

If $B:[a,b]\to\mathcal{L}(X)$ is strongly continuous on $[a,b]$, then we have the inequality:

$$\begin{aligned}&\left\|\left(\int_a^b B(s)\,ds\right)f(t)-\int_a^b B(s)\,f(s)\,ds\right\| \qquad (2.9)\\ \le\;& K\int_a^b|t-s|^{\beta}\|B(s)\|\,ds\\ \le\;& K\times\begin{cases}\frac{(b-t)^{\beta+1}+(t-a)^{\beta+1}}{\beta+1}\,|||B|||_{[a,b],\infty} & \textit{if}\quad \|B(\cdot)\|\in L_\infty([a,b];\mathbb{R}_+);\\[2mm] \left[\frac{(b-t)^{q\beta+1}+(t-a)^{q\beta+1}}{q\beta+1}\right]^{\frac1q}|||B|||_{[a,b],p} & \textit{if}\quad p>1,\ \frac1p+\frac1q=1\\ & \textit{and}\quad \|B(\cdot)\|\in L_p([a,b];\mathbb{R}_+);\\[2mm] \left[\frac12(b-a)+\left|t-\frac{a+b}{2}\right|\right]^{\beta}|||B|||_{[a,b],1}\end{cases}\end{aligned}$$

for any $t\in[a,b]$.

The following corollary holds.

Corollary 2. *Assume that f and B are as in Theorem 3. If, in addition, $\int_a^b B(s)\,ds$ is invertible in $\mathcal{L}(X)$, then we have the inequality:*

$$\begin{aligned}&\left\|f(t)-\left(\int_a^b B(s)\,ds\right)^{-1}\int_a^b B(s)\,f(s)\,ds\right\|\\ \le\;& K\left\|\left(\int_a^b B(s)\,ds\right)^{-1}\right\|\int_a^b|t-s|^{\beta}\|B(s)\|\,ds\end{aligned}\tag{2.10}$$

for any $t\in[a,b]$.

Remark 3. *It is obvious that the inequality (2.10) contains as a particular case what is the so called Ostrowski's inequality for weighted integrals (see (1.2)).*

3. Inequalities for Linear Operators

Let $0\le a<b<\infty$ and $A\in\mathcal{L}(X)$. We recall that the operatorial norm of A is given by

$$\|A\|=\sup\{\|Ax\|:\|x\|\le1\}.$$

The *resolvent set* of A (denoted by $\rho(A)$) is the set of all complex scalars λ for which $\lambda I - A$ is an invertible operator. Here I is the identity operator in $\mathcal{L}(X)$. The complementary set of $\rho(A)$ in the complex plane, denoted by $\sigma(A)$, is the *spectrum* of A. It is known that $\sigma(A)$ is a compact set in $\mathbb{C}$. The series $\left(\sum_{n\geq 0}\frac{(tA)^n}{n!}\right)$ converges absolutely and locally uniformly for $t\in\mathbb{R}$. If we denote by e^{tA} its sum, then

$$\left\|e^{tA}\right\| \leq e^{|t|\|A\|},\ \ t\in\mathbb{R}.$$

Proposition 1. *Let X be a real or complex Banach space, $A\in\mathcal{L}(X)$ and β be a non-null real number such that $-\beta\in\rho(A)$. Then for all $0\leq a<b<\infty$ and each $s\in[a,b]$, we have*

$$\left\|\frac{e^{\beta b}-e^{\beta a}}{\beta}\cdot e^{sA}-(\beta I+A)^{-1}\left[e^{b(\beta I+A)}-e^{a(\beta I+A)}\right]\right\| \tag{3.1}$$
$$\leq \|A\|\,e^{b\|A\|}\cdot\left[\frac{1}{4}(b-a)^2+\left(s-\frac{a+b}{2}\right)^2\right]\cdot\max\left\{e^{\beta b},e^{\beta a}\right\}.$$

Proof. We apply the second inequality from Corollary 1 in the following particular case.

$$B(\tau):=e^{\tau A},\ \ f(\tau)=e^{\beta\tau}x,\ \ \tau\in[a,b],\ x\in X.$$

For all $\xi,\eta\in[a,b]$ there exists an α between ξ and η such that

$$\begin{aligned}\|B(\xi)-B(\eta)\| &= \left\|\sum_{n=1}^{\infty}\frac{(\xi^n-\eta^n)}{n!}A^n\right\| = \left\|(\xi-\eta)A\sum_{n=0}^{\infty}\frac{(\alpha A)^n}{n!}\right\|\\ &\leq \|A\|\left\|e^{\alpha A}\right\|\cdot|\xi-\eta|\leq\|A\|\,e^{b\|A\|}\cdot|\xi-\eta|.\end{aligned}$$

The function $\tau\mapsto e^{\tau A}$ is thus Lipschitzian on $[a,b]$ with the constant $L:=\|A\|\,e^{b\cdot\|A\|}$. On the other hand we have

$$\begin{aligned}\int_a^b e^{\tau A}\left(e^{\beta\tau}x\right)d\tau &= \int_a^b e^{\tau A}\left(e^{\beta\tau}Ix\right)d\tau=\int_a^b e^{\tau(A+\beta I)}x\,d\tau\\ &= (A+\beta I)^{-1}\left[e^{b(A+\beta I)}-e^{a(A+\beta I)}\right]x,\end{aligned}$$

and

$$|||f|||_{[a,b],\infty}=\sup_{\tau\in[a,b]}\left\|e^{\tau\beta}x\right\|=\max\left\{e^{\beta b},e^{\beta a}\right\}\cdot\|x\|.$$

Placing all the above results in the second inequality from (2.3) and taking the supremum for all $x\in X$, we will obtain the desired inequality (3.1). ∎

Remark 4. *Let $A\in\mathcal{L}(X)$ such that $0\in\rho(A)$. Taking the limit as $\beta\to 0$ in (3.1), we get the inequality*

$$\left\|(b-a)\,e^{sA}-A^{-1}\left[e^{bA}-e^{aA}\right]\right\|$$
$$\leq \|A\|\,e^{b\|A\|}\cdot\left[\frac{1}{4}(b-a)^2+\left(s-\frac{a+b}{2}\right)^2\right],$$

where a,b and s are as in Proposition 1.

Proposition 2. *Let $A \in \mathcal{L}(X)$ be an invertible operator, $t \geq 0$ and $0 \leq s \leq t$. Then the following inequality holds:*

$$\left\| \frac{t^2}{2} \sin(sA) - A^{-2} [\sin(tA) - tA\cos(tA)] \right\| \leq \frac{2s^3 + 2t^3 - 3st^2}{6} \|A\| . \tag{3.2}$$

In particular, if $X = \mathbb{R}$, $A = 1$ and $s = 0$ it follows the scalar inequality

$$|\sin t - t\cos t| \leq \frac{t^3}{3}, \quad \textit{for all } t \geq 0.$$

Proof. We apply the inequality from (2.3) in the following particular case:

$$B(\tau) = \sin(\tau A) := \sum_{n=0}^{\infty} (-1)^n \frac{(\tau A)^{2n+1}}{(2n+1)!}, \quad \tau \geq 0,$$

and

$$f(\tau) = \tau \cdot x, \quad \text{for fixed } x \in X. \tag{3.3}$$

For each $\xi, \eta \in [0, t]$, we have

$$\begin{aligned} \|B(\xi) - B(\eta)\| &= \left\| A \left(\sum_{n=0}^{\infty} (-1)^n \frac{(\xi - \eta)\alpha^{2n}}{(2n)!} A^{2n} \right) \right\| \\ &\leq \|A\| |\xi - \eta| \cdot \|\cos(\alpha A)\| \leq \|A\| |\xi - \eta| , \end{aligned}$$

where α is a real number between ξ and η, i.e., the function $\tau \longmapsto B(\tau) : \mathbb{R}_+ \to \mathcal{L}(X)$ is $\|A\|$ −Lipschitzian.

Moreover, it is easy to see that

$$\int_0^t B(\tau) f(\tau) d\tau = A^{-2} [\sin(tA) - tA\cos(tA)] x$$

and

$$\int_0^t |s - \tau| |f(\tau)| d\tau = \frac{2s^3 + 2t^3 - 3st^2}{6} \|x\| . \tag{3.4}$$

Applying the first inequality from (2.3) and taking the supremum for $x \in X$ with $\|x\| \leq 1$, we get (3.2). ∎

4. Quadrature Formulae

Consider the division of the interval $[a, b]$ given by

$$I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \tag{4.1}$$

and $h_i := t_{i+1} - t_i$, $\nu(h) := \max_{i=\overline{0,n-1}} h_i$. For the intermediate points $\boldsymbol{\xi} := (\xi_0, \ldots, \xi_{n-1})$ with $\xi_i \in [t_i, t_{i+1}]$, $i = \overline{0, n-1}$, define the sum

$$S_n^{(1)}(B, f; I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} B(\xi_i) \int_{t_i}^{t_{i+1}} f(s) ds. \tag{4.2}$$

Then we may state the following result in approximating the integral

$$\int_a^b B(s) f(s) ds,$$

based on Theorem 2.

Theorem 4. *Assume that $B:[a,b]\to\mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e., it satisfies the condition (2.1) and $f:[a,b]\to X$ is Bochner integrable on $[a,b]$. Then we have the representation*

$$\int_a^b B(s)\,f(s)\,ds = S_n^{(1)}(B,f;I_n,\boldsymbol{\xi}) + R_n^{(1)}(B,f;I_n,\boldsymbol{\xi}), \tag{4.3}$$

where $S_n^{(1)}(B,f;I_n,\boldsymbol{\xi})$ is as given by (4.2) and the remainder $R_n^{(1)}(B,f;I_n,\boldsymbol{\xi})$ satisfies the estimate

$$\begin{aligned}
&\left\|R_n^{(1)}(B,f;I_n,\boldsymbol{\xi})\right\| \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}|||f|||_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right] \\[2ex]
\dfrac{1}{(q\alpha+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\left\{\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]\right\}^{\frac{1}{q}}, \\
\qquad\qquad\qquad\qquad\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\[2ex]
\left[\dfrac{1}{2}\nu(h)+\displaystyle\max_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[a,b],1}
\end{cases} \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}|||f|||_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}h_i^{\alpha+1} \\[2ex]
\dfrac{1}{(q\alpha+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\left(\displaystyle\sum_{i=0}^{n-1}h_i^{q\alpha+1}\right)^{\frac{1}{q}}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\[2ex]
\left[\dfrac{1}{2}\nu(h)+\displaystyle\max_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[a,b],1}
\end{cases} \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}|||f|||_{[a,b],\infty}[\nu(h)]^{\alpha} \\[2ex]
\dfrac{(b-a)^{\frac{1}{q}}}{(q\alpha+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}[\nu(h)]^{\alpha} \\[2ex]
|||f|||_{[a,b],1}[\nu(h)]^{\alpha}.
\end{cases}
\end{aligned}$$

Proof. Applying Theorem 4 on $[x_i, x_{i+1}]$ $\left(i=\overline{0,n-1}\right)$, we may write that

$$\begin{aligned}
&\left\|\int_{t_i}^{t_{i+1}} B(s) f(s)\,ds - B(\xi_i)\int_{t_i}^{t_{i+1}} f(s)\,ds\right\| \\
&\leq H\times\begin{cases}
\left[\dfrac{(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}}{\alpha+1}\right]|||f|||_{[t_i,t_{i+1}],\infty} \\[2ex]
\left[\dfrac{(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}}|||f|||_{[t_i,t_{i+1}],p} \\[2ex]
\left[\dfrac{1}{2}(t_{i+1}-t_i)+\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[t_i,t_{i+1}],1}
\end{cases}.
\end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalised triangle inequality we get

$$\begin{aligned}
&\left\|R_n^{(1)}(B,f;I_n,\boldsymbol{\xi})\right\| \\
&\leq \sum_{i=0}^{n-1}\left\|\int_{t_i}^{t_{i+1}} B(s) f(s)\,ds - B(\xi_i)\int_{t_i}^{t_{i+1}} f(s)\,ds\right\| \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}\sum\limits_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right]|||f|||_{[t_i,t_{i+1}],\infty} \\[2ex]
\dfrac{1}{(q\alpha+1)^{\frac{1}{q}}}\left[\sum\limits_{i=0}^{n-1}(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]^{\frac{1}{q}}|||f|||_{[t_i,t_{i+1}],p} \\[2ex]
\sum\limits_{i=0}^{n-1}\left[\dfrac{1}{2}h_i+\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[t_i,t_{i+1}],1}
\end{cases}.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right]|||f|||_{[t_i,t_{i+1}],\infty} \\
&\leq |||f|||_{[a,b],\infty}\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right] \\
&\leq |||f|||_{[a,b],\infty}\sum_{i=0}^{n-1}h_i^{\alpha+1}\leq|||f|||_{[a,b],\infty}(b-a)[\nu(h)]^{\alpha}.
\end{aligned}$$

Using the discrete Hölder inequality, we may write that

$$\begin{aligned}
&\left[\sum_{i=0}^{n-1}(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]^{\frac{1}{q}}|||f|||_{[t_i,t_{i+1}],p} \\
&\leq \left[\sum_{i=0}^{n-1}\left(\left[(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]^{\frac{1}{q}}\right)^{q}\right]^{\frac{1}{q}}\times\left[\sum_{i=0}^{n-1}|||f|||_{[t_i,t_{i+1}],p}^{p}\right]^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{\sum_{i=0}^{n-1}\left[\left(t_{i+1}-\xi_i\right)^{q\alpha+1}+\left(\xi_i-t_i\right)^{q\alpha+1}\right]\right\}^{\frac{1}{q}}\left(\int_a^b \|f(t)\|^p\,ds\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=0}^{n-1} h_i^{q\alpha+1}\right)^{\frac{1}{q}} |||f|||_{[a,b],p}^{p} \leq (b-a)^{\frac{1}{q}} |||f|||_{[a,b],p} \left[\nu(h)\right]^{\alpha}.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]^{\alpha} |||f|||_{[t_i,t_{i+1}],1} \\
&\leq \left[\frac{1}{2}\max_{i=\overline{0,n-1}} h_i+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]^{\alpha} |||f|||_{[a,b],1} \\
&\leq \left[\nu(h)\right]^{\alpha} |||f|||_{[a,b],1}
\end{aligned}
$$

and the theorem is proved. ∎

The following corollary holds.

Corollary 3. *If B is Lipschitzian with the constant L, then we have the representation (4.3) and the remainder $R_n^{(1)}(B,f;I_n,\boldsymbol{\xi})$ satisfies the estimates:*

$$
\left\|R_n^{(1)}(B,f;I_n,\boldsymbol{\xi})\right\| \tag{4.4}
$$

$$
\leq L\times\begin{cases}
|||f|||_{[a,b],\infty}\left[\frac{1}{4}\sum\limits_{i=0}^{n-1}h_i^2+\sum\limits_{i=0}^{n-1}\left(\xi_i-\frac{t_{i+1}+t_i}{2}\right)^2\right] \\[2ex]
\frac{1}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\left\{\sum\limits_{i=0}^{n-1}\left[\left(t_{i+1}-\xi_i\right)^{q+1}+\left(\xi_i-t_i\right)^{q+1}\right]\right\}^{\frac{1}{q}}, \\
\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\[2ex]
\left[\frac{1}{2}\nu(h)+\max\limits_{i=\overline{0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]|||f|||_{[a,b],1}
\end{cases}
$$

$$
\leq L\times\begin{cases}
\frac{1}{2}|||f|||_{[a,b],\infty}\sum\limits_{i=0}^{n-1}h_i^2 \\[2ex]
\frac{1}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\left(\sum\limits_{i=0}^{n-1}h_i^{q+1}\right)^{\frac{1}{q}} \\[2ex]
\left[\frac{1}{2}\nu(h)+\max\limits_{i=\overline{0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]|||f|||_{[a,b],1}
\end{cases}
$$

$$
\leq L\times\begin{cases}
\frac{1}{2}|||f|||_{[a,b],\infty}(b-a)\,\nu(h) \\[2ex]
\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\,\nu(h) \\[2ex]
|||f|||_{[a,b],1}\,\nu(h)
\end{cases}.
$$

The second possibility we have for approximating the integral $\int_a^b B(s) f(s)\, ds$ is embodied in the following theorem based on Theorem 3.

Theorem 5. *Assume that $f:[a,b]\to X$ is Hölder continuous, i.e., the condition (2.8) holds. If $B:[a,b]\to\mathcal{L}(X)$ is strongly continuous on $[a,b]$, then we have the representation:*

$$\int_a^b B(s) f(s)\, ds = S_n^{(2)}(B,f;I_n,\boldsymbol{\xi}) + R_n^{(2)}(B,f;I_n,\boldsymbol{\xi}), \tag{4.5}$$

where

$$S_n^{(2)}(B,f;I_n,\boldsymbol{\xi}) := \sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}} B(s)\, ds\right) f(\xi_i) \tag{4.6}$$

and the remainder $R_n^{(2)}(B,f;I_n,\boldsymbol{\xi})$ satisfies the estimate:

$$\left\|R_n^{(2)}(B,f;I_n,\boldsymbol{\xi})\right\| \tag{4.7}$$

$$\leq K\times\begin{cases}\dfrac{1}{\beta+1}\,|||B|||_{[a,b],\infty}\sum\limits_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\beta+1}+(\xi_i-t_i)^{\beta+1}\right]\\[2ex] \dfrac{1}{(q\beta+1)^{\frac{1}{q}}}\,|||B|||_{[a,b],p}\left\{\sum\limits_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q\beta+1}+(\xi_i-t_i)^{q\beta+1}\right]\right\}^{\frac{1}{q}}, \\ \hfill p>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] \left[\dfrac{1}{2}\nu(h)+\max\limits_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\beta}|||B|||_{[a,b],1}\end{cases}$$

$$\leq K\times\begin{cases}\dfrac{1}{\beta+1}\,|||B|||_{[a,b],\infty}\sum\limits_{i=0}^{n-1}h_i^{\beta+1}\\[2ex] \dfrac{1}{(q\beta+1)^{\frac{1}{q}}}\,|||B|||_{[a,b],p}\left\{\sum\limits_{i=0}^{n-1}h_i^{q\beta+1}\right\}^{\frac{1}{q}}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] \left[\dfrac{1}{2}\nu(h)+\max\limits_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\beta}|||B|||_{[a,b],1}\end{cases}$$

$$\leq K\times\begin{cases}\dfrac{1}{\beta+1}\,|||B|||_{[a,b],\infty}(b-a)\left[\nu(h)\right]^{\beta}\\[2ex] \dfrac{(b-a)^{\frac{1}{q}}}{(q\beta+1)^{\frac{1}{q}}}\,|||B|||_{[a,b],p}\left[\nu(h)\right]^{\beta}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] |||B|||_{[a,b],1}\left[\nu(h)\right]^{\beta}\ .\end{cases}$$

If we consider the quadrature

$$M_n^{(1)}(B,f;I_n) := \sum_{i=0}^{n-1} B\left(\frac{t_i+t_{i+1}}{2}\right)\int_{t_i}^{t_{i+1}} f(s)\, ds, \tag{4.8}$$

then we have the representation

$$\int_a^b B(s) f(s)\, ds = M_n^{(1)}(B,f;I_n) + R_n^{(1)}(B,f;I_n), \tag{4.9}$$

and the remainder $R_n^{(1)}(B,f;I_n)$ satisfies the estimate:

$$\left\| R_n^{(1)}(B,f;I_n) \right\| \tag{4.10}$$

$$\leq \quad H \times \begin{cases} \dfrac{1}{2^{\alpha}(\alpha+1)} \left|\left|\left|f\right|\right|\right|_{[a,b],\infty} \sum\limits_{i=0}^{n-1} h_i^{\alpha+1} \\[2ex] \dfrac{1}{2^{\alpha}(q\alpha+1)^{\frac{1}{q}}} \left|\left|\left|f\right|\right|\right|_{[a,b],p} \left[\sum\limits_{i=0}^{n-1} h_i^{q\alpha+1}\right]^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{1}{2^{\alpha}} \left[\nu(h)\right]^{\alpha} \left|\left|\left|f\right|\right|\right|_{[a,b],1} \end{cases}$$

$$\leq \quad H \times \begin{cases} \dfrac{1}{2^{\alpha}(\alpha+1)} (b-a) \left|\left|\left|f\right|\right|\right|_{[a,b],\infty} \left[\nu(h)\right]^{\alpha} \\[2ex] \dfrac{(b-a)^{\frac{1}{q}}}{2^{\alpha}(q\alpha+1)^{\frac{1}{q}}} \left|\left|\left|f\right|\right|\right|_{[a,b],p} \left[\nu(h)\right]^{\alpha}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{1}{2^{\alpha}} \left|\left|\left|f\right|\right|\right|_{[a,b],1} \left[\nu(h)\right]^{\alpha} \end{cases},$$

provided that B and f are as in Theorem 4.

Now, if we consider the quadrature

$$M_n^{(2)}(B,f;I_n) := \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} B(s)\, ds \right) f\left(\frac{t_i + t_{i+1}}{2} \right), \tag{4.11}$$

then we also have

$$\int_a^b B(s) f(s)\, ds = M_n^{(2)}(B,f;I_n) + R_n^{(2)}(B,f;I_n), \tag{4.12}$$

and in this case the remainder satisfies the bound

$$\left\| R_n^{(2)}(B,f;I_n) \right\| \tag{4.13}$$

$$\leq \quad K \times \begin{cases} \dfrac{1}{2^{\beta}(\beta+1)} \left|\left|\left|B\right|\right|\right|_{[a,b],\infty} \sum\limits_{i=0}^{n-1} h_i^{\beta+1} \\[2ex] \dfrac{1}{2^{\beta}(q\beta+1)^{\frac{1}{q}}} \left|\left|\left|B\right|\right|\right|_{[a,b],p} \left(\sum\limits_{i=0}^{n-1} h_i^{q\beta+1}\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{1}{2^{\beta}} \left[\nu(h)\right]^{\beta} \left|\left|\left|B\right|\right|\right|_{[a,b],1} \end{cases}$$

$$\leq K\times\begin{cases}\dfrac{1}{2^{\beta}\left(\beta+1\right)}\left(b-a\right)\left\|\left|B\right|\right\|_{[a,b],\infty}\left[\nu\left(h\right)\right]^{\beta}\\ \\ \dfrac{\left(b-a\right)^{\frac{1}{q}}}{2^{\beta}\left(q\beta+1\right)^{\frac{1}{q}}}\left\|\left|B\right|\right\|_{[a,b],p}\left[\nu\left(h\right)\right]^{\beta}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \dfrac{1}{2^{\beta}}\left\|\left|B\right|\right\|_{[a,b],1}\left[\nu\left(h\right)\right]^{\beta}\end{cases}, \tag{4.14}$$

provided B and f satisfy the hypothesis of Theorem 5.

Now, if we consider the equidistant partitioning of $[a,b]$,

$$E_n : t_i := a+\left(\frac{b-a}{n}\right)\cdot i, \quad i=\overline{0,n},$$

then $M_n^{(1)}\left(B,f;E_n\right)$ becomes

$$M_n^{(1)}\left(B,f\right):=\sum_{i=0}^{n-1}B\left(a+\left(i+\frac{1}{2}\right)\cdot\frac{b-a}{n}\right)\int_{a+\frac{b-a}{n}\cdot i}^{a+\frac{b-a}{n}\cdot(i+1)}f\left(s\right)ds \tag{4.15}$$

and then

$$\int_a^b B\left(s\right)f\left(s\right)ds=M_n^{(1)}\left(B,f\right)+R_n^{(1)}\left(B,f\right), \tag{4.16}$$

where the remainder satisfies the bound

$$\left\|R_n^{(1)}\left(B,f\right)\right\|\leq H\times\begin{cases}\dfrac{\left(b-a\right)^{\alpha+1}}{2^{\alpha}\left(\alpha+1\right)n^{\alpha}}\left\|\left|f\right|\right\|_{[a,b],\infty}\\ \\ \dfrac{\left(b-a\right)^{\alpha+\frac{1}{q}}}{2^{\alpha}\left(\alpha+1\right)n^{\alpha}}\left\|\left|f\right|\right\|_{[a,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \dfrac{\left(b-a\right)^{\alpha}}{2^{\alpha}n^{\alpha}}\left\|\left|f\right|\right\|_{[a,b],1}\end{cases}. \tag{4.17}$$

Also, we have

$$\int_a^b B\left(s\right)f\left(s\right)ds=M_n^{(2)}\left(B,f\right)+R_n^{(2)}\left(B,f\right), \tag{4.18}$$

where

$$M_n^{(2)}\left(B,f\right):=\sum_{i=0}^{n-1}\left(\int_{a+\frac{b-a}{n}\cdot i}^{a+\frac{b-a}{n}\cdot(i+1)}B\left(s\right)ds\right)f\left(a+\left(i+\frac{1}{2}\right)\cdot\frac{b-a}{n}\right),$$

and the remainder $R_n^{(2)}\left(B,f\right)$ satisfies the estimate

$$\left\|R_n^{(2)}\left(B,f\right)\right\|\leq K\times\begin{cases}\dfrac{\left(b-a\right)^{\beta+1}}{2^{\beta}\left(\beta+1\right)n^{\beta}}\left\|B\right\|_{[a,b],\infty}\\ \\ \dfrac{\left(b-a\right)^{\beta+\frac{1}{q}}}{2^{\beta}\left(\beta+1\right)n^{\beta}}\left\|\left|B\right|\right\|_{[a,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \dfrac{\left(b-a\right)^{\beta}}{2^{\beta}n^{\beta}}\left\|\left|B\right|\right\|_{[a,b],1}\end{cases}. \tag{4.19}$$

5. Application for Differential Equations in Banach Spaces

We recall that a family of operators $\mathcal{U} = \{U(t,s) : t \geq s\} \subset \mathcal{L}(X)$ with $t,s \in \mathbb{R}$ or $t,s \in \mathbb{R}_+$ is called an *evolution family* if:

(i) $U(t,t) = I$ and $U(t,s)U(s,\tau) = U(t,\tau)$ for all $t \geq s \geq \tau$; and
(ii) for each $x \in X$, the function $(t,s) \longmapsto U(t,s)x$ is continuous for $t \geq s$.

Here I is the identity operator in $\mathcal{L}(X)$.

An evolution family $\{U(t,s) : t \geq s\}$ is said to be *exponentially bounded* if, in addition,

(iii) there exist the constants $M \geq 1$ and $\omega > 0$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \ \ t \geq s. \tag{5.1}$$

Evolution families appear as solutions for abstract Cauchy problems of the form

$$\dot{u}(t) = A(t)u(t), \ u(s) = x_s, \ x_s \in \mathcal{D}(A(s)), \ t \geq s, \ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+), \tag{5.2}$$

where the domain $\mathcal{D}(A(s))$ of the linear operator $A(s)$ is assumed to be dense in X. An evolution family is said to solve the abstract Cauchy problem (5.2) if for each $s \in \mathbb{R}$ there exists a dense subset $Y_s \subseteq D(A(s))$ such that for each $x_s \in Y_s$ the function

$$t \longmapsto u(t) := U(t,s)x_s : [s,\infty) \to X,$$

is differentiable, $u(t) \in D(A(t))$ for all $t \geq s$ and

$$\frac{d}{dt}u(t) = A(t)u(t), \ \ t \geq s.$$

This later definition can be found in [15]. In this definition the operators $A(t)$ can be unbounded. The Cauchy problem (5.2) is called *well-posed* if there exists an evolution family $\{U(t,s) : t \geq s\}$ which solves it.

It is known that the well-posedness of (5.2) can be destroyed by a bounded and continuous perturbation [13]. Let $f : \mathbb{R} \to X$ be a locally integrable function. Consider the inhomogeneous Cauchy problem:

$$\dot{u}(t) = A(t)u(t) + f(t), \ u(s) = x_s \in X, \ t \geq s, \ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+). \tag{5.3}$$

A continuous function $t \longmapsto u(t) : [s,\infty) \to X$ is said to a *mild solution* of the Cauchy problem (5.3) if $u(s) = x_s$ and there exists an evolution family $\{U(t,\tau) : t \geq \tau\}$ such that

$$u(t) = U(t,s)x_s + \int_s^t U(t,\tau)f(\tau)\,d\tau, \ t \geq s, \ x_s \in X, \ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+). \tag{5.4}$$

The following theorem holds.

Theorem 6. *Let $\mathcal{U} = \{U(\nu,\eta) : \nu \geq \eta\} \subset \mathcal{L}(X)$ be an evolution family and $f : \mathbb{R} \to X$ be a locally Bochner integrable and locally bounded function. We assume that for all $\nu \in \mathbb{R}$ (or $\mathbb{R}_+$) the function $\eta \longmapsto U(\nu,\eta) : [\nu,\infty) \to \mathcal{L}(X)$ is locally Hölder continuous (i.e. for all $a,b \geq \nu$, $a < b$, there exist $\alpha \in (0,1]$ and $H > 0$ such that*

$$\|U(\nu,t) - U(\nu,s)\| \leq H|t-s|^{\alpha}, \ \ \textit{for all } t,s \in [a,b] \textit{)}.$$

We use the notations in Section 4 for $a = 0$ and $b = t > 0$. The map $u(\cdot)$ from (5.4) can be represented as

$$u(t) = U(t,0)x_0 + \sum_{i=0}^{n-1} U(t,\xi_i)\int_{t_i}^{t_{i+1}} f(s)\,ds + R_n^{(1)}(\mathcal{U},f,I_n,\boldsymbol{\xi}) \tag{5.5}$$

where the remainder $R_n^{(1)}(\mathcal{U},f,I_n,\boldsymbol{\xi})$ *satisfies the estimate*

$$\left\|R_n^{(1)}(\mathcal{U},f,I_n,\boldsymbol{\xi})\right\| \leq \frac{H}{\alpha+1}\,|||f|||_{[0,t],\infty}\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right].$$

Proof. It follows by representation (4.3) and the first estimate after it. ∎

Moreover, if n is a natural number, $i\in\{0,\dots,n\}$, $t_i:=\frac{t\cdot i}{n}$ and $\xi_i:=\frac{(2i+1)t}{2n}$, then

$$u(t)=U(t,0)\,x_0+\sum_{i=0}^{n-1}U\left(t,\frac{(2i+1)\,t}{2n}\right)\int_{\frac{t\cdot i}{n}}^{\frac{t\cdot(i+1)}{n}}f(s)\,ds+R_n^{(1)} \tag{5.6}$$

and the remainder $R_n^{(1)}$ satisfies the estimate

$$\left\|R_n^{(1)}\right\|\leq\frac{H}{\alpha+1}\cdot\frac{t^{\alpha+1}}{2^{\alpha}\cdot n^{\alpha}}\,|||f|||_{[0,t],\infty}. \tag{5.7}$$

The following theorem also holds.

Theorem 7. *Let* $\mathcal{U}=\{U(\nu,\eta):\nu\geq\eta\}\subset\mathcal{L}(X)$ *be an exponentially bounded evolution family of bounded linear operators acting on the Banach space* X *and* $f:\mathbb{R}\to X$ *be a locally Hölder continuous function, i.e., for all* $a,b\in\mathbb{R}$, $a<b$ *there exist* $\beta\in(0,1]$ *and* $K>0$ *such that (2.8) holds. We use the notations of Section 4 for* $a=0$ *and* $b=t>0$. *The map* $u(\cdot)$ *from (5.4) can be represented as*

$$u(t)=U(t,0)\,x_0+\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}U(t,\tau)\,d\tau\right)(f(\xi_i))+R_n^{(2)}(\mathcal{U},f,I_n,\boldsymbol{\xi}) \tag{5.8}$$

where the remainder $R_n^{(2)}(\mathcal{U},f,I_n,\boldsymbol{\xi})$ *satisfies the estimate*

$$\left\|R_n^{(2)}(\mathcal{U},f,I_n,\boldsymbol{\xi})\right\|\leq\frac{KM}{\beta+1}e^{\omega t}\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\beta+1}+(\xi_i-t_i)^{\beta+1}\right].$$

Proof. It follows from the first estimate in (4.7) for $B(s):=U(t,s)$, using the fact that

$$|||B(\cdot)|||_{[0,t],\infty}=\sup_{\tau\in[0,t]}\|U(t,\tau)\|\leq\sup_{\tau\in[0,t]}Me^{\omega(t-\tau)}\leq Me^{\omega t}.$$

∎

Moreover, if n is a natural number, $i\in\{0,\dots,n\}$, $t_i:=\frac{t\cdot i}{n}$ and $\xi_i:=\frac{(2i+1)t}{2n}$ then

$$u(t)=U(t,0)\,x_0+\sum_{i=0}^{n-1}\left(\int_{\frac{t\cdot i}{n}}^{\frac{t\cdot(i+1)}{n}}U(t,\tau)\,d\tau\right)f\left(\frac{(2i+1)\,t}{2n}\right)+R_n^{(2)} \tag{5.9}$$

and the remainder $R_n^{(2)}$ satisfies the estimate

$$\left\|R_n^{(2)}\right\|\leq\frac{KM}{\beta+1}e^{\omega t}\cdot\frac{t^{\beta+1}}{2^{\beta}\cdot n^{\beta}}. \tag{5.10}$$

6. Some Numerical Examples

1. Let $X=\mathbb{R}^2$, $x=(\xi,\eta)\in\mathbb{R}^2$, $\|x\|_2=\sqrt{\xi^r+\eta^2}$. We consider the linear 2-dimensional system

$$\begin{cases} \dot{u}_1(t)=\left(-1-\sin^2 t\right)u_1(t)+(-1+\sin t\cos t)\,u_2(t)+e^{-t}; \\ \dot{u}_2(t)=(1+\sin t\cos t)\,u_1(t)+\left(-1-\cos^2 t\right)u_2(t)+e^{-2t}; \\ u_1(0)=u_2(0)=0. \end{cases} \tag{6.1}$$

If we denote

$$A(t):=\begin{pmatrix} -1-\sin^2 t & -1+\sin t\cos t \\ 1+\sin t\cos t & -1-\cos^2 t \end{pmatrix},\quad f(t)=\left(e^{-t},e^{-2t}\right),\quad x=(0,0)$$

and we identify (ξ,η) with $\begin{pmatrix}\xi\\ \eta\end{pmatrix}$, then the above system is a Cauchy problem. The evolution family associated with $A(t)$ is

$$U(t,s)=P(t)\,P^{-1}(s),\quad t\geq s,\ t,s\in\mathbb{R},$$

where

$$P(t)=\begin{pmatrix} e^{-t}\cos t & e^{-2t}\sin t \\ -e^{-t}\sin t & e^{-2t}\cos t \end{pmatrix},\quad t\in\mathbb{R}. \tag{6.2}$$

The exact solution of the system (6.1) is $u=(u_1,u_2)$, where

$$u_1(t)=\left(e^{-t}\cos t\right)E_1(t)+\left(e^{-2t}\sin t\right)E_2(t)$$

$$u_2(t)=-\left(e^{-t}\sin t\right)E_1(t)+\left(e^{-2t}\cos t\right)E_2(t),\quad t\in\mathbb{R},$$

and

$$\begin{aligned} E_1(t) &= \sin t+\frac{1}{2}e^{-t}(\cos t+\sin t)-\frac{1}{2}, \\ E_2(t) &= \sin t+\frac{1}{2}(\sin t-\cos t)\cdot e^{t}+\frac{1}{2}, \end{aligned}$$

see [2, Section 4] for details. The function $t\mapsto A(t)$ is bounded on $\mathbb{R}$ and therefore there exist $M\geq 1$ and $\omega>0$

$$\|U(t,s)\|\leq Me^{\omega|t-s|},\quad \text{for all } t,s\in\mathbb{R}.$$

Let $\xi\geq 0$ be fixed and $t,s\geq\xi$. Then there exists a real number μ between t and s such that

$$\|U(\xi,t)-U(\xi,s)\|=|t-s|\,\|U(\xi,\mu)\,A(\mu)\|\leq Me^{\omega\mu}\,\|\|A(\cdot)\|\|_\infty\cdot|t-s|,$$

that is, the function $\eta\mapsto U(\xi,\eta)$ is locally Lipschitz continuous on $[\xi,\infty)$.

Using (6.2), it follows

$$U(t,s)=\begin{pmatrix} a_{11}(t,s) & a_{12}(t,s) \\ a_{21}(t,s) & a_{22}(t,s) \end{pmatrix},$$

where

$$\begin{aligned}
a_{11}(t,s) &= e^{(s-t)}\cos t\cos s + e^{2(s-t)}\sin t\sin s;\\
a_{12}(t,s) &= -e^{(s-t)}\cos t\sin s + \frac{1}{2}e^{2(s-t)}\sin t\cos s;\\
a_{21}(t,s) &= -e^{(s-t)}\sin t\cos s + e^{2(s-t)}\cos t\sin s;\\
a_{22}(t,s) &= e^{(s-t)}\sin t\sin s + \frac{1}{2}e^{2(s-t)}\cos t\cos s.
\end{aligned}$$

Then from (5.6) we obtain the following approximating formula for $u(\cdot)$:

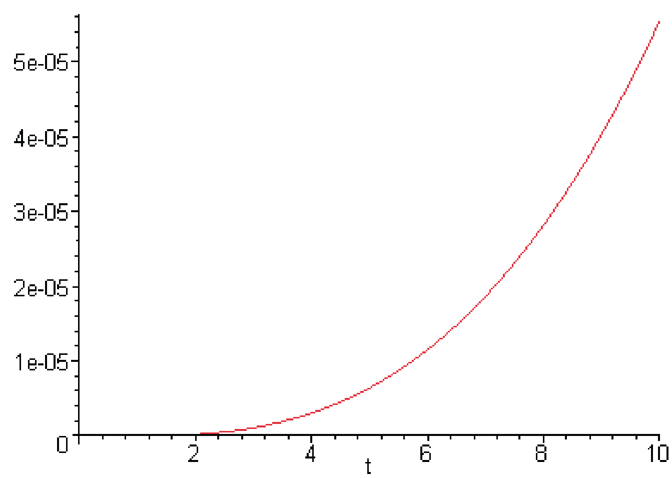
$$\begin{aligned}
u_1(t) &= -\sum_{i=0}^{n-1}\left[a_{11}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{t(i+1)}{n}} - e^{-\frac{ti}{n}}\right)\right.\\
&\quad\left. + \frac{1}{2}a_{12}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{2t(i+1)}{n}} - e^{-\frac{2ti}{n}}\right)\right] + R_{1,n}^{(1)}
\end{aligned}$$

and

$$\begin{aligned}
u_2(t) &= -\sum_{i=0}^{n-1}\left[a_{21}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{t(i+1)}{n}} - e^{-\frac{ti}{n}}\right)\right.\\
&\quad\left. + \frac{1}{2}a_{22}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{2t(i+1)}{n}} - e^{-\frac{2ti}{n}}\right)\right] + R_{2,n}^{(1)},
\end{aligned}$$

where the remainder $R_n^{(1)} = \left(R_{1,n}^{(1)}, R_{2,n}^{(1)}\right)$ satisfies the estimate (5.7) with $\alpha = 1$, $H = Me^{\omega t}\,|||A(\cdot)|||_\infty$ and $|||f|||_{[0,t],\infty} \le 2$.

The Figure 1 contains the behaviour of the error $\varepsilon_n(t) := \left\|\left(R_{1,n}^{(1)}, R_{2,n}^{(1)}\right)\right\|_2$ for $n = 200$.



2. Let $X = \mathbb{R}$ and $U(t,s) := \frac{t+1}{s+1},\ t \ge s \ge 0$. It is clear that the family $\{U(t,s) : t \ge s \ge 0\} \subset \mathcal{L}(\mathbb{R})$ is an exponentially bounded evolution family which solves the Cauchy problem

$$\dot{u}(t) = \frac{1}{t+1}u(t),\quad u(s) = x_s \in \mathbb{R},\quad t \ge s \ge 0.$$

Consider the inhomogeneous Cauchy problem

$$(6.3)\qquad \begin{cases} \dot{u}(t) = \frac{1}{t+1} u(t) + \cos[\ln(t+1)], \quad t \geq 0 \\ \\ u(0) = 0. \end{cases}$$

The solution of (6.3) is given by

$$u(t) = \int_0^t \frac{t+1}{\tau+1} \cos(\ln(\tau+1))\, d\tau = (t+1)\sin[\ln(t+1)],\ t \geq 0.$$

From (5.9) we obtain the approximating formula for $u(\cdot)$ as,

$$u(t) = (t+1) \sum_{i=0}^{n-1} \ln\left[\frac{n+ti+t}{n+ti}\right] \cos\left\{\ln\left[1 + \frac{(2i+1)\,t}{2n}\right]\right\} + R_n,$$

where R_n satisfies the estimate (5.10) with $K = M = \omega = \beta = 1$. Indeed,

$$\frac{t+1}{s+1} \leq e^t, \ \text{ for all } t \geq s \geq 0$$

and

$$|\cos[\ln(t+1)] - \cos[\ln(s+1)]| = |t-s| \left| \frac{1}{c+1} \sin[\ln(c+1)] \right| \leq |t-s|$$

for all $t \geq s \geq 0$, where c is some real number between s and t.

The Figure 2 contains the behaviour of the error $\varepsilon_n(t) := |R_n|$ for $n = 400$.

References

[1] G.A. ANASTASSIOU, Ostrowski type inequalities, *Proc. of Amer. Math. Soc.,* **123**(12) (1999), 3775-3781.

[2] C. BUŞE, S.S. DRAGOMIR and A. SOFO, Ostrowski's inequality for vector-valued functions of bounded semivariation and applications, in press.

[3] S.S. DRAGOMIR, N.S. BARNETT and P. CERONE, An $n-$dimensional version of Ostrowski's inequality for mappings of Hölder type, *Kyungpook Math. J.,* **40**(1) (2000), 65-75.

[4] S.S. DRAGOMIR, P. CERONE, J. ROUMELIOTIS and S. WANG, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sc. Math. Roumanie,* **42**(90)(4) (1992), 301-304.

[5] A.M. FINK, Bounds on the derivation of a function from its averages, *Czech. Math. J.,* **42** (1992), 289-310.

[6] E. HILLE and R.S. PHILIPS, *Functional Analysis and Semi-Groups,* Amer. Math. Soc. Colloq. Publ., Vol. **31**, Providence, R.I., (1957).

[7] M. MATIĆ, J.E. PEČARIĆ and N. UJEVIĆ, Generalisation of weighted version of Ostrowski's inequality and some related results, *J. of Ineq. & App.,* **5** (2000), 639-666.

[8] G.V. MILOVANOVIĆ, On some integral inequalities, *Univ. Beograd Publ. Elektrotehn Fak. Ser. Mat. Fiz.,* No. 498-541 (1975), 119-124.

[9] G.V. MILOVANOVIĆ, O nekim funkcionalnim nejednakostima, *Univ. Beograd Publ. Elektrotehn Fak. Ser. Mat. Fiz.,* No. 599 (1977), 1-59.

[10] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives,* Kluwer Academic Publishers, Dordrecht, 1991.

[11] A. OSTROWSKI, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

[12] J.E. PEČARIĆ and B. SAVIĆ, O novom postupku razvijanja funkcija u red i nekim primjenama, *Zbornik radova V A KoV (Beograd),* **9** (1983), 171-202.

[13] R.S. PHILIPS, Perturbation theory from semi-groups of linear operators, *Trans. Amer. Math. Soc.,* **74** (1953), 199-221.

[14] J. ROUMELIOTIS, P. CERONE and S.S. DRAGOMIR, An Ostrowski type inequality for weighted mappings with bounded second derivatives, *J. KSIAM,* **3**(2) (1999), 107–119. *RGMIA Res. Rep. Coll.,* **1**(1) (1998), `http://rgmia.vu.edu.au/v1n1/index.html`.

[15] R. SCHNAUBELT, *Exponential bounds and hyperbolicity of evolution families,* Dissertation, Tubingen, 1996.

(N.S. Barnett, P. Cerone and S.S. Dragomir), School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melburne City MC 8001,, Victoria, Australia.

E-mail address: `neil@matilda.vu.edu.au`
URL: `http://sci.vu.edu.au/staff/neilb.html`

(C. Buşe), Department of Mathematics, West University of Timişoara, Bd. V. Parvan 4, 1900 Timişoara, România.

E-mail address: `buse@hilbert.math.uvt.ro`
URL: `http://rgmia.vu.edu.au/BuseCVhtml/`

E-mail address: `pc@matilda.vu.edu.au`
URL: `http://sci.vu.edu.au/staff/peterc.html`

E-mail address: `sever@matilda.vu.edu.au`
URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`