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POWER INEQUALITIES FOR THE NUMERICAL RADIUS OF A PRODUCT OF TWO OPERATORS IN HILBERT SPACES

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ABSTRACT. Some power inequalities for the numerical radius of a product of two operators in Hilbert spaces with applications for commutators and self-commutators are given.

1. INTRODUCTION

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers \mathbb{C} given by [11, p. 1]:

$$W(T) = \{ \langle Tx, x \rangle, x \in H, \|x\| = 1 \}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [11, p. 8]:

$$(1.1) \quad w(T) = \sup \{ |\lambda|, \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle|, \|x\| = 1 \}.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \rightarrow H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [11, p. 9]:

$$(1.2) \quad w(T) \leq \|T\| \leq 2w(T),$$

for any $T \in B(H)$

For other results on numerical radii, see [12], Chapter 11.

If A, B are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then

$$(1.3) \quad w(AB) \leq 4w(A)w(B).$$

In the case that $AB = BA$, then

$$(1.4) \quad w(AB) \leq 2w(A)w(B).$$

The following results are also well known [11, p. 38]:

If A is a unitary operator that commutes with another operator B , then

$$(1.5) \quad w(AB) \leq w(B).$$

If A is an isometry and $AB = BA$, then (1.5) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. If the operators A and B double commute, then [11, p. 38]

$$(1.6) \quad w(AB) \leq w(B)\|A\|.$$

As a consequence of the above, we have [11, p. 39]:

Let A be a normal operator commuting with B , then

$$(1.7) \quad w(AB) \leq w(A)w(B).$$

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For other results and historical comments on the above see [11, p. 39–41]. For recent inequalities involving the numerical radius, see [1]–[9], [13], [14]–[16] and [17].

2. INEQUALITIES FOR A PRODUCT OF TWO OPERATORS

Theorem 1. *For any $A, B \in B(H)$ and $r \geq 1$, we have the inequality:*

$$(2.1) \quad w^r(B^*A) \leq \frac{1}{2} \|(A^*A)^r + (B^*B)^r\|.$$

The constant $\frac{1}{2}$ is best possible.

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ we have:

$$(2.2) \quad |\langle B^*Ax, x \rangle| = |\langle Ax, Bx \rangle| \leq \|Ax\| \cdot \|Bx\| \\ = \langle A^*Ax, x \rangle^{1/2} \cdot \langle B^*Bx, x \rangle^{1/2}, \quad x \in H.$$

Utilising the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \geq 1$, we have successively,

$$(2.3) \quad \langle A^*Ax, x \rangle^{1/2} \cdot \langle B^*Bx, x \rangle^{1/2} \leq \frac{\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle}{2} \\ \leq \left(\frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right)^{\frac{1}{r}}$$

for any $x \in H$.

It is known that if P is a positive operator then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [15])

$$(2.4) \quad \langle Px, x \rangle^r \leq \langle P^r x, x \rangle.$$

Applying this property to the positive operator A^*A and B^*B , we deduce that

$$(2.5) \quad \left(\frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right)^{\frac{1}{r}} \leq \left(\frac{\langle (A^*A)^r x, x \rangle + \langle (B^*B)^r x, x \rangle}{2} \right)^{\frac{1}{r}} \\ = \left(\frac{\langle [(A^*A)^r + (B^*B)^r] x, x \rangle}{2} \right)^{\frac{1}{r}}$$

for any $x \in H$, $\|x\| = 1$.

Now, on making use of the inequalities (2.2), (2.3) and (2.5), we get the inequality:

$$(2.6) \quad |\langle (B^*A)^r x, x \rangle|^r \leq \frac{1}{2} \langle [(A^*A)^r + (B^*B)^r] x, x \rangle$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.6) and since the operator $[(A^*A)^r + (B^*B)^r]$ is self-adjoint, we deduce the desired inequality (2.1).

For $r = 1$ and $B = A$, we get in both sides of (2.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1). \square

Corollary 1. *For any $A \in B(H)$ and $r \geq 1$ we have the inequalities:*

$$(2.7) \quad w^r(A) \leq \frac{1}{2} \|(A^*A)^r + I\|$$

and

$$(2.8) \quad w^r(A^2) \leq \frac{1}{2} \|(A^*A)^r + (AA^*)^r\|,$$

respectively.

A different approach is considered in the following result:

Theorem 2. *For any $A, B \in B(H)$ and any $\alpha \in (0, 1)$ and $r \geq 1$, we have the inequality:*

$$(2.9) \quad w^{2r}(B^*A) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) (B^*B)^{\frac{r}{1-\alpha}} \right\|.$$

Proof. By Schwarz's inequality, we have:

$$(2.10) \quad \begin{aligned} |\langle (B^*A)x, x \rangle|^2 &\leq \langle (A^*A)x, x \rangle \cdot \langle (B^*B)x, x \rangle \\ &= \left\langle \left[(A^*A)^{\frac{1}{\alpha}} \right]^\alpha x, x \right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} x, x \right\rangle, \end{aligned}$$

for any $x \in H$.

It is well known that (see for instance [15]) if P is a positive operator and $q \in (0, 1]$ then for any $u \in H$, $\|u\| = 1$, we have

$$(2.11) \quad \langle P^q u, u \rangle \leq \langle Pu, u \rangle^q.$$

Applying this property to the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$ ($\alpha \in (0, 1)$), we have

$$(2.12) \quad \begin{aligned} \left\langle \left[(A^*A)^{\frac{1}{\alpha}} \right]^\alpha x, x \right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} x, x \right\rangle \\ \leq \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^\alpha \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^{1-\alpha}, \end{aligned}$$

for any $x \in H$, $\|x\| = 1$.

Now, utilising the weighted arithmetic mean - geometric mean inequality, i.e., $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b$, $\alpha \in (0, 1)$, $a, b \geq 0$, we get

$$(2.13) \quad \begin{aligned} \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^\alpha \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^{1-\alpha} \\ \leq \alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle \end{aligned}$$

for any $x \in H$, $\|x\| = 1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t) = t^r$, $r \geq 1$, namely

$$\alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}, \quad \alpha \in (0, 1), \quad a, b \geq 0,$$

we deduce that

$$(2.14) \quad \begin{aligned} \alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle \\ \leq \left[\alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^r + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^r \right]^{\frac{1}{r}} \\ \leq \left[\alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}} x, x \right\rangle \right]^{\frac{1}{r}}, \end{aligned}$$

for any $x \in H$, $\|x\| = 1$, where, for the last inequality we used the inequality (2.4) for the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$.

Now, on making use of the inequalities (2.10), (2.12), (2.13) and (2.14), we get

$$(2.15) \quad |\langle (B^*A)x, x \rangle|^{2r} \leq \left\langle \left[\alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) (B^*B)^{\frac{r}{1-\alpha}} \right] x, x \right\rangle$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.15) produces the desired inequality (2.9). \square

Remark 1. The particular case $\alpha = \frac{1}{2}$ produces the inequality

$$(2.16) \quad w^{2r}(B^*A) \leq \frac{1}{2} \left\| (A^*A)^{2r} + (B^*B)^{2r} \right\|,$$

for $r \geq 1$. Notice that $\frac{1}{2}$ is best possible in (2.16) since for $r = 1$ and $B = A$ we get in both sides of (2.16) the same quantity $\|A\|^4$.

Corollary 2. For any $A \in B(H)$ and $\alpha \in (0, 1)$, $r \geq 1$, we have the inequalities

$$(2.17) \quad w^{2r}(A) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1 - \alpha) I \right\|$$

and

$$(2.18) \quad w^{2r}(A^2) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1 - \alpha) (AA^*)^{\frac{r}{1-\alpha}} \right\|,$$

respectively.

Moreover, we have

$$(2.19) \quad \|A\|^{4r} \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1 - \alpha) (A^*A)^{\frac{r}{1-\alpha}} \right\|.$$

3. INEQUALITIES FOR THE SUM OF TWO PRODUCTS

The following result may be stated:

Theorem 3. For any $A, B, C, D \in B(H)$ and $r, s \geq 1$ we have:

$$(3.1) \quad w^2 \left(\frac{B^*A + D^*C}{2} \right) \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(B^*B)^s + (D^*D)^s}{2} \right\|^{\frac{1}{s}}.$$

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ we have:

$$(3.2) \quad \begin{aligned} & |\langle (B^*A + D^*C)x, x \rangle|^2 \\ &= |\langle B^*Ax, x \rangle + \langle D^*Cx, x \rangle|^2 \\ &\leq [|\langle B^*Ax, x \rangle| + |\langle D^*Cx, x \rangle|]^2 \\ &\leq \left[\langle A^*Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^*Bx, x \rangle^{\frac{1}{2}} + \langle C^*Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^*Dx, x \rangle^{\frac{1}{2}} \right]^2, \end{aligned}$$

for any $x \in H$.

Now, on utilising the elementary inequality:

$$(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2), \quad a, b, c, d \in \mathbb{R},$$

we then conclude that:

$$(3.3) \quad \begin{aligned} & \langle A^*Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^*Bx, x \rangle^{\frac{1}{2}} + \langle C^*Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^*Dx, x \rangle^{\frac{1}{2}} \\ & \leq (\langle A^*Ax, x \rangle + \langle C^*Cx, x \rangle) \cdot (\langle B^*Bx, x \rangle + \langle D^*Dx, x \rangle), \end{aligned}$$

for any $x \in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for $r, s \geq 1$ that

$$(3.4) \quad (\langle A^*Ax, x \rangle + \langle C^*Cx, x \rangle) \cdot (\langle B^*Bx, x \rangle + \langle D^*Dx, x \rangle) \\ \leq 4 \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + (D^*D)^s}{2} \right] x, x \right\rangle^{\frac{1}{s}}$$

for any $x \in H$, $\|x\| = 1$.

Consequently, by (3.2) – (3.4) we have:

$$(3.5) \quad \left| \left\langle \left[\frac{B^*A + D^*C}{2} \right] x, x \right\rangle \right|^2 \\ \leq \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + (D^*D)^s}{2} \right] x, x \right\rangle^{\frac{1}{s}}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ we deduce the desired inequality (3.1). \square

Remark 2. If $s = r$, then the inequality (3.1) is equivalent with:

$$(3.6) \quad w^{2r} \left(\frac{B^*A + D^*C}{2} \right) \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(B^*B)^r + (D^*D)^r}{2} \right\|.$$

Corollary 3. For any $A, C \in B(H)$ we have:

$$(3.7) \quad w^{2r} \left(\frac{A + C}{2} \right) \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|,$$

where $r \geq 1$. Also, we have

$$(3.8) \quad w^2 \left(\frac{A^2 + C^2}{2} \right) \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(AA^*)^s + (CC^*)^s}{2} \right\|^{\frac{1}{s}}$$

for all $r, s \geq 1$, and in particular

$$(3.9) \quad w^{2r} \left(\frac{A^2 + C^2}{2} \right) \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(AA^*)^r + (CC^*)^r}{2} \right\|$$

for $r \geq 1$.

The inequality (3.7) follows from (3.1) for $B = D = I$, while the inequality (3.8) is obtained from the same inequality (3.1) for $B = A^*$ and $D = C^*$.

Another particular result of interest is the following one:

Corollary 4. For any $A, B \in B(H)$ we have:

$$(3.10) \quad \left\| \frac{B^*A + A^*B}{2} \right\|^2 \leq \left\| \frac{(A^*A)^r + (B^*B)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(A^*A)^s + (B^*B)^s}{2} \right\|^{\frac{1}{s}}$$

for $r, s \geq 1$ and, in particular,

$$(3.11) \quad \left\| \frac{B^*A + A^*B}{2} \right\|^r \leq \left\| \frac{(A^*A)^r + (B^*B)^r}{2} \right\|$$

for any $r \geq 1$.

The inequality (3.9) follows from (3.1) for $D = A$ and $C = B$ and taking into account that the operator $\frac{1}{2}(B^*A + A^*B)$ is self-adjoint and that

$$w \left[\frac{1}{2} (B^*A + A^*B) \right] = \left\| \frac{B^*A + A^*B}{2} \right\|.$$

Another particular case that might be of interest is the following one.

Corollary 5. *For any $A, D \in B(H)$ we have:*

$$(3.12) \quad w^2 \left(\frac{A+D}{2} \right) \leq \left\| \frac{(A^*A)^r + I}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(DD^*)^s + I}{2} \right\|^{\frac{1}{s}},$$

where $r, s \geq 1$. In particular

$$(3.13) \quad w^2(A) \leq \left\| \frac{(A^*A)^r + I}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(AA^*)^s + I}{2} \right\|^{\frac{1}{s}}.$$

Moreover, for any $r \geq 1$ we have

$$w^{2r}(A) \leq \left\| \frac{(A^*A)^r + I}{2} \right\| \cdot \left\| \frac{(AA^*)^r + I}{2} \right\|.$$

The proof is obvious by the inequality (3.1) on choosing $B = I$, $C = I$ and writing the inequality for D^* instead of D .

Remark 3. *If $T \in B(H)$ and $T = A + iC$, i.e., A and C are its Cartesian decomposition, then we get from (3.7) that*

$$w^{2r}(T) \leq 2^{2r-1} \|A^{2r} + C^{2r}\|,$$

for any $r \geq 1$.

Also, since $A = \operatorname{Re}(T) = \frac{T+T^*}{2}$ and $C = \operatorname{Im}(T) = \frac{T-T^*}{2i}$, then from (3.7) we get the following inequalities as well:

$$\|\operatorname{Re}(T)\|^{2r} \leq \left\| \frac{(T^*T)^r + (TT^*)^r}{2} \right\|$$

and

$$\|\operatorname{Im}(T)\|^{2r} \leq \left\| \frac{(T^*T)^r + (TT^*)^r}{2} \right\|$$

for any $r \geq 1$.

In terms of the *Euclidean radius* of two operators $w_e(\cdot, \cdot)$, where, as in [1],

$$w_e(T, U) := \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + |\langle Ux, x \rangle|^2 \right)^{\frac{1}{2}},$$

we have the following result as well.

Theorem 4. *For any $A, B, C, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have the inequality:*

$$(3.14) \quad w_e^2(B^*A, D^*C) \leq \|(A^*A)^p + (C^*C)^p\|^{1/p} \cdot \|(B^*B)^q + (D^*D)^q\|^{1/q}.$$

Proof. For any $x \in H$, $\|x\| = 1$ we have the inequalities

$$\begin{aligned}
& |\langle B^*Ax, x \rangle|^2 + |\langle D^*Cx, x \rangle|^2 \\
& \leq \langle A^*Ax, x \rangle \cdot \langle B^*Bx, x \rangle + \langle C^*Cx, x \rangle \cdot \langle D^*Dx, x \rangle \\
& \leq (\langle A^*Ax, x \rangle^p + \langle C^*Cx, x \rangle^p)^{1/p} \cdot (\langle B^*Bx, x \rangle^q + \langle D^*Dx, x \rangle^q)^{1/q} \\
& \leq (\langle (A^*A)^p x, x \rangle + \langle (C^*C)^p x, x \rangle)^{1/p} \cdot (\langle (B^*B)^q x, x \rangle + \langle (D^*D)^q x, x \rangle)^{1/q} \\
& \leq \langle [(A^*A)^p + (C^*C)^p] x, x \rangle^{1/p} \cdot \langle [(B^*B)^q + (D^*D)^q] x, x \rangle^{1/q}.
\end{aligned}$$

Taking the supremum over $x \in H$, $\|x\| = 1$ and noticing that the operators $(A^*A)^p + (C^*C)^p$ and $(B^*B)^q + (D^*D)^q$ are self-adjoint, we deduce the desired inequality (3.14). \square

The following particular case is of interest.

Corollary 6. *For any $A, C \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$w_e^2(A, C) \leq 2^{1/q} \|(A^*A)^p + (C^*C)^p\|^{1/p}.$$

The proof follows from (3.14) for $B = D = I$.

Corollary 7. *For any $A, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$w_e^2(A, D) \leq \|(A^*A)^p + I\|^{1/p} \cdot \|(D^*D)^q + I\|^{1/q}.$$

4. VECTOR INEQUALITIES FOR THE COMMUTATOR

The commutator of two bounded linear operators T and U is the operator $TU - UT$. For the usual norm $\|\cdot\|$ and for any two operators T and U , by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality:

$$(4.1) \quad \|TU - UT\| \leq 2\|T\|\|U\|.$$

In [10], the following result has been obtained as well

$$(4.2) \quad \|TU - UT\| \leq 2 \min\{\|T\|, \|U\|\} \min\{\|T - U\|, \|T + U\|\}.$$

By utilising Theorem 3 we can state the following result for the numerical radius of the commutator.

Proposition 1. *For any $T, U \in B(H)$ and $r, s \geq 1$ we have*

$$(4.3) \quad w^2(TU - UT) \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(T^*T)^r + (U^*U)^r\|^{\frac{1}{r}} \cdot \|(TT^*)^s + (UU^*)^s\|^{\frac{1}{s}}.$$

Proof. Follows by Theorem 3 on choosing $B = T^*$, $A = U$, $D = -U^*$ and $C = T$. \square

Remark 4. *In particular, for $r = s$ we get from (4.3) that*

$$(4.4) \quad w^{2r}(TU - UT) \leq 2^{2r-2} \|(T^*T)^r + (U^*U)^r\| \cdot \|(TT^*)^r + (UU^*)^r\|$$

and for $r = 1$ we get

$$(4.5) \quad w^2(TU - UT) \leq \|T^*T + U^*U\| \cdot \|TT^* + UU^*\|.$$

For a bounded linear operator $T \in B(H)$, the self-commutator is the operator $T^*T - TT^*$. Observe that the operator $V := -i(T^*T - TT^*)$ is self-adjoint and $w(V) = \|V\|$, i.e.,

$$w(T^*T - TT^*) = \|T^*T - TT^*\|.$$

Now, utilising (4.3) for $U = T^*$ we can state the following corollary.

Corollary 8. *For any $T \in B(H)$ we have the inequality:*

$$(4.6) \quad \|T^*T - TT^*\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(T^*T)^r + (TT^*)^r\|^{\frac{1}{r}} \cdot \|(T^*T)^s + (TT^*)^s\|^{\frac{1}{s}}.$$

In particular, we have

$$(4.7) \quad \|T^*T - TT^*\|^r \leq 2^{r-1} \|(T^*T)^r + (TT^*)^r\|,$$

for any $r \geq 1$.

Moreover, for $r = 1$ we have

$$(4.8) \quad \|T^*T - TT^*\| \leq \|T^*T + TT^*\|.$$

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