Power Inequalities for the Numerical Radius of a Product of Two Operators in Hilbert Spaces

POWER INEQUALITIES FOR THE NUMERICAL RADIUS OF A PRODUCT OF TWO OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR

Abstract. Some power inequalities for the numerical radius of a product of two operators in Hilbert spaces with applications for commutators and self-commutators are given.

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [11, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, \ x \in H, \ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [11, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x \rangle|, \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [11, p. 9]:

$$w(T) \leq \|T\| \leq 2w(T), \tag{1.2}$$

for any $T \in B(H)$

For other results on numerical radii, see [12], Chapter 11.

If A, B are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then

$$w(AB) \leq 4w(A)w(B). \tag{1.3}$$

In the case that $AB = BA$, then

$$w(AB) \leq 2w(A)w(B). \tag{1.4}$$

The following results are also well known [11, p. 38]:

If A is a unitary operator that commutes with another operator B, then

$$w(AB) \leq w(B). \tag{1.5}$$

If A is an isometry and $AB = BA$, then (1.5) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. If the operators A and B double commute, then [11, p. 38]

$$w(AB) \leq w(B)\|A\|. \tag{1.6}$$

As a consequence of the above, we have [11, p. 39]:

Let A be a normal operator commuting with B, then

$$w(AB) \leq w(A)w(B). \tag{1.7}$$

Date: August 06, 2008.

1991 *Mathematics Subject Classification.* 47A12 (47A30 47A63 47B15).

For other results and historical comments on the above see [11, p. 39–41]. For recent inequalities involving the numerical radius, see [1]-[9], [13], [14]-[16] and [17].

2. Inequalities for a Product of Two Operators

Theorem 1. *For any $A, B \in B(H)$ and $r \geq 1$, we have the inequality:*

$$w^r (B^* A) \leq \frac{1}{2} \|(A^* A)^r + (B^* B)^r\| . \tag{2.1}$$

The constant $\frac{1}{2}$ is best possible.

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle .,.\rangle)$ we have:

$$\begin{aligned} |\langle B^* Ax, x\rangle| &= |\langle Ax, Bx\rangle| \leq \|Ax\| \cdot \|Bx\| \\ &= \langle A^* Ax, x\rangle^{1/2} \cdot \langle B^* Bx, x\rangle^{1/2}, \qquad x \in H. \end{aligned} \tag{2.2}$$

Utilising the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \geq 1$, we have successively,

$$\begin{aligned} \langle A^* Ax, x\rangle^{1/2} \cdot \langle B^* Bx, x\rangle^{1/2} &\leq \frac{\langle A^* Ax, x\rangle + \langle B^* Bx, x\rangle}{2} \\ &\leq \left(\frac{\langle A^* Ax, x\rangle^r + \langle B^* Bx, x\rangle^r}{2} \right)^{\frac{1}{r}} \end{aligned} \tag{2.3}$$

for any $x \in H$.

It is known that if P is a positive operator then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [15])

$$\langle Px, x\rangle^r \leq \langle P^r x, x\rangle . \tag{2.4}$$

Applying this property to the positive operator $A^* A$ and $B^* B$, we deduce that

$$\begin{aligned} \left(\frac{\langle A^* Ax, x\rangle^r + \langle B^* Bx, x\rangle^r}{2} \right)^{\frac{1}{r}} &\leq \left(\frac{\langle (A^* A)^r x, x\rangle + \langle (B^* B)^r x, x\rangle}{2} \right)^{\frac{1}{r}} \\ &= \left(\frac{\langle [(A^* A)^r + (B^* B)^r] x, x\rangle}{2} \right)^{\frac{1}{r}} \end{aligned} \tag{2.5}$$

for any $x \in H$, $\|x\| = 1$.

Now, on making use of the inequalities (2.2), (2.3) and (2.5), we get the inequality:

$$|\langle (B^* A)^r x, x\rangle|^r \leq \frac{1}{2} \langle [(A^* A)^r + (B^* B)^r] x, x\rangle \tag{2.6}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.6) and since the operator $[(A^* A)^r + (B^* B)^r]$ is self-adjoint, we deduce the desired inequality (2.1).

For $r = 1$ and $B = A$, we get in both sides of (2.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1). □

Corollary 1. *For any $A \in B(H)$ and $r \geq 1$ we have the inequalities:*

$$w^r (A) \leq \frac{1}{2} \|(A^* A)^r + I\| \tag{2.7}$$

and

$$w^r \left(A^2\right) \leq \frac{1}{2} \|(A^* A)^r + (AA^*)^r\| , \tag{2.8}$$

respectively.

A different approach is considered in the following result:

Theorem 2. *For any $A,B\in B(H)$ and any $\alpha\in(0,1)$ and $r\geq 1$, we have the inequality:*

$$w^{2r}(B^*A)\leq\left\|\alpha(A^*A)^{\frac{r}{\alpha}}+(1-\alpha)(B^*B)^{\frac{r}{1-\alpha}}\right\|. \tag{2.9}$$

Proof. By Schwarz's inequality, we have:

$$\begin{aligned}|\langle(B^*A)x,x\rangle|^2&\leq\langle(A^*A)x,x\rangle\cdot\langle(B^*B)x,x\rangle\\&=\left\langle\left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha}x,x\right\rangle\cdot\left\langle\left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}x,x\right\rangle,\end{aligned}\tag{2.10}$$

for any $x\in H$.

It is well known that (see for instance [15]) if P is a positive operator and $q\in(0,1]$ then for any $u\in H$, $\|u\|=1$, we have

$$\langle P^q u,u\rangle\leq\langle Pu,u\rangle^q. \tag{2.11}$$

Applying this property to the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$ $(\alpha\in(0,1))$, we have

$$\begin{aligned}&\left\langle\left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha}x,x\right\rangle\cdot\left\langle\left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}x,x\right\rangle\\&\qquad\leq\left\langle(A^*A)^{\frac{1}{\alpha}}x,x\right\rangle^{\alpha}\cdot\left\langle(B^*B)^{\frac{1}{1-\alpha}}x,x\right\rangle^{1-\alpha},\end{aligned}\tag{2.12}$$

for any $x\in H$, $\|x\|=1$.

Now, utilising the weighted arithmetic mean - geometric mean inequality, i.e., $a^{\alpha}b^{1-\alpha}\leq\alpha a+(1-\alpha)b$, $\alpha\in(0,1)$, $a,b\geq 0$, we get

$$\begin{aligned}&\left\langle(A^*A)^{\frac{1}{\alpha}}x,x\right\rangle^{\alpha}\cdot\left\langle(B^*B)^{\frac{1}{1-\alpha}}x,x\right\rangle^{1-\alpha}\\&\qquad\leq\alpha\left\langle(A^*A)^{\frac{1}{\alpha}}x,x\right\rangle+(1-\alpha)\left\langle(B^*B)^{\frac{1}{1-\alpha}}x,x\right\rangle\end{aligned}\tag{2.13}$$

for any $x\in H$, $\|x\|=1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t)=t^r$, $r\geq 1$, namely

$$\alpha a+(1-\alpha)b\leq(\alpha a^r+(1-\alpha)b^r)^{\frac{1}{r}},\qquad\alpha\in(0,1),\ a,b\geq 0,$$

we deduce that

$$\begin{aligned}&\alpha\left\langle(A^*A)^{\frac{1}{\alpha}}x,x\right\rangle+(1-\alpha)\left\langle(B^*B)^{\frac{1}{1-\alpha}}x,x\right\rangle\\&\leq\left[\alpha\left\langle(A^*A)^{\frac{1}{\alpha}}x,x\right\rangle^{r}+(1-\alpha)\left\langle(B^*B)^{\frac{1}{1-\alpha}}x,x\right\rangle^{r}\right]^{\frac{1}{r}}\\&\leq\left[\alpha\left\langle(A^*A)^{\frac{r}{\alpha}}x,x\right\rangle+(1-\alpha)\left\langle(B^*B)^{\frac{r}{1-\alpha}}x,x\right\rangle\right]^{\frac{1}{r}},\end{aligned}\tag{2.14}$$

for any $x\in H$, $\|x\|=1$, where, for the last inequality we used the inequality (2.4) for the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$.

Now, on making use of the inequalities (2.10), (2.12), (2.13) and (2.14), we get

$$|\langle(B^*A)x,x\rangle|^{2r}\leq\left\langle\left[\alpha(A^*A)^{\frac{r}{\alpha}}+(1-\alpha)(B^*B)^{\frac{r}{1-\alpha}}\right]x,x\right\rangle \tag{2.15}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.15) produces the desired inequality (2.9). □

Remark 1. *The particular case* $\alpha = \frac{1}{2}$ *produces the inequality*

$$w^{2r}\left(B^{*}A\right) \leq \frac{1}{2}\left\|\left(A^{*}A\right)^{2r} + \left(B^{*}B\right)^{2r}\right\|, \tag{2.16}$$

for $r \geq 1$. *Notice that* $\frac{1}{2}$ *is best possible in (2.16) since for* $r = 1$ *and* $B = A$ *we get in both sides of (2.16) the same quantity* $\|A\|^{4}$.

Corollary 2. *For any* $A \in B(H)$ *and* $\alpha \in (0,1)$, $r \geq 1$, *we have the inequalities*

$$w^{2r}(A) \leq \left\|\alpha\left(A^{*}A\right)^{\frac{r}{\alpha}} + (1-\alpha)I\right\| \tag{2.17}$$

and

$$w^{2r}\left(A^{2}\right) \leq \left\|\alpha\left(A^{*}A\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(AA^{*}\right)^{\frac{r}{1-\alpha}}\right\|, \tag{2.18}$$

respectively.

Moreover, we have

$$\|A\|^{4r} \leq \left\|\alpha\left(A^{*}A\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(A^{*}A\right)^{\frac{r}{1-\alpha}}\right\|. \tag{2.19}$$

3. Inequalities for the Sum of Two Products

The following result may be stated:

Theorem 3. *For any* $A, B, C, D \in B(H)$ *and* $r, s \geq 1$ *we have:*

$$w^{2}\left(\frac{B^{*}A + D^{*}C}{2}\right) \leq \left\|\frac{\left(A^{*}A\right)^{r} + \left(C^{*}C\right)^{r}}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{\left(B^{*}B\right)^{s} + \left(D^{*}D\right)^{s}}{2}\right\|^{\frac{1}{s}}. \tag{3.1}$$

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ we have:

$$\begin{aligned} &\left|\left\langle\left(B^{*}A + D^{*}C\right)x, x\right\rangle\right|^{2} \\ &= \left|\left\langle B^{*}Ax, x\right\rangle + \left\langle D^{*}Cx, x\right\rangle\right|^{2} \\ &\leq \left[\left|\left\langle B^{*}Ax, x\right\rangle\right| + \left|\left\langle D^{*}Cx, x\right\rangle\right|\right]^{2} \\ &\leq \left[\left\langle A^{*}Ax, x\right\rangle^{\frac{1}{2}} \cdot \left\langle B^{*}Bx, x\right\rangle^{\frac{1}{2}} + \left\langle C^{*}Cx, x\right\rangle^{\frac{1}{2}} \cdot \left\langle D^{*}Dx, x\right\rangle^{\frac{1}{2}}\right]^{2}, \end{aligned} \tag{3.2}$$

for any $x \in H$.

Now, on utilising the elementary inequality:

$$(ab + cd)^{2} \leq \left(a^{2} + c^{2}\right)\left(b^{2} + d^{2}\right), \qquad a, b, c, d \in \mathbb{R},$$

we then conclude that:

$$\begin{aligned} &\left\langle A^{*}Ax, x\right\rangle^{\frac{1}{2}} \cdot \left\langle B^{*}Bx, x\right\rangle^{\frac{1}{2}} + \left\langle C^{*}Cx, x\right\rangle^{\frac{1}{2}} \cdot \left\langle D^{*}Dx, x\right\rangle^{\frac{1}{2}} \\ &\qquad \leq \left(\left\langle A^{*}Ax, x\right\rangle + \left\langle C^{*}Cx, x\right\rangle\right) \cdot \left(\left\langle B^{*}Bx, x\right\rangle + \left\langle D^{*}Dx, x\right\rangle\right), \end{aligned} \tag{3.3}$$

for any $x \in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for $r,s\geq 1$ that

$$(\langle A^{*}Ax,x\rangle+\langle C^{*}Cx,x\rangle)\cdot(\langle B^{*}Bx,x\rangle+\langle D^{*}Dx,x\rangle)\leq 4\left\langle\left[\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right]x,x\right\rangle^{\frac{1}{r}}\cdot\left\langle\left[\frac{(B^{*}B)^{s}+(D^{*}D)^{s}}{2}\right]x,x\right\rangle^{\frac{1}{s}} \tag{3.4}$$

for any $x\in H$, $\|x\|=1$.

Consequently, by (3.2) – (3.4) we have:

$$\left|\left\langle\left[\frac{B^{*}A+D^{*}C}{2}\right]x,x\right\rangle\right|^{2}\leq\left\langle\left[\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right]x,x\right\rangle^{\frac{1}{r}}\cdot\left\langle\left[\frac{(B^{*}B)^{s}+(D^{*}D)^{s}}{2}\right]x,x\right\rangle^{\frac{1}{s}} \tag{3.5}$$

for any $x\in H$, $\|x\|=1$.

Taking the supremum over $x\in H$, $\|x\|=1$ we deduce the desired inequality (3.1). □

Remark 2. *If $s=r$, then the inequality (3.1) is equivalent with:*

$$w^{2r}\left(\frac{B^{*}A+D^{*}C}{2}\right)\leq\left\|\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right\|\cdot\left\|\frac{(B^{*}B)^{r}+(D^{*}D)^{r}}{2}\right\|. \tag{3.6}$$

Corollary 3. *For any $A,C\in B(H)$ we have:*

$$w^{2r}\left(\frac{A+C}{2}\right)\leq\left\|\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right\|, \tag{3.7}$$

where $r\geq 1$. Also, we have

$$w^{2}\left(\frac{A^{2}+C^{2}}{2}\right)\leq\left\|\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(AA^{*})^{s}+(CC^{*})^{s}}{2}\right\|^{\frac{1}{s}} \tag{3.8}$$

for all $r,s\geq 1$, and in particular

$$w^{2r}\left(\frac{A^{2}+C^{2}}{2}\right)\leq\left\|\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right\|\cdot\left\|\frac{(AA^{*})^{r}+(CC^{*})^{r}}{2}\right\| \tag{3.9}$$

for $r\geq 1$.

The inequality (3.7) follows from (3.1) for $B=D=I$, while the inequality (3.8) is obtained from the same inequality (3.1) for $B=A^{*}$ and $D=C^{*}$.

Another particular result of interest is the following one:

Corollary 4. *For any $A,B\in B(H)$ we have:*

$$\left\|\frac{B^{*}A+A^{*}B}{2}\right\|^{2}\leq\left\|\frac{(A^{*}A)^{r}+(B^{*}B)^{r}}{2}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(A^{*}A)^{s}+(B^{*}B)^{s}}{2}\right\|^{\frac{1}{s}} \tag{3.10}$$

for $r,s\geq 1$ and, in particular,

$$\left\|\frac{B^{*}A+A^{*}B}{2}\right\|^{r}\leq\left\|\frac{(A^{*}A)^{r}+(B^{*}B)^{r}}{2}\right\| \tag{3.11}$$

for any $r\geq 1$.

The inequality (3.9) follows from (3.1) for $D = A$ and $C = B$ and taking into account that the operator $\frac{1}{2}(B^*A + A^*B)$ is self-adjoint and that

$$w\left[\frac{1}{2}(B^*A + A^*B)\right] = \left\|\frac{B^*A + A^*B}{2}\right\|.$$

Another particular case that might be of interest is the following one.

Corollary 5. *For any $A, D \in B(H)$ we have:*

$$w^2\left(\frac{A+D}{2}\right) \le \left\|\frac{(A^*A)^r + I}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(DD^*)^s + I}{2}\right\|^{\frac{1}{s}}, \tag{3.12}$$

where $r, s \ge 1$. In particular

$$w^2(A) \le \left\|\frac{(A^*A)^r + I}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(AA^*)^s + I}{2}\right\|^{\frac{1}{s}}. \tag{3.13}$$

Moreover, for any $r \ge 1$ we have

$$w^{2r}(A) \le \left\|\frac{(A^*A)^r + I}{2}\right\| \cdot \left\|\frac{(AA^*)^r + I}{2}\right\|.$$

The proof is obvious by the inequality (3.1) on choosing $B = I$, $C = I$ and writing the inequality for D^* instead of D.

Remark 3. *If $T \in B(H)$ and $T = A + iC$, i.e., A and C are its Cartesian decomposition, then we get from (3.7) that*

$$w^{2r}(T) \le 2^{2r-1}\left\|A^{2r} + C^{2r}\right\|,$$

for any $r \ge 1$.

Also, since $A = \operatorname{Re}(T) = \frac{T+T^*}{2}$ and $C = \operatorname{Im}(T) = \frac{T-T^*}{2i}$, then from (3.7) we get the following inequalities as well:

$$\|\operatorname{Re}(T)\|^{2r} \le \left\|\frac{(T^*T)^r + (TT^*)^r}{2}\right\|$$

and

$$\|\operatorname{Im}(T)\|^{2r} \le \left\|\frac{(T^*T)^r + (TT^*)^r}{2}\right\|$$

for any $r \ge 1$.

In terms of the *Euclidean radius* of two operators $w_e(\cdot,\cdot)$, where, as in [1],

$$w_e(T,U) := \sup_{\|x\|=1}\left(|\langle Tx, x\rangle|^2 + |\langle Ux, x\rangle|^2\right)^{\frac{1}{2}},$$

we have the following result as well.

Theorem 4. *For any $A, B, C, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have the inequality:*

$$w_e^2(B^*A, D^*C) \le \left\|(A^*A)^p + (C^*C)^p\right\|^{1/p} \cdot \left\|(B^*B)^q + (D^*D)^q\right\|^{1/q}. \tag{3.14}$$

Proof. For any $x \in H$, $\|x\| = 1$ we have the inequalities

$$\begin{aligned} & |\langle B^* A x, x\rangle|^2 + |\langle D^* C x, x\rangle|^2 \\ & \leq \langle A^* A x, x\rangle \cdot \langle B^* B x, x\rangle + \langle C^* C x, x\rangle \cdot \langle D^* D x, x\rangle \\ & \leq \left(\langle A^* A x, x\rangle^p + \langle C^* C x, x\rangle^p\right)^{1/p} \cdot \left(\langle B^* B x, x\rangle^q + \langle D^* D x, x\rangle^q\right)^{1/q} \\ & \leq \left(\langle (A^* A)^p x, x\rangle + \langle (C^* C)^p x, x\rangle\right)^{1/p} \cdot \left(\langle (B^* B)^q x, x\rangle + \langle (D^* D)^q x, x\rangle\right)^{1/q} \\ & \leq \langle [(A^* A)^p + (C^* C)^p] x, x\rangle^{1/p} \cdot \langle [(B^* B)^q + (D^* D)^q] x, x\rangle^{1/q}. \end{aligned}$$

Taking the supremum over $x \in H$, $\|x\| = 1$ and noticing that the operators $(A^*A)^p + (C^*C)^p$ and $(B^*B)^q + (D^*D)^q$ are self-adjoint, we deduce the desired inequality (3.14). □

The following particular case is of interest.

Corollary 6. *For any $A, C \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$w_e^2(A, C) \leq 2^{1/q} \left\|(A^*A)^p + (C^*C)^p\right\|^{1/p}.$$

The proof follows from (3.14) for $B = D = I$.

Corollary 7. *For any $A, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$w_e^2(A, D) \leq \left\|(A^*A)^p + I\right\|^{1/p} \cdot \left\|(D^*D)^q + I\right\|^{1/q}.$$

4. Vector Inequalities for the Commutator

The commutator of two bounded linear operators T and U is the operator $TU - UT$. For the usual norm $\|\cdot\|$ and for any two operators T and U, by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality:

$$\|TU - UT\| \leq 2 \|T\| \|U\|. \tag{4.1}$$

In [10], the following result has been obtained as well

$$\|TU - UT\| \leq 2 \min\{\|T\|, \|U\|\} \min\{\|T - U\|, \|T + U\|\}. \tag{4.2}$$

By utilising Theorem 3 we can state the following result for the numerical radius of the commutator.

Proposition 1. *For any $T, U \in B(H)$ and $r, s \geq 1$ we have*

$$w^2(TU - UT) \leq 2^{2 - \frac{1}{r} - \frac{1}{s}} \left\|(T^*T)^r + (U^*U)^r\right\|^{\frac{1}{r}} \cdot \left\|(TT^*)^s + (UU^*)^s\right\|^{\frac{1}{s}}. \tag{4.3}$$

Proof. Follows by Theorem 3 on choosing $B = T^*$, $A = U$, $D = -U^*$ and $C = T$. □

Remark 4. *In particular, for $r = s$ we get from (4.3) that*

$$w^{2r}(TU - UT) \leq 2^{2r-2} \left\|(T^*T)^r + (U^*U)^r\right\| \cdot \left\|(TT^*)^r + (UU^*)^r\right\| \tag{4.4}$$

and for $r = 1$ we get

$$w^2(TU - UT) \leq \|T^*T + U^*U\| \cdot \|TT^* + UU^*\|. \tag{4.5}$$

For a bounded linear operator $T \in B(H)$, the self-commutator is the operator $T^*T - TT^*$. Observe that the operator $V := -i(T^*T - TT^*)$ is self-adjoint and $w(V) = \|V\|$, i.e.,

$$w(T^*T - TT^*) = \|T^*T - TT^*\|.$$

Now, utilising (4.3) for $U = T^*$ we can state the following corollary.

Corollary 8. *For any $T \in B(H)$ we have the inequality:*

$$\|T^*T - TT^*\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(T^*T)^r + (TT^*)^r\|^{\frac{1}{r}} \cdot \|(T^*T)^s + (TT^*)^s\|^{\frac{1}{s}}. \tag{4.6}$$

In particular, we have

$$\|T^*T - TT^*\|^r \leq 2^{r-1} \|(T^*T)^r + (TT^*)^r\|, \tag{4.7}$$

for any $r \geq 1$.

Moreover, for $r = 1$ we have

$$\|T^*T - TT^*\| \leq \|T^*T + TT^*\|. \tag{4.8}$$

References

[1] S. S. Dragomir, Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces. *Linear Algebra Appl.* **419** (2006), no. 1, 256–264.

[2] S.S. Dragomir, Reverse inequalities for the numerical radius of linear operators in Hilbert spaces. *Bull. Austral. Math. Soc.* **73** (2006), no. 2, 255–262.

[3] S.S. Dragomir, A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces. *Banach J. Math. Anal.* **1** (2007), no. 2, 154–175..

[4] S.S. Dragomir, Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. *Demonstratio Math.* **40** (2007), no. 2, 411–417.

[5] S.S. Dragomir, Norm and numerical radius inequalities for sums of bounded linear operators in Hilbert spaces. *Facta Univ. Ser. Math. Inform.* **22** (2007), no. 1, 61–75.

[6] S.S. Dragomir, Inequalities for some functionals associated with bounded linear operators in Hilbert spaces. *Publ. Res. Inst. Math.* Sci. 43 (2007), No. 4, 1095–1110.

[7] S. S. Dragomir, The hypo-Euclidean norm of an n-tuple of vectors in inner product spaces and applications. *J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 52, 22 pp.

[8] S.S. Dragomir, New inequalities of the Kantorovich type for bounded linear operators in Hilbert spaces. *Linear Algebra Appl.* **428** (2008), no. 11-12, 2750–2760.

[9] S.S. Dragomir, Inequalities for the numerical radius, the norm and the maximum of the real part of bounded linear operators in Hilbert spaces. *Linear Algebra Appl.* **428** (2008), no. 11-12, 2980–2994.

[10] S.S. Dragomir, Some inequalities for commutators of bounded linear operators in Hilbert spaces, Preprint, *RGMIA Res. Rep. Coll.*, **11**(2008), No. 1, Article 7, [Online http://www.staff.vu.edu.au/rgmia/v11n1.asp].

[11] K.E. Gustafson and D.K.M. Rao, *Numerical Range,* Springer-Verlag, New York, Inc., 1997.

[12] P.R. Halmos, *A Hilbert Space Problem Book,* Springer-Verlag, New York, Heidelberg, Berlin, Second edition, 1982.

[13] M. El-Haddad, and F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II. *Studia Math.* **182** (2007), no. 2, 133–140.

[14] F. Kittaneh, Notes on some inequalities for Hilbert space operators, *Publ. Res. Inst. Math. Sci.* **24** (1988), 283–293.

[15] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Studia Math.,* **158**(1) (2003), 11-17.

[16] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *Studia Math.,* **168**(1) (2005), 73-80.

[17] T. Yamazaki, On upper and lower bounds for the numerical radius and an equality condition. *Studia Math.* **178** (2007), no. 1, 83–89.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, VIC, Australia. 8001
E-mail address: sever.dragomir@vu.edu.au
URL: http://www.staff.vu.edu.au/rgmia/dragomir/