The Strengthened Hardy Inequalities and its New Generalizations

THE STRENGTHENED HARDY INEQUALITIES AND ITS NEW GENERALIZATIONS

Sever Silvestru Dragomir and Young-Ho Kim

Abstract. In this article, using the properties of power mean, new generalizations of the strengthened Hardy Inequalities are proved.

1. Introduction

It is well known that the following Hardy's Inequality (see [4, Theorem 326]): *if* $p > 1$ *and* $a_n \geq 0$, *then*

$$\sum \Big(\frac{a_1 + a_2 + ... + a_n}{n}\Big)^p < \Big(\frac{p}{p-1}\Big)^p \sum a_n^p, \tag{1.1}$$

unless all the a are zero. The constant is the best possible.

This theorem was discovered in the course of attempts to simplify the proofs then known of Hilbert's double series theorems (see [4, Theorem 315]). Hilbert's double series theorem was completed by the above inequality. This inequality was first proved by Hardy [3], except that Hardy was unable to fit the constant in inequality (1.1). If in inequality (1.1) we write a_n for a_n^p, we obtain

$$\sum \Big(\frac{a_1^{1/p} + a_2^{1/p} + \cdots + a_n^{1/p}}{n}\Big)^p < \Big(\frac{p}{p-1}\Big)^p \sum a_n. \tag{1.2}$$

If we make $p \to \infty$, and use the elementary mean values

$$\lim_{p\to 0}\Big(\sum_{i=1}^n \frac{1}{n} a_i^p\Big)^{1/p} = \Big(\prod_{i=1}^n a_i\Big)^{1/n},$$

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we obtain

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty} a_n,$$

and this suggests the more complete theorem which follow;

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty} a_n, \tag{1.3}$$

unless (a_n) *is null. The constant is the best possible.*

The inequality given in (1.3) which later went by the name of *Carleman's inequality,* led to a great many papers dealing with alternative proofs, various generalizations, and numerous variants and applications in analysis. It is natural to attempt to prove the complete inequality by means of following

$$\Big(\prod_{i=1}^{n} a_i\Big)^{1/n} < \sum_{i=1}^{n}\frac{1}{n}a_i, \tag{1.4}$$

unless all the a_i are equal. But a direct application of inequality (1.4) to the left-hand side of the inequality (1.2) is insufficient. To remedy this, we apply inequality (1.4) not to $a_1, a_2, ..., a_n$ but to $c_1a_1, c_2a_2, ..., c_na_n$, and choose the c so that when $\sum a_n$ is near the boundary of convergence, these numbers shall be 'roughly equal'. This requires that c_n shall be roughly of the order of n.

By Hardy (see, [4, Theorem 349]), the Carleman's inequality was generalized as follows:

If $a_n \geq 0, \lambda_n > 0,\ \Lambda_n = \sum_{m=1}^{n}\lambda_m (n \in N)$ *and* $0 < \sum_{n=1}^{\infty}\lambda_n a_n < \infty$*, then*

$$\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n} < e\sum_{n=1}^{\infty}\lambda_n a_n. \tag{1.5}$$

Recently, Z. Xie and Y. Zhong [7] gave an improvement of the inequality (1.5) as follows: *If* $a_n \geq 0, 0 < \lambda_{n+1} \leq \lambda_n,\ \Lambda_n = \sum_{m=1}^{n}\lambda_m (n \in N)$ *and* $0 < \sum_{n=1}^{\infty}\lambda_n a_n < \infty$*, then*

$$\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n} < e\sum_{n=1}^{\infty}\Big(1-\frac{6\lambda_n}{12\Lambda_n+11\lambda_n}\Big)\lambda_n a_n. \tag{1.6}$$

Most recently, Z. Yang [11] obtained the strengthened Carleman's inequality as follows: *If* $a_n \geq 0$, $n = 1, 2, ...,$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\begin{aligned} &\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \\ &< e \sum_{n=1}^{\infty} \Big(1 - \frac{1}{2(1+n)} - \frac{1}{24(1+n)^2} - \frac{1}{48(1+n)^3}\Big) a_n. \end{aligned} \tag{1.7}$$

It is immediate from the proof of inequality (1.6) and the inequality (1.7) that we can deduce the following new strengthened Hardy's inequality:

$$\begin{aligned} &\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &< e \sum_{n=1}^{\infty} \Big(1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3}\Big) \lambda_n a_n. \end{aligned} \tag{1.8}$$

But we know that the inequality (1.8) is a better improvement of the inequality (1.6), as a result of following

$$\Big(1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3}\Big) < \Big(1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\Big)$$

for $\Lambda_n/\lambda_n \geq 1$.

The purpose of this paper is to prove new extension of the strengthened Hardy's inequality in the spirit of the strict monotonicity of the power mean of n distinct positive numbers.

For any positive values $a_1, a_2, \ldots, a_n$ and positive weights $\alpha_1, \alpha_2 \ldots, \alpha_n$, $\sum_{i=1}^{n} \alpha_i = 1$, and any real $p \neq 0$, we defined the power mean, or the mean of order p of the value a with weights α by

$$M_p(a; \alpha) = M_p(a_1, a_2, \ldots, a_n; \alpha_1, \alpha_2 \ldots, \alpha_n) = \Big(\sum_{i=1}^{n} \alpha_i a_i^p\Big)^{1/p}.$$

An easy application of L'Hospital's rule shows that

$$\lim_{p\to 0} M_p(a;\alpha) = \prod_{i=1}^{n} a_i^{\alpha_i},$$

the geometric mean. Accordingly, we define $M_0(a;\alpha) = \prod_{i=1}^{n} a_i^{\alpha_i}$. It is well known that $M_p(a;\alpha)$ is a nondecreasing function of p for $-\infty \le p \le \infty$, and is strictly increasing unless all the a_i are equal (cf. [1]).

2. Strengthened Hardy's Inequalities

The main results of this paper are presented as follows:

Lemma 2.1 [7]. *Let $x \ge 1$, then we have the following inequality:*

$$\frac{12x+11}{12x+5}\Big(1+\frac{1}{x}\Big)^x < e < \frac{14x+12}{14x+5}\Big(1+\frac{1}{x}\Big)^x. \tag{2.1}$$

We can deduce the following improvement results of the inequality (1.6):

Theorem 2.2. *Let $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m (\Lambda_n \ge 1)$, $a_n \ge 0 (n \in N)$, $0 < p \le 1$ and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. Then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\Big)^p \lambda_n (a_n)^p \Lambda_n^{p-1} \Big(\sum_{k=1}^{n} \lambda_k (c_k a_k)^p\Big)^{(1-p)/p}. \tag{2.2}$$

where $c_k^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n}/(\Lambda_n)^{\Lambda_{n-1}}$.

Proof. By the power mean inequality, we have

$$\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n} \le \Big(\sum_{m=1}^{n} q_m (\alpha_m)^p\Big)^{1/p},$$

for $\alpha_m \ge 0$, $p \ge 0$ and $q_m > 0 (m = 1, 2, \ldots, n)$ with $\sum_{m=1}^{n} q_m = 1$. Setting $c_m > 0, \alpha_m = c_m a_m$ and $q_m = \lambda_m/\Lambda_n$, we obtain

$$(c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n} \le \Big(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p\Big)^{1/p}.$$

Using the above inequality, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\
&= \sum_{n=1}^{\infty} \lambda_{n+1} \frac{(c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \qquad (2{,}3)\\
&\le \sum_{n=1}^{\infty} \Big[\frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}}\Big] \Big(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p\Big)^{1/p}.
\end{aligned}
$$

By using the following inequality (see [2], [6]),

$$
\Big(\sum_{m=1}^{n} z_m\Big)^t \le t \sum_{m=1}^{n} z_m \Big(\sum_{k=1}^{m} z_k\Big)^{t-1},
$$

where $t \ge 1$ is constant and $z_m \ge 0 (m = 1, 2, \cdots)$, it is easy to observe that

$$
\begin{aligned}
&\Big(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p\Big)^{1/p} \\
&\le \frac{1}{\Lambda_n} \Big(\sum_{m=1}^{n} \lambda_m (c_m a_m)^p\Big)^{1/p} \qquad (2.4)\\
&\le \frac{1}{p\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \Big(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p\Big)^{(1-p)/p}
\end{aligned}
$$

for $\Lambda_n \ge 1$ and $0 < p \le 1$. Then, by (2.3) and (2.4), we obtain

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\
&\le \frac{1}{p} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \Big(\frac{\lambda_{n+1}}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}}\Big) \Big(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p\Big)^{(1-p)/p}
\end{aligned}
$$

Choosing $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n} \quad (n \in N)$ and setting $\Lambda_0 = 0$, from $\lambda_{n+1} \le \lambda_n$, it follows that

$$
\begin{aligned}
c_n &= \Big[\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}\Big]^{1/\lambda_n} = \Big(1 + \frac{\lambda_{n+1}}{\Lambda_n}\Big)^{\Lambda_n/\lambda_n} \cdot \Lambda_n \\
&\le \Big(1 + \frac{\lambda_n}{\Lambda_n}\Big)^{\Lambda_n/\lambda_n} \cdot \Lambda_n.
\end{aligned}
$$

This implies that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n}\\
&\le \frac{1}{p}\sum_{m=1}^{n}\lambda_m(c_m a_m)^p\sum_{n=m}^{\infty}\frac{\lambda_{n+1}}{\Lambda_n\Lambda_{n+1}}\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p}\\
&= \frac{1}{p}\sum_{m=1}^{n}\lambda_m(c_m a_m)^p\sum_{n=m}^{\infty}\Big(\frac{1}{\Lambda_n}-\frac{1}{\Lambda_{n+1}}\Big)\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p}\\
&= \frac{1}{p}\sum_{m=1}^{n}\lambda_m(c_m a_m)^p\frac{1}{\Lambda_m}\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p}\\
&\le \frac{1}{p}\sum_{m=1}^{\infty}\Big(1+\frac{1}{\Lambda_m/\lambda_m}\Big)^{p\Lambda_m/\lambda_m}\lambda_m(a_m)^p\Lambda_m^{p-1}\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p}.
\end{aligned}$$

Hence, by the above inequality and Lemma 2.1, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n}\\
&< \frac{e^p}{p}\sum_{n=1}^{\infty}\Big(1-\frac{6\lambda_n}{12\Lambda_n+11\lambda_n}\Big)^p\lambda_n(a_n)^p\Lambda_n^{p-1}\Big(\sum_{k=1}^{n}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p}.
\end{aligned}$$

Thus Theorem 2.2 is proved.

Setting $p\equiv 1$ in Theorem 2.2, then, form inequality (2.2) we have the inequality (1.6). Also assuming that $\lambda_n = 1$ in the Theorem, we have an extension of the strengthened Carleman's inequality as following:

Corollary 2.3. *Let $a_n \ge 0 (n\in N)$, $0<p\le 1$ and $0<\sum_{n=1}^{\infty}a_n<\infty$. Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}\\
&< \frac{e^p}{p}\sum_{n=1}^{\infty}\Big(1-\frac{6}{12n+11}\Big)^p(a_n)^p n^{p-1}\Big(\sum_{k=1}^{n}(c_k a_k)^p\Big)^{(1-p)/p}.
\end{aligned}$$

where $c_k = (1+1/k)^k\cdot k$.

Similarly to Theorem 2.2, we can consider a generalization version of the inequality (1.8) as following theorem:

Theorem 2.4. *Let* $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$, $a_n \ge 0 (n \in N)$, $0 < p \le 1$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. *Then*

$$\begin{aligned} &\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &< \frac{e}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3}\Big)^p \\ &\quad \times \lambda_n (a_n)^p \Lambda_n^{p-1} \Big(\sum_{k=1}^{n} \lambda_k (c_k a_k)^p\Big)^{(1-p)/p}. \end{aligned} \tag{2.5}$$

The proof is almost the same as in proving Theorem 2.2. We here only need to note that

$$\Big(1 + \frac{1}{x}\Big)^x < e\Big(1 - \frac{1}{2(1+x)} - \frac{1}{2(1+x)^2} - \frac{1}{2(1+x)^3}\Big)$$

for $x > 0$, which proved in [11, Lemma 1].

Corollary 2.5. *Let* $a_n \ge 0 (n \in N)$, $0 < p \le 1$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\begin{aligned} &\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \\ &< \frac{e^p}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{1}{2(1+n)} - \frac{1}{24(1+n)^2} - \frac{1}{48(1+n)^3}\Big)^p \\ &\quad \times (a_n)^p n^{p-1} \Big(\sum_{k=1}^{n} (c_k a_k)^p\Big)^{(1-p)/p}. \end{aligned}$$

where $c_k = (1 + 1/k)^k \cdot k$.

Lemma 2.6. *If* $a_1, a_2, \ldots, a_n > 0$ *and* $\alpha_1, \alpha_2 \ldots, \alpha_n > 0$ *with* $\sum_{i=1}^{n} \alpha_i = 1$, *then we have the following inequality:*

$$\Big(\prod_{i=1}^{n} a_i^{\alpha_i}\Big)^k \le \Big(\sum_{i=1}^{n} \alpha_i (a_i)^p\Big)^{k/p}$$

for $0 < k, p$ *with the equality holding if and only if all* a_i *are same.*

Note that Lemma 2.6 is easily deduced form the fact that $M_p(a; \alpha)$ is a continuous strictly increasing function of p.

Now, we are ready to introduce the following new general strengthened Hardy's inequality.

Theorem 2.7. *Let* $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m (\Lambda_n \ge 1)$, $a_n \ge 0 (n \in N)$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n (a_n)^t < \infty$ *for* $0 < p \le t < \infty$. *Then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} < \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\Big)^{p/t} \times \lambda_n (a_n)^p \Lambda_n^{(p-t)/t} \Big(\sum_{k=1}^{n} \lambda_k c_k a_k\Big)^{(t-p)/p}. \tag{2.6}$$

Proof. The proof is almost the same as in Theorem 2.2. By Lemma 2.6, we have

$$(\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n})^t \le \Big(\sum_{m=1}^{n} q_m (\alpha_m)^p\Big)^{t/p}, \quad p, t \ge 0,$$

where $\alpha_m \ge 0$ and $q_m > 0 (m = 1, 2, \ldots, n)$ with $\sum_{m=1}^{n} q_m = 1$. Setting $c_m > 0, \alpha_m = c_m a_m$ and $q_m = \lambda_m / \Lambda_n$, we obtain

$$\big((c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n}\big)^t \le \Big(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p\Big)^{t/p}.$$

Using the above inequality, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} \\ &\le \sum_{n=1}^{\infty} \Big[\frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{t/\Lambda_n}}\Big] \frac{1}{\Lambda_n} \Big(\sum_{m=1}^{n} \lambda_m (c_m a_m)^p\Big)^{t/p} \end{aligned} \tag{2.7}$$

for $\Lambda_n \ge 1$ and $t \ge p$. By using the following inequality (see [2], [6]),

$$\Big(\sum_{m=1}^{n} z_m\Big)^t \le t \sum_{m=1}^{n} z_m \Big(\sum_{k=1}^{m} z_k\Big)^{t-1},$$

where $t \ge 1$ is constant and $z_m \ge 0 (m = 1, 2, \cdots)$, it is easy to observe that

$$\Big(\sum_{m=1}^{n} \lambda_m (c_m a_m)^p\Big)^{t/p} \le \frac{t}{p} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \Big(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p\Big)^{(t-p)/p}. \tag{2.8}$$

for $\Lambda_n \geq 1$ and $t \geq p$. Then, by (2.7) and (2.8), we obtain

$$\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} \leq \sum_{n=1}^{\infty} \Big[\frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{t/\Lambda_n}}\Big] \frac{1}{\Lambda_n} \frac{t}{p} \times \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \Big(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p\Big)^{(t-p)/p}. \tag{2.9}$$

Choosing $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n/t} \quad (n \in N)$ and setting $\Lambda_0 = 0$, from $\lambda_{n+1} \leq \lambda_n$, we have

$$c_n = \Big[\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}\Big]^{1/t\lambda_n} = \Big(1 + \frac{\lambda_{n+1}}{\Lambda_n}\Big)^{\Lambda_n/t\lambda_n} \cdot \Lambda_n^{1/t} \leq \Big(1 + \frac{\lambda_n}{\Lambda_n}\Big)^{\Lambda_n/t\lambda_n} \cdot \Lambda_n^{1/t}.$$

This implies that

$$\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} \leq \frac{t}{p} \sum_{m=1}^{\infty} \Big[\Big(1 + \frac{1}{\Lambda_m/\lambda_m}\Big)^{\Lambda_m/\lambda_m}\Big]^{p/t} \lambda_m (a_m)^p \Lambda^{(p-t)/t} \Big(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p\Big)^{(t-p)/p}.$$

Hence, by the above inequality and Lemma 2.1, we have

$$\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} < \frac{te^{p/t}}{p} \sum_{m=1}^{\infty} \Big(1 - \frac{6\lambda_m}{12\Lambda_m + 11\lambda_m}\Big)^{p/t} \times \lambda_m (a_m)^p \Lambda_m^{(p-t)/t} \Big(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p\Big)^{(t-p)/p}.$$

Thus the inequality (2.6) is proved.

Remark. Setting $t \equiv 1$ in Theorem 2.7, then from (2.6), we obtain the inequality (2.2) in Theorem 2.2. Hence the inequality (2.6) is a new generalization of Hardy's inequality.

Moreover, we can consider a generalization version of the inequality (2.5) as following theorem:

Theorem 2.8. *Let* $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m (\Lambda_n \geq 1)$, $a_n \geq 0 (n \in N)$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n (a_n)^t < \infty$ *for* $0 < p \leq t < \infty$. *Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} \\
&< \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3}\Big)^{p/t} \\
&\quad \times \lambda_n (a_n)^p \Lambda_n^{(p-t)/t} \Big(\sum_{k=1}^{n} \lambda_k c_k a_k\Big)^{(t-p)/p}.
\end{aligned}$$

Proof. The proof is similar to the proof of theorem 2.7.

Also assuming that $\lambda_n = 1$ in the Theorem 2.7 and Theorem 2.8, we have further extension of the strengthened Carleman's inequality as following:

Corollary 2.9. *Let* $a_n \geq 0 (n \in N)$, $0 < p \leq 1$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{t/n} \\
&< \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{6}{12n + 11}\Big)^{p/t} (a_n)^p n^{(p-t)/t} \Big(\sum_{k=1}^{n} (c_k a_k)^p\Big)^{(t-p)/p}.
\end{aligned}$$

where $c_k = (1 + 1/k)^k \cdot k$.

Corollary 2.10. *Let* $a_n \geq 0 (n \in N)$, $0 < p \leq 1$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{t/n} \\
&< \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{1}{2(1+n)} - \frac{1}{24(1+n)^2} - \frac{1}{48(1+n)^3}\Big)^{p/t} \\
&\quad \times (a_n)^p n^{(p-t)/t} \Big(\sum_{k=1}^{n} (c_k a_k)^p\Big)^{(t-p)/p}.
\end{aligned}$$

where $c_k = (1 + 1/k)^k \cdot k$.

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School of Communications and Informatics Victoria University of Technology PO Box 14428, MCMC Melbourne, Victoria 8001, Australia
E-mail address: sever.dragomir@vu.edu.au

Department of Applied Mathematics(or, Brain Korea 21 Project Corps), Changwon National University,Changwon 641-773, Korea
E-mail address: yhkim@sarim.changwon.ac.kr