Monotonicity of Sequences Involving Convex and Concave Functions

MONOTONICITY OF SEQUENCES INVOLVING CONVEX AND CONCAVE FUNCTIONS

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ABSTRACT. Let f be an increasing and convex (concave) function on $[0,1)$ and ϕ a positive increasing concave function on $[0,\infty)$ such that $\phi(0)=0$ and the sequence $\bigl\{\phi(i+1)\bigl(\frac{\phi(i+1)}{\phi(i)}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases $\bigl($the sequence $\bigl\{\phi(i)\bigl(\frac{\phi(i)}{\phi(i+1)}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases$\bigr)$. Then the sequence $\bigl\{\frac{1}{\phi(n)}\sum_{i=0}^{n-1}f\bigl(\frac{\phi(i)}{\phi(n)}\bigr)\bigr\}_{n\in\mathbb{N}}$ is increasing.

1. Introduction

Let f be a strictly increasing convex (or concave) function in $(0,1]$, J.-Ch. Kuang in [8] verified that

$$\frac{1}{n}\sum_{k=1}^{n}f\Bigl(\frac{k}{n}\Bigr)>\frac{1}{n+1}\sum_{k=1}^{n+1}f\Bigl(\frac{k}{n+1}\Bigr)>\int_0^1 f(x)\,\mathrm{d}x. \tag{1}$$

In [15], the second author generalized the results in [8] and obtained the following main result and some corollaries:

Let f be a strictly increasing convex (or concave) function in $(0,1]$, then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k}f\bigl(\frac{i}{n+k}\bigr)$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)\,\mathrm{d}t$, that is,

$$\frac{1}{n}\sum_{i=k+1}^{n+k}f\Bigl(\frac{i}{n+k}\Bigr)>\frac{1}{n+1}\sum_{i=k+1}^{n+k+1}f\Bigl(\frac{i}{n+k+1}\Bigr)>\int_0^1 f(t)\,\mathrm{d}t, \tag{2}$$

where k is a nonnegative integer, n a natural number.

With the help of these conclusions, we can deduce Alzer's inequality (see [8]), Minc-Sathre's inequality (see [16]), and other inequalities involving the sum of

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powers of positive numbers or the ratios of the arithmetic means of n numbers (see [18, 22]). These inequalities have been investigated by many mathematicians. For more information, please refer to the references in this paper. Some results in another direction can be found in [3] and the book online [4, pp. 20–26].

Considering the convexity of a given function or sequence and using the Hermite-Hadamard inequality in [7, 11], the following results were obtained in [19].

Theorem A. *Let f be an increasing and convex (concave) function defined on $[0,1]$, $\{a_i\}_{i\in\mathbb{N}}$ an increasing positive sequence such that $\bigl\{i\bigl(\frac{a_i}{a_{i+1}}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases $\bigl($the sequence $\bigl\{i\bigl(\frac{a_{i+1}}{a_i}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases$\bigr)$, then the sequence $\bigl\{\frac{1}{n}\sum_{i=1}^{n} f\bigl(\frac{a_i}{a_n}\bigr)\bigr\}_{n\in\mathbb{N}}$ is decreasing. That is*

$$\frac{1}{n}\sum_{i=1}^{n} f\Bigl(\frac{a_i}{a_n}\Bigr) \ge \frac{1}{n+1}\sum_{i=1}^{n+1} f\Bigl(\frac{a_i}{a_{n+1}}\Bigr) \ge \int_0^1 f(t)\,\mathrm{d}t. \tag{3}$$

Theorem B. *Let f be an increasing and convex (concave) positive function defined on $[0,1]$, and φ be an increasing convex positive function defined on $[0,\infty)$ such that $\varphi(0)=0$ and $\bigl\{\varphi(i)\bigl[\frac{\varphi(i)}{\varphi(i+1)}-1\bigr]\bigr\}_{i\in\mathbb{N}}$ decreases, then $\bigl\{\frac{1}{\varphi(n)}\sum_{i=1}^{n} f\bigl(\frac{\varphi(i)}{\varphi(n)}\bigr)\bigr\}_{n\in\mathbb{N}}$ is decreasing. That is*

$$\frac{1}{\varphi(n)}\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) \ge \frac{1}{\varphi(n+1)}\sum_{i=1}^{n+1} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right). \tag{4}$$

Taking particular sequences $\{a_i\}_{i\in\mathbb{N}}$ and special functions f and φ in Theorem A and Theorem B, many new inequalities between ratios of mean values are obtained. Further, Alzer's inequality, Minc-Sathre's inequality, and the like, may be recovered under the current setting.

In this article, using a similar approach to that in [19], the following theorems are obtained.

Theorem 1. *Let f be an increasing and convex (concave) function defined on $[0,1]$. Then the sequences $\bigl\{\frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n})\bigr\}_{n\in\mathbb{N}}$ decreases and $\bigl\{\frac{1}{n}\sum_{i=0}^{n-1} f(\frac{i}{n})\bigr\}_{n\in\mathbb{N}}$ increases, and*

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f\Bigl(\frac{i}{n}\Bigr) \ge \frac{1}{n+1}\sum_{i=1}^{n+1} f\Bigl(\frac{i}{n+1}\Bigr) \ge \int_0^1 f(t)\,\mathrm{d}t \\ \ge \frac{1}{n+1}\sum_{i=0}^{n} f\Bigl(\frac{i}{n+1}\Bigr) \ge \frac{1}{n}\sum_{i=0}^{n-1} f\Bigl(\frac{i}{n}\Bigr).\end{aligned} \tag{5}$$

Theorem 2. *Let f be an increasing and convex (concave) function defined on $[0,1)$, the sequence $\{a_i\}_{i\in\mathbb{N}}$ be a positive increasing sequence such that the sequence $\bigl\{i\bigl(\frac{a_{i+1}}{a_i}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases $\bigl($the sequence $\bigl\{i\bigl(\frac{a_i}{a_{i+1}}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases$\bigr)$. Then the sequence $\bigl\{\frac{1}{n}\sum_{i=1}^{n-1}f\bigl(\frac{a_i}{a_n}\bigr)\bigr\}_{n\in\mathbb{N}}$ is increasing, and*

$$\int_0^1 f(t)\,\mathrm{d}t \ge \frac{1}{n+1}\sum_{i=0}^{n} f\Bigl(\frac{a_i}{a_{n+1}}\Bigr) \ge \frac{1}{n}\sum_{i=0}^{n-1} f\Bigl(\frac{a_i}{a_n}\Bigr), \tag{6}$$

where $a_0=0$.

Theorem 3. *Let f be an increasing and convex (concave) function defined on $[0,1]$ and ϕ be a positive increasing concave function defined on $[0,\infty)$ such that $\phi(0)=0$ and the sequence $\bigl\{\phi(i+1)\bigl(\frac{\phi(i+1)}{\phi(i)}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases $\bigl($the sequence $\bigl\{\phi(i)\bigl(\frac{\phi(i)}{\phi(i+1)}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases$\bigr)$. Then the sequence $\bigl\{\frac{1}{\phi(n)}\sum_{i=0}^{n-1}f\bigl(\frac{\phi(i)}{\phi(n)}\bigr)\bigr\}_{n\in\mathbb{N}}$ is increasing, that is,*

$$\frac{1}{\phi(n+1)}\sum_{i=0}^{n} f\left(\frac{\phi(i)}{\phi(n+1)}\right) \ge \frac{1}{\phi(n)}\sum_{i=0}^{n-1} f\left(\frac{\phi(i)}{\phi(n)}\right). \tag{7}$$

2. Proofs of Theorems

Proof of Theorem 1. The first inequality in (5) is equivalent to inequality (1). Now we will prove the last inequality in (5).

The last inequality in (5) is equivalent to

$$\begin{aligned}
(n+1)\sum_{i=0}^{n-1} f\Bigl(\frac{i}{n}\Bigr) &\le n\sum_{i=0}^{n} f\Bigl(\frac{i}{n+1}\Bigr),\\
f(0)+(n+1)\sum_{i=1}^{n-1} f\Bigl(\frac{i}{n}\Bigr) &\le n\sum_{i=1}^{n} f\Bigl(\frac{i}{n+1}\Bigr),\\
\sum_{i=1}^{n}\Bigl[if\Bigl(\frac{i-1}{n}\Bigr)+(n-i)f\Bigl(\frac{i}{n}\Bigr)\Bigr] &\le n\sum_{i=1}^{n} f\Bigl(\frac{i}{n+1}\Bigr),\\
\sum_{i=1}^{n}\Bigl[\frac{i}{n}f\Bigl(\frac{i-1}{n}\Bigr)+\Bigl(1-\frac{i}{n}\Bigr)f\Bigl(\frac{i}{n}\Bigr)\Bigr] &\le \sum_{i=1}^{n} f\Bigl(\frac{i}{n+1}\Bigr).
\end{aligned} \tag{8}$$

It is easy to see that

$$\frac{i(i-1)+(n-i)i}{n^2} < \frac{i}{n+1}, \tag{9}$$

$$\frac{(i+1)^2+(n-i)i}{(n+1)^2} \ge \frac{i}{n}. \tag{10}$$

Since the function f is increasing, from (9) and (10), it follows that

$$f\left(\frac{i(i-1)+(n-i)i}{n^2}\right) \le f\Big(\frac{i}{n+1}\Big), \tag{11}$$

$$f\left(\frac{(i+1)^2+(n-i)i}{(n+1)^2}\right) \ge f\Big(\frac{i}{n}\Big). \tag{12}$$

If f is concave, then we have

$$\frac{i}{n}f\Big(\frac{i-1}{n}\Big)+\Big(1-\frac{i}{n}\Big)f\Big(\frac{i}{n}\Big) \le f\left(\frac{i(i-1)+(n-i)i}{n^2}\right). \tag{13}$$

Combining of (11) with (13) yields

$$\frac{i}{n}f\Big(\frac{i-1}{n}\Big)+\Big(1-\frac{i}{n}\Big)f\Big(\frac{i}{n}\Big) \le f\Big(\frac{i}{n+1}\Big). \tag{14}$$

This implies that the last line in (8) is valid.

If f is convex, then

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[\frac{i}{n+1}f\Big(\frac{i}{n+1}\Big)+\frac{n-i+1}{n+1}f\Big(\frac{i-1}{n+1}\Big)\Big]\\
&\ge \sum_{i=1}^{n} f\Big(\frac{i}{n+1}\cdot\frac{i}{n+1}+\frac{n-i+1}{n+1}\cdot\frac{i-1}{n+1}\Big)\\
&=\sum_{i=0}^{n-1} f\left(\frac{(i+1)^2+(n-i)i}{(n+1)^2}\right).
\end{aligned} \tag{15}$$

Combining (12) with (15) yields

$$\begin{aligned}
\frac{n}{n+1}\sum_{i=0}^{n} f\Big(\frac{i}{n+1}\Big) &= \frac{n}{n+1}f(0)+\frac{n}{n+1}\sum_{i=1}^{n} f\Big(\frac{i}{n+1}\Big)\\
&=\sum_{i=1}^{n}\Big[\frac{i}{n+1}f\Big(\frac{i}{n+1}\Big)+\frac{n-i+1}{n+1}f\Big(\frac{i-1}{n+1}\Big)\Big]\\
&\ge \sum_{i=0}^{n-1} f\Big(\frac{i}{n}\Big).
\end{aligned} \tag{16}$$

The proof is complete. □

Proof of Theorem 2. The right inequality in (6) can be rewritten as

$$\begin{aligned}(n+1)\sum_{i=0}^{n-1} f\Big(\frac{a_i}{a_n}\Big) &\le n\sum_{i=0}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big),\\ f(0)+(n+1)\sum_{i=1}^{n-1} f\Big(\frac{a_i}{a_n}\Big) &\le n\sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big),\\ \sum_{i=1}^{n}\Big[if\Big(\frac{a_{i-1}}{a_n}\Big)+(n-i)f\Big(\frac{a_i}{a_n}\Big)\Big] &\le n\sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big),\\ \sum_{i=1}^{n}\Big[\frac{i}{n}f\Big(\frac{a_{i-1}}{a_n}\Big)+\Big(1-\frac{i}{n}\Big)f\Big(\frac{a_i}{a_n}\Big)\Big] &\le \sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big).\end{aligned} \tag{17}$$

If the sequence $\big\{i\big(\frac{a_{i+1}}{a_i}-1\big)\big\}_{i\in\mathbb{N}}$ is decreasing, then

$$\frac{(i+1)a_{i+1}+(n-i)a_i}{(n+1)a_{n+1}} \ge \frac{a_i}{a_n}. \tag{18}$$

In fact, inequality (18) is equivalent to

$$(i+1)\Big(\frac{a_{i+1}}{a_i}-1\Big) \ge (n+1)\Big(\frac{a_{n+1}}{a_n}-1\Big).$$

Let $x_i = i\big(\frac{a_{i+1}}{a_i}-1\big)$, then $\{x_i\}_{i\in\mathbb{N}}$ decreases, therefore

$$\begin{aligned}&(i+1)\Big(\frac{a_{i+1}}{a_i}-1\Big)-(n+1)\Big(\frac{a_{n+1}}{a_n}-1\Big)\\ &=\frac{(i+1)x_i}{i}-\frac{(n+1)x_n}{n}\\ &=(x_i-x_n)+\Big(\frac{x_i}{i}-\frac{x_n}{n}\Big)\\ &\ge 0.\end{aligned}$$

On the other hand, if the sequence $\big\{i\big(\frac{a_i}{a_{i+1}}-1\big)\big\}_{i\in\mathbb{N}}$ increases, then

$$i\Big(\frac{a_{i-1}}{a_i}-1\Big) \le n\Big(\frac{a_n}{a_{n+1}}-1\Big). \tag{19}$$

In fact, we have

$$i\Big(\frac{a_{i-1}}{a_i}-1\Big) \le (i-1)\Big(\frac{a_{i-1}}{a_i}-1\Big) \le n\Big(\frac{a_n}{a_{n+1}}-1\Big).$$

The inequality (19) can be rewritten as

$$\frac{ia_{i-1}+(n-i)a_i}{na_n} \le \frac{a_i}{a_{n+1}}. \tag{20}$$

Since the function f is increasing, it follows from inequalities (18) and (20) that

$$f\left(\frac{(i+1)a_{i+1}+(n-i)a_i}{(n+1)a_{n+1}}\right) \ge f\Big(\frac{a_i}{a_n}\Big) \tag{21}$$

and

$$f\Big(\frac{ia_{i-1}+(n-i)a_i}{na_n}\Big)\le f\Big(\frac{a_i}{a_{n+1}}\Big), \tag{22}$$

respectively.

If f is a positive increasing convex function and the sequence $\big\{i\big(\frac{a_{i+1}}{a_i}-1\big)\big\}_{i\in\mathbb{N}}$ decreases, then from (17) and (21),

$$\begin{aligned}
\frac{n}{n+1}\sum_{i=0}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big) &= \frac{n}{n+1}f(0)+\frac{n}{n+1}\sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big)\\
&=\sum_{i=1}^{n}\Big[\frac{i}{n+1}f\Big(\frac{a_i}{a_{n+1}}\Big)+\frac{n-i+1}{n+1}f\Big(\frac{a_{i-1}}{a_{n+1}}\Big)\Big]\\
&\ge\sum_{i=1}^{n} f\left(\frac{ia_i+(n-i+1)a_{i-1}}{(n+1)a_{n+1}}\right)\\
&=\sum_{i=0}^{n-1} f\left(\frac{(i+1)a_{i+1}+(n-i)a_i}{(n+1)a_{n+1}}\right)\\
&\ge\sum_{i=0}^{n-1} f\Big(\frac{a_i}{a_n}\Big),
\end{aligned} \tag{23}$$

where we define $a_0=0$.

If f is a positive increasing concave function and the sequence $\big\{i\big(\frac{a_i}{a_{i+1}}-1\big)\big\}_{i\in\mathbb{N}}$ increases, then from (17) and (22),

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[\frac{i}{n}f\Big(\frac{a_{i-1}}{a_n}\Big)+\Big(1-\frac{i}{n}\Big)f\Big(\frac{a_i}{a_n}\Big)\Big]\\
&\qquad\le\sum_{i=1}^{n} f\Big(\frac{ia_{i-1}+(n-i)a_i}{na_n}\Big)\le\sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big).
\end{aligned} \tag{24}$$

The proof is complete. □

Proof of Theorem 3. Firstly, suppose that the function f is an increasing convex function and the sequence $\big\{\phi(i+1)\big(\frac{\phi(i+1)}{\phi(i)}-1\big)\big\}_{i\in\mathbb{N}}$ is decreasing. Then

$$\phi(i+1)\left(\frac{\phi(i+1)}{\phi(i)}-1\right)\ge\phi(n+1)\left(\frac{\phi(n+1)}{\phi(n)}-1\right), \tag{25}$$

which is equivalent to

$$\frac{\phi^2(i+1)+[\phi(n+1)-\phi(i+1)]\phi(i)}{\phi^2(n+1)}\ge\frac{\phi(i)}{\phi(n)}. \tag{26}$$

Therefore

$$f\left(\frac{\phi^2(i+1)+[\phi(n+1)-\phi(i+1)]\phi(i)}{\phi^2(n+1)}\right)\geq f\left(\frac{\phi(i)}{\phi(n)}\right), \tag{27}$$

since the function f is increasing.

Further, by standard convexity arguments, it follows that

$$\begin{aligned}
&\sum_{i=1}^{n}\left[\frac{\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i)}{\phi(n+1)}\right)+\frac{\phi(n+1)-\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i-1)}{\phi(n+1)}\right)\right]\\
&\geq\sum_{i=1}^{n}f\left(\frac{\phi^2(i)+[\phi(n+1)-\phi(i)]\phi(i-1)}{\phi^2(n+1)}\right)\\
&=\sum_{i=0}^{n-1}f\left(\frac{\phi^2(i+1)+[\phi(n+1)-\phi(i+1)]\phi(i)}{\phi^2(n+1)}\right)\\
&\geq\sum_{i=0}^{n-1}f\left(\frac{\phi(i)}{\phi(n)}\right),
\end{aligned}\tag{28}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}\left[\frac{\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i)}{\phi(n+1)}\right)+\frac{\phi(n+1)-\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i-1)}{\phi(n+1)}\right)\right]\\
&=\sum_{i=0}^{n-1}\frac{\phi(n+1)-\phi(i+1)+\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i)}{\phi(n+1)}\right)+\frac{\phi(n)}{\phi(n+1)}f\left(\frac{\phi(n)}{\phi(n+1)}\right)\\
&\leq\frac{\phi(n)}{\phi(n+1)}\sum_{i=0}^{n-1}f\left(\frac{\phi(i)}{\phi(n+1)}\right)+\frac{\phi(n)}{\phi(n+1)}f\left(\frac{\phi(n)}{\phi(n+1)}\right)\\
&=\frac{\phi(n)}{\phi(n+1)}\sum_{i=0}^{n}f\left(\frac{\phi(i)}{\phi(n+1)}\right).
\end{aligned}\tag{29}$$

Combining of (28) with (29) yields

$$\frac{\phi(n)}{\phi(n+1)}\sum_{i=0}^{n}f\left(\frac{\phi(i)}{\phi(n+1)}\right)\geq\sum_{i=0}^{n-1}f\left(\frac{\phi(i)}{\phi(n)}\right)$$

and so inequality (7) holds.

Secondly, let f be an increasing concave function and the sequence $\big\{\phi(i)\big(\frac{\phi(i)}{\phi(i+1)}-1\big)\big\}_{i\in\mathbb{N}}$ be increasing. Then

$$\phi(n)\left(\frac{\phi(n)}{\phi(n+1)}-1\right)\geq\phi(i-1)\left(\frac{\phi(i-1)}{\phi(i)}-1\right), \tag{30}$$

which is equivalent to

$$\frac{\phi(i)}{\phi(n+1)}\geq\frac{\phi^2(i-1)+[\phi(n)-\phi(i-1)]\phi(i)}{\phi^2(n)}, \tag{31}$$

and hence

$$f\left(\frac{\phi(i)}{\phi(n+1)}\right) \geq f\left(\frac{\phi^2(i-1)+[\phi(n)-\phi(i-1)]\phi(i)}{\phi^2(n)}\right). \tag{32}$$

Thus from (32)

$$\begin{aligned}
\sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n+1)}\right) &\geq \sum_{i=1}^{n} f\left(\frac{\phi^2(i-1)+[\phi(n)-\phi(i-1)]\phi(i)}{\phi^2(n)}\right)\\
&\geq \sum_{i=1}^{n}\left[\frac{\phi(i-1)}{\phi(n)} f\left(\frac{\phi(i-1)}{\phi(n)}\right)+\frac{\phi(n)-\phi(i-1)}{\phi(n)} f\left(\frac{\phi(i)}{\phi(n)}\right)\right], \text{ (since } f \text{ is concave)},\\
&\geq \sum_{i=1}^{n}\left[\frac{\phi(i-1)}{\phi(n)} f\left(\frac{\phi(i-1)}{\phi(n)}\right)+\frac{\phi(n+1)-\phi(i)}{\phi(n)} f\left(\frac{\phi(i)}{\phi(n)}\right)\right], \text{ (since } \phi \text{ is concave)}.
\end{aligned} \tag{33}$$

Inequality (33) can be rewritten as

$$\begin{aligned}
&\phi(n) \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n+1)}\right)\\
&\geq \sum_{i=1}^{n}\left[\phi(i-1) f\left(\frac{\phi(i-1)}{\phi(n)}\right)+[\phi(n+1)-\phi(i)] f\left(\frac{\phi(i)}{\phi(n)}\right)\right]\\
&= \phi(n+1) \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n)}\right)-\phi(n) f(1),
\end{aligned} \tag{34}$$

which is equivalent to

$$\begin{gathered}
\phi(n+1) \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n)}\right) \leq \phi(n) \sum_{i=1}^{n+1} f\left(\frac{\phi(i)}{\phi(n+1)}\right),\\
\frac{1}{\phi(n)} \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n)}\right) \leq \frac{1}{\phi(n+1)} \sum_{i=1}^{n+1} f\left(\frac{\phi(i)}{\phi(n+1)}\right).
\end{gathered} \tag{35}$$

Therefore

$$\begin{aligned}
&\frac{1}{\phi(n+1)} \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n+1)}\right)-\frac{1}{\phi(n)} \sum_{i=1}^{n-1} f\left(\frac{\phi(i)}{\phi(n)}\right)\\
&\geq\left[\frac{1}{\phi(n)}-\frac{1}{\phi(n+1)}\right] f(1)\\
&\geq\left[\frac{1}{\phi(n)}-\frac{1}{\phi(n+1)}\right] f(0),
\end{aligned} \tag{36}$$

which implies the inequality (7).

The proof is complete. □

3. Corollaries

From these theorems, we can obtain many new inequalities related to Alzer's inequality and others or, similar inequalities to those in [19].

If $f(x)=x^r$ for $x\in[0,1]$ and $r>0$, then it follows from Theorem 1 that

Corollary 1. *Let $n\in\mathbb{N}$, then, for all real number $r>0$, we have*

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n-1}i^r}{\frac{1}{n+1}\sum_{i=1}^{n}i^r}\right)^{1/r}\le\frac{n}{n+1}\le\left(\frac{\frac{1}{n}\sum_{i=1}^{n}i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1}i^r}\right)^{1/r}. \tag{37}$$

The right hand inequality in (37) is called Alzer's inequality.

Taking $f(x)=\ln(1+x)$ and $f(x)=\ln\frac{x}{1+x}$ for $x\in[0,1]$ in Theorem 2 produces

Corollary 2. *If $\{a_i\}_{i\ge0}$ is a positive increasing sequence such that $a_0=0$ and the sequence $\big\{i\big(\frac{a_i}{a_{i+1}}-1\big)\big\}_{i\in\mathbb{N}}$ increases, then*

$$\frac{a_n}{a_{n+1}}\ge\frac{\sqrt[n]{\prod_{i=0}^{n-1}(a_i+a_n)}}{\sqrt[n+1]{\prod_{i=0}^{n}(a_i+a_{n+1})}}\ge\frac{\sqrt[n]{\prod_{i=0}^{n-1}a_i}}{\sqrt[n+1]{\prod_{i=0}^{n}a_i}}. \tag{38}$$

Similarly, if $f(x)=\ln(1+x)$ for $x\in[0,1]$, we have from Theorem 3

Corollary 3. *Let ϕ be a positive increasing cancave function defined on $[0,\infty)$ such that $\phi(0)=0$ and the sequence $\big\{\phi(i)\big(\frac{\phi(i)}{\phi(i+1)}-1\big)\big\}_{i\in\mathbb{N}}$ increases, then*

$$\frac{[\phi(n)]^{n/\phi(n)}}{[\phi(n+1)]^{(n+1)/\phi(n+1)}}\ge\frac{\sqrt[\phi(n)]{\prod_{i=0}^{n-1}[\phi(n)-\phi(i)]}}{\sqrt[\phi(n+1)]{\prod_{i=0}^{n}[\phi(n+1)-\phi(i)]}}. \tag{39}$$

Remark 1. Theorem A and Theorem 2 together give upper and lower bounds for integral $\int_0^1 f(t)\,dt$. Further, Theorem B and Theorem 3 may be combined to give, with the stated conditions holding,

$$\begin{aligned}\frac{\phi(n+1)}{\phi(n)}\sum_{i=0}^{n-1}f\left(\frac{\phi(i)}{\phi(n)}\right)-f(0)&\le\sum_{i=1}^{n}f\left(\frac{\phi(i)}{\phi(n+1)}\right)\\&\le\frac{\phi(n+1)}{\phi(n)}\sum_{i=1}^{n}f\left(\frac{\phi(i)}{\phi(n)}\right)-f(1).\end{aligned} \tag{40}$$

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