Some Osrowski Type Inequalities for Double Integrals of Functions whose Partial Derivatives Satisfy Certain Convexity Properties

SOME OSROWSKI TYPE INEQUALITIES FOR DOUBLE INTEGRALS OF FUNCTIONS WHOSE PARTIAL DERIVATIVES SATISFY CERTAIN CONVEXITY PROPERTIES

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Abstract. In this paper we point out some new two-dimensional integral inequalities, for functions whose partial derivatives satisfy certain convexity properties.

1. Introduction

Barnett and Dragomir [1], Milovanović [5], Pachpatte [6] and Hanna et al. [4] developed two dimensional integral inequalities whose error bounds were expressed in term of Lebesgue norms of the integrand partial derivatives. Here we consider a function whose first partial derivatives exist and satisfy certain convexity properties over a given rectangular region. The work in this paper is presented in the following order. In Section 2, a two variable integral identity for first differentiable mapping is developed. In Section 3, we derive some double integral inequalities. Error bounds are expressed in term of the L_p norms of the first and first mixed partial derivatives of the integrand.

2. Integral Identity

The following identity is interesting in itself as well:

Lemma 1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a function so that the partial derivatives $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial^2 f}{\partial x\partial y}$ to be denoted by D_x, D_y and D_{xy} herein respectively, are continuous on $[a,b]\times[c,d]$, then for any $(x_0,y_0)\in[a,b]\times[c,d]$ we have the identity:*

$$
\begin{aligned}
(2.1)\quad f(x_0,y_0) &= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt \\
&+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y\left[t,(1-\mu)y_0+\mu s\right]d\mu\right)dsdt \\
&+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x\left[(1-\lambda)x_0+\lambda t,s\right]d\lambda\right)dsdt
\end{aligned}
$$

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$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s) \times\left(\int_0^1\int_0^1 D_{xy}\left[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s\right]d\mu d\lambda\right)dsdt.$$

Proof. We will use the following identity, which has been established in [3]

$$Q(x)=\frac{1}{b-a}\int_a^b Q(t) + \frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 Q'\left[(1-\lambda)x+\lambda t\right]d\lambda\right)dt, \tag{2.2}$$

where $x\in[a,b]$ and $Q:[a,b]\to\mathbb{R}$ is an absolutely continuous function on $[a,b]$. For the sake of completeness we give here a short proof for (2.2).

For any $x,t\in[a,b]$, $x\neq t$,

$$\frac{Q(x)-Q(t)}{x-t}=\frac{\int_t^x Q'(u)du}{x-t}=\int_0^1 Q'[(1-\lambda)t+\lambda x]d\lambda$$

where we used the change of variables,

$$u=(1-\lambda)t+\lambda x, \qquad u\in[0,1].$$

Then

$$Q(x)=Q(t)+(x-t)\int_0^1 Q'[(1-\lambda)t+\lambda x]d\lambda$$

and this holds for any $x,t\in[a,b]$. Integrating over $t\in[a,b]$ and dividing by $(b-a)$ we get (2.2).

Now, fix $y_0\in[c,d]$, then by (2.2) we have the equality

$$f(x_0,y_0)=\frac{1}{b-a}\int_a^b f(t,y_0)\,dy + \frac{1}{b-a}\int_a^b (x_0-t)\left(\int_0^1 D_x\left[(1-\lambda)x_0+\lambda t,y_0\right]d\lambda\right)dt \tag{2.3}$$

for any $t\in[a,b]$. Applying (2.2) over the second variable, we may write:

$$f(t,y_0)=\frac{1}{d-c}\int_a^b f(t,y_0)\,dy + \frac{1}{d-c}\int_c^d (y_0-s)\left(\int_0^1 D_y\left[t,(1-\mu)y_0+\mu s\right]d\mu\right)dt. \tag{2.4}$$

Integrating (2.4) over $t\in[a,b]$ and using Fubini's theorem we deduce

$$\frac{1}{b-a}\int_a^b f(t,y_0)\,dy = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y\left[t,(1-\mu)y_0+\mu s\right]d\mu\right)dsdt. \tag{2.5}$$

If we fix $x_0, t \in [a,b]$, then applying (2.2) again, we obtain

$$D_x[(1-\lambda)x_0 + \lambda t, y_0] = \frac{1}{d-c}\int_c^d D_x[(1-\lambda)x_0+\lambda t, s]\,ds + \frac{1}{d-c}\int_c^d (y_0 - s)\left(\int_0^1 D_{xy}[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s]\,d\mu\right)ds. \tag{2.6}$$

We then integrate (2.6) over $\lambda \in [0,1]$, and, on inverting the order of integrals, we get that

$$\int_0^1 D_x[(1-\lambda)x_0+\lambda t, y_0]\,d\lambda = \frac{1}{d-c}\int_c^d D_x[(1-\lambda)x_0+\lambda t, s]\,ds + \frac{1}{d-c}\int_c^d (y_0-s)\left(\int_0^1\int_0^1 D_{xy}[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s]\,d\mu d\lambda\right)ds. \tag{2.7}$$

Consequently, we deduce

$$\begin{aligned} &\frac{1}{(b-a)}\int_c^d (x_0-t)\left(\int_0^1 D_x[(1-\lambda)x_0+\lambda t, y_0]\,d\lambda\right)dt \\ &= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x[(1-\lambda)x_0+\lambda t, s]\,d\lambda\right)dsdt \\ &\quad + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s) \\ &\quad \times\left(\int_0^1\int_0^1 D_{xy}[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s]\,d\mu d\lambda\right)dsdt. \end{aligned} \tag{2.8}$$

Finally, using (2.3), (2.5) and (2.8) we obtain the desired equality (2.1). ∎

3. Some Integral Inequalities

3.1. Mapping Whose First Derivative Belongs to $L_\infty[[a,b]\times[c,d]]$.

In this section we tap the equalities of the previous section and develop inequalities for the depiction of the two dimensional integral of a function with respect to the derivatives at a multiple number of points over the given region.

We start with the following result.

Theorem 1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a function such that the partial derivatives D_x, D_y and D_{xy} exist and are continuous on $[a,b]\times[c,d]$. If $|D_x|$ is convex over first direction, $|D_y|$ is convex over the second direction and $|D_{xy}|$ is convex in both* *directions, then we have the inequality*

$$\begin{aligned}(3.1)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \\ &\leq \frac{\left[\|D_x(x_0,.)\|_\infty + \|D_x\|_\infty\right]}{2(b-a)}\left[\frac{1}{4}(b-a)^2 + \left(x_0 - \frac{a+b}{2}\right)^2\right] \\ &+ \frac{\left[\|D_y(.,y_0)\|_\infty + \|D_y\|_\infty\right]}{2(d-c)}\left[\frac{1}{4}(d-c)^2 + \left(y_0 - \frac{c+d}{2}\right)^2\right] \\ &+ \frac{\left[\|D_{xy}(x_0,.)\|_\infty + |D_{xy}(x_0,y_0)| + \|D_{xy}(.,y_0)\|_\infty + \|D_{xy}\|_\infty\right]}{4(b-a)(d-c)} \\ &\times\left[\frac{1}{4}(b-a)^2 + \left(x_0 - \frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2 + \left(y_0 - \frac{c+d}{2}\right)^2\right].\end{aligned}$$

for all $(x_0, y_0) \in [a,b] \times [c,d]$ *, where*

$$\|D_x(x_0,.)\|_\infty := \sup_{s\in[c,d]}|D_x(x_0,s)| < \infty, \quad \|D_y(.,y_0)\|_\infty := \sup_{t\in[a,b]}|D_y(t,y_0)| < \infty,$$

$$\|D_x\|_\infty := \sup_{(t,s)\in[a,b]\times[c,d]}|D_x(t,s)| < \infty,$$

$$\|D_{xy}(x_0,.)\|_\infty := \sup_{s\in[c,d]}|D_{xy}(x_0,s)| < \infty, \quad \|D_{xy}(.,y_0)\|_\infty := \sup_{t\in[a,b]}|D_{xy}(t,y_0)| < \infty$$

and

$$\|D_{xy}\|_\infty := \sup_{(t,s)\in[a,b]\times[c,d]}|D_{xy}(t,s)| < \infty.$$

Proof. Using Lemma 1 we get from (2.1)

$$\begin{aligned}(3.2)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \\ &= \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x[(1-\lambda)x_0+\lambda t, s]\,d\lambda\right)dsdt\right. \\ &+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y[t,(1-\mu)y_0+\mu s]\,d\mu\right)dsdt \\ &+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s) \\ &\left.\times\left(\int_0^1\int_0^1 D_{xy}[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s]\,d\mu d\lambda\right)dsdt\right|\end{aligned}$$

By the triangle inequality it follows that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \tag{3.3}$$
$$\leq \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x[(1-\lambda)x_0+\lambda t,s]\,d\lambda\right)dsdt\right|$$
$$+\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y[t,(1-\mu)y_0+\mu s]\,d\mu\right)dsdt\right|$$
$$+\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s)\right.$$
$$\left.\times\left(\int_0^1\int_0^1 D_{xy}[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s]\,d\mu d\lambda\right)dsdt\right|.$$

Now, since $|D_x(.,s)|$ is convex over the $t-$direction for any $s\in[c,d]$, using the properties of the modulus, we then have that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x[(1-\lambda)x_0+\lambda t,s]\,d\lambda\right)dsdt\right| \tag{3.4}$$
$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |x_0-t|\left(\int_0^1 |D_x[(1-\lambda)x_0+\lambda t,s]|\,d\lambda\right)dsdt$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |x_0-t| \tag{3.5}$$
$$\times\left(\int_0^1 [(1-\lambda)|D_x(x_0,s)|+\lambda|D_x(t,s)|]\,d\lambda\right)dsdt$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \left|\frac{x_0-t}{2}\right|(\|D_x(x_0,.)\|_\infty+\|D_x\|_\infty)\,dsdt \tag{3.6}$$
$$=\frac{[\|D_x(x_0,.)\|_\infty+\|D_x\|_\infty]}{2(b-a)(d-c)}\int_a^b\int_c^d |x_0-t|\,dsdt$$
$$=\frac{[\|D_x(x_0,.)\|_\infty+\|D_x\|_\infty]}{2(b-a)}\left[\frac{1}{4}(b-a)^2+\left(x_0-\frac{a+b}{2}\right)^2\right]$$

since

$$\begin{aligned}\int_a^b\int_c^d |x_0-t|\,dsdt &= \int_c^d ds\int_a^b |x_0-t|\,dt\\ &= (d-c)\left[\int_a^{x_0}(x_0-t)\,dt+\int_{x_0}^b (t-x_0)\,dt\right].\end{aligned}$$

In a similar manner, since $|D_x(t,.)|$ is convex over the $s-$direction for any $t \in (a,b)$, we obtain

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y[t,(1-\mu)y_0+\mu s]\,d\mu\right)dsdt\right| \le \frac{[\|D_y(.,y_0)\|_\infty+\|D_y\|_\infty]}{2(d-c)}\left[\frac{1}{4}(d-c)^2+\left(y_0-\frac{c+d}{2}\right)^2\right]. \tag{3.7}$$

Now, taking into account that $|D_{xy}(.,.)|$ is also convex over both t and s we can prove that

$$\begin{aligned}\Bigg|\frac{1}{(b-a)(d-c)}&\int_a^b\int_c^d (x_0-t)(y_0-s) \\ &\times\left(\int_0^1\int_0^1 D_{xy}[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s]\,d\mu d\lambda\right)dsdt\Bigg| \\ &\le \frac{[\|D_{xy}(x_0,.)\|_\infty+|D_{xy}(x_0,y_0)|+\|D_{xy}(.,y_0)\|_\infty+\|D_{xy}\|_\infty]}{4(b-a)(d-c)} \\ &\quad\times\left[\frac{1}{4}(b-a)^2+\left(x_0-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y_0-\frac{c+d}{2}\right)^2\right].\end{aligned} \tag{3.8}$$

Utilizing (3.6), (3.7), (3.8) and substituting into (3.2), we obtain the desired result. Thus, the theorem is proved. ∎

Taking into account that x_0 and y_0 are free parameters, we can produce "mid-point" and "boundary-point" type results by choosing appropriate values for x_0 and y_0.

The following corollary will give the best result in the class.

Corollary 1. *Under the assumptions of Theorem 1, we have the inequality:*

$$\begin{aligned}\left|\int_a^b\int_c^d f(t,s)dsdt-(b-a)(d-c)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right| &\le \frac{\|D_x(\frac{a+b}{2},.)\|_\infty+\|D_x\|_\infty}{8}(b-a)^2(d-c) \\ &+\frac{\|D_y(.,\frac{c+d}{2})\|_\infty+\|D_y\|_\infty}{8}(b-a)(d-c)^2 \\ &+\frac{(b-a)^2(d-c)^2}{64}\Bigg[\|D_{xy}(\frac{a+b}{2},.)\|_\infty \\ &+|D_{xy}(\frac{a+b}{2},\frac{c+d}{2})|+\|D_{xy}(.,\frac{c+d}{2})\|_\infty+\|D_{xy}\|_\infty\Bigg].\end{aligned} \tag{3.9}$$

3.2. Mappings whose First Derivative Belongs to $L_p[[a,b]\times[c,d]]$.

In this section we point out an inequality for double integrals in terms of the $\|\cdot\|_p-$norm of the first derivatives.

Theorem 2. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a function such that the partial derivatives D_x, D_y, D_{xy} exist and are continuous on $[a,b]\times[c,d]$. If $|D_x|$ is convex over first* *direction,* $|D_y|$ *is convex over the second direction and* $|D_{xy}|$ *is convex in both directions, then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right|$$

$$\leq \frac{\left[(b-a)^{\frac{1}{p}}\|D_x(x_0,.)\|_p + \|D_x\|_p\right]}{2(b-a)(d-c)^{\frac{q-1}{q}}}\left[\frac{(x_0-a)^{q+1}+(b-x_0)^{q+1}}{(q+1)}\right]^{\frac{1}{q}}.$$

$$+\frac{\left[(d-c)^{\frac{1}{p}}\|D_y(.,y_0)\|_p + \|D_y\|_p\right]}{2(b-a)^{\frac{q-1}{q}}(d-c)}\left[\frac{(y_0-c)^{q+1}+(d-y_0)^{q+1}}{(q+1)}\right]^{\frac{1}{q}}$$

$$+\frac{\left[G_1(x_0)+G_2(x_0,y_0)+G_3(y_0)+\|D_{xy}\|_p\right]}{4(b-a)(d-c)}$$

$$\times\left[\frac{(x_0-a)^{q+1}+(b-x_0)^{q+1}}{(q+1)}\right]^{\frac{1}{q}}\left[\frac{(y_0-c)^{q+1}+(d-y_0)^{q+1}}{(q+1)}\right]^{\frac{1}{q}}.$$

for all $(x_0,y_0)\in[a,b]\times[c,d]$ *where*

$$G_1(x_0) := (b-a)^{\frac{1}{p}}\|D_{xy}(x_0,.)\|_p\,,$$

$$G_2(x_0,y_0) := (b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}}|D_{xy}(x_0,y_0)|,$$

$$G_3(y_0) := (d-c)^{\frac{1}{p}}\|D_{xy}(.,y_0)\|_p\,,$$

$$\|D_x(x_0,.)\|_p := \left(\int_c^d |D_x(x_0,s)|^p\,ds\right)^{\frac{1}{p}} < \infty,$$

$$\|D_y(.,y_0)\|_p := \left(\int_a^b |D_y(t,y_0)|^p\,dt\right)^{\frac{1}{p}} < \infty,$$

$$\|D_x\|_p := \left(\int_a^b\int_c^d |D_x(t,s)|^p\,dsdt\right)^{\frac{1}{p}} < \infty,$$

$$\|D_{xy}(x_0,.)\|_p := \left(\int_c^d |D_{xy}(x_0,s)|^p\,ds\right)^{\frac{1}{p}} < \infty,$$

$$\|D_{xy}(.,y_0)\|_p := \left(\int_a^b |D_{xy}(t,y_0)|^p\,dt\right)^{\frac{1}{p}} < \infty,$$

and

$$\|D_{xy}\|_p := \left(\int_a^b\int_c^d |D_{xy}(t,s)|^p\,dsdt\right)^{\frac{1}{p}} < \infty.$$

where $\frac{1}{p}+\frac{1}{q}=1$ *for all* $(1<p<\infty)$.

Proof. We can write (3.3) as

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \leq M_1(x_0)+M_2(y_0)+M_3(x_0,y_0)\,,$$

where

$$
\begin{aligned}
M_1(x_0) &:= \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (x_0 - t) \left(\int_0^1 D_x [(1-\lambda) x_0 + \lambda t, s] \, d\lambda \right) ds dt \right|, \\
M_2(y_0) &:= \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (y_0 - s) \left(\int_0^1 D_y [t, (1-\mu) y_0 + \mu s] \, d\mu \right) ds dt \right|
\end{aligned}
$$

and

$$
\begin{aligned}
M_3(x_0, y_0) &:= \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (x_0 - t)(y_0 - s) \right. \\
&\quad \left. \times \left(\int_0^1 \int_0^1 D_{xy} [(1-\lambda) x_0 + \lambda t, (1-\mu) y_0 + \mu s] \, d\mu d\lambda \right) ds dt \right|.
\end{aligned}
$$

Now

$$
\begin{aligned}
M_1(x_0) &= \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (x_0 - t) \left(\int_0^1 D_x [(1-\lambda) x_0 + \lambda t, s] \, d\lambda \right) ds dt \right| \\
&\leq \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d |x_0 - t| \left(|D_x(x_0, s)| + |D_x(t, s)| \right) ds dt
\end{aligned}
$$

then

$$
M_1(x_0) \leq \frac{1}{2(b-a)(d-c)} \left[I_{x_0}^{(1)} + I_{x_0}^{(2)} \right], \tag{3.10}
$$

where

$$
I_{x_0}^{(1)} := \int_a^b \int_c^d |x_0 - t| \, |D_x(x_0, s)| \, ds dt, \tag{3.11}
$$

$$
I_{x_0}^{(2)} := \int_a^b \int_c^d |x_0 - t| \, |D_x(t, s)| \, ds dt. \tag{3.12}
$$

Applying the Hölder inequality for double integrals for $I_{x_0}^{(1)}$, we find that

$$
\begin{aligned}
I_{x_0}^{(1)} &\leq \left(\int_a^b \int_c^d |x_0 - t|^q \, ds dt \right)^{\frac{1}{q}} \left(\int_a^b \int_c^d |D_x(x_0, s)|^p \, ds dt \right)^{\frac{1}{p}} \\
&= \left((d-c) \int_a^b |x_0 - t|^q \, dt \right)^{\frac{1}{q}} \left((b-a) \int_c^d |D_x(x_0, s)|^p \, ds \right)^{\frac{1}{p}} \\
&= (d-c)^{\frac{1}{q}} \left[\int_a^{x_0} (x_0 - t)^q \, dt + \int_{x_0}^b (t - x_0)^q \, dt \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{p}} \left\| D_x(x_0, .) \right\|_p \\
&= (d-c)^{\frac{1}{q}} \left[\left. \frac{-(x_0 - t)^{q+1}}{q+1} \right|_a^{x_0} + \left. \frac{(t - x_0)^{q+1}}{q+1} \right|_{x_0}^b \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{p}} \left\| D_x(x_0, .) \right\|_p,
\end{aligned}
$$

then we get

$$
I_{x_0}^{(1)} \leq (b-a)^{\frac{1}{p}} (d-c)^{\frac{1}{q}} \left[\frac{(x_0 - a)^{q+1} + (b - x_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} \left\| D_x(x_0, .) \right\|_p. \tag{3.13}
$$

Similarly

$$I_{x_0}^{(2)} \leq (d-c)^{\frac{1}{q}} \left[\frac{(x_0-a)^{q+1} + (b-x_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} \|D_x\|_p . \tag{3.14}$$

Using (3.13) and (3.14) and substituting in (3.10), M_1 becomes

$$M_1(x_0) \leq \frac{\left[(b-a)^{\frac{1}{p}} \|D_x(x_0,.)\|_p + \|D_x\|_p\right]}{2(b-a)(d-c)^{\frac{q-1}{q}}} \left[\frac{(x_0-a)^{q+1} + (b-x_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} . \tag{3.15}$$

In similar way

$$M_2(y_0) \leq \frac{\left[(d-c)^{\frac{1}{p}} \|D_y(.,y_0)\|_p + \|D_y\|_p\right]}{2(b-a)^{\frac{q-1}{q}}(d-c)} \left[\frac{(y_0-c)^{q+1} + (d-y_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}}$$

and

$$\begin{aligned} M_3(x_0,y_0) \quad &\leq \quad \frac{[G_1(x_0) + G_2(x_0,y_0) + G_3(y_0) + \|D_{xy}\|_p]}{4(b-a)(d-c)} \\ &\times \left[\frac{(x_0-a)^{q+1} + (b-x_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} \left[\frac{(y_0-c)^{q+1} + (d-y_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} . \end{aligned}$$

Thus the theorem is proved. ∎

The best inequality in the class can be produced at $x_0 = \frac{a+b}{2}$ and $y_0 = \frac{c+d}{2}$ as in the following corollary

Corollary 2. *Taking into account the conditions and assumptions in Theorem 2, the following inequality holds*

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,dsdt - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \\ \leq \quad &\frac{\left[(b-a)^{\frac{1}{p}} \left\|D_x\left(\frac{a+b}{2},.\right)\right\|_p + \|D_x\|_p\right]}{4(d-c)^{\frac{q-1}{q}}} \left[\frac{(b-a)}{q+1}\right]^{\frac{1}{q}} \\ &+ \frac{\left[(d-c)^{\frac{1}{p}} \left\|D_y\left(.,\frac{c+d}{2}\right)\right\|_p + \|D_y\|_p\right]}{4(b-a)^{\frac{q-1}{q}}} \left[\frac{(d-c)}{q+1}\right]^{\frac{1}{q}} \\ &+ \frac{\left[G_1(\frac{a+b}{2}) + G_2(\frac{a+b}{2},\frac{d+c}{2}) + G_3(\frac{d+c}{2}) + \|D_{xy}\|_p\right]}{16} \\ &\times \left[\frac{(b-a)}{q+1}\right]^{\frac{1}{q}} \left[\frac{(d-c)}{q+1}\right]^{\frac{1}{q}} \end{aligned} \tag{3.16}$$

where

$$G_1\left(\frac{a+b}{2}\right) := (b-a)^{\frac{1}{p}} \left\| D_{xy}\left(\frac{a+b}{2},.\right) \right\|_p ,$$

$$G_2\left(\frac{a+b}{2},\frac{d+c}{2}\right) := (b-a)^{\frac{1}{p}} (d-c)^{\frac{1}{p}} \left| D_{xy}\left(\frac{a+b}{2},\frac{d+c}{2}\right) \right| ,$$

and

$$G_3(\frac{d+c}{2}) := (d-c)^{\frac{1}{p}} \left\| D_{xy}\left(.,\frac{d+c}{2}\right)\right\|_p .$$

3.3. Mappings whose First Derivative Belongs to $L_1\left[[a,b]\times[c,d]\right]$.

In this section an inequality of Ostrowski type involving two-dimensional integrals for functions whose partial derivatives belong to L_1 and satisfy certain convexity properties can be produced as shown in the following theorem.

Theorem 3. *With the assumption in Theorem 1, we have*

$$\begin{aligned}(3.17)\quad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \\ & \le \frac{[(b-a)\|D_x(x_0,.)\|_1+\|D_x\|_1]}{2(b-a)(d-c)}\left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right] \\ & + \frac{\left[(d-c)\|D_y(.,y_0)\|_1+\|D_y\|_1\right]}{2(b-a)(d-c)}\left[\frac{1}{2}(d-c)+\left|y_0-\frac{c+d}{2}\right|\right] \\ & + \frac{\left[S_1(x_0)+S_2(x_0,y_0)+S_3(y_0)+\|D_{xy}\|_1\right]}{4(b-a)(d-c)} \\ & \times\left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y_0-\frac{c+d}{2}\right|\right].\end{aligned}$$

for all $(x_0,y_0)\in[a,b]\times[c,d]$, *where*

$$S_1(x_0) := (b-a)\|D_{xy}(x_0,.)\|_1,$$

$$S_2(x_0,y_0) := (b-a)(d-c)|D_{xy}(x_0,y_0)|,$$

$$S_3(y_0) := (d-c)\|D_{xy}(.,y_0)\|_1,$$

$$\|D_x(x_0,.)\|_1 := \int_c^d |D_x(x_0,s)|ds<\infty,\quad \|D_y(.,y_0)\|_1 := \int_a^b |D_y(t,y_0)|dt<\infty,$$

$$\|D_x\|_1 := \int_a^b\int_c^d |D_x(t,s)|\,dsdt<\infty,$$

$$\|D_{xy}(x_0,.)\|_1 := \int_c^d |D_{xy}(x_0,s)|ds<\infty,\quad \|D_{xy}(.,y_0)\|_1 := \int_a^b |D_{xy}(t,y_0)|dt<\infty,$$

and

$$\|D_{xy}\|_1 := \int_a^b\int_c^d |D_x(t,s)|\,dsdt<\infty.$$

Proof. Utilizing the equations (3.2) to (3.5) we can write

$$\begin{aligned}(3.18)\quad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x[(1-\lambda)x_0+\lambda t,s]\,d\lambda\right)dsdt\right| \\ & \le \frac{[(b-a)\|D_x(x_0,.)\|_1+\|D_x\|_1]}{2(b-a)(d-c)}\sup_{t\in[a,b]}|x_0-t| \\ & \le \frac{[(b-a)\|D_x(x_0,.)\|_1+\|D_x\|_1]}{2(b-a)(d-c)}\left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right]\end{aligned}$$

since

$$\sup_{t\in[a,b]}|x_0-t|=\max\{x_0-a,b-x_0\}=\left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right]$$

where we have used the fact that

$$\max\{X,Y\}=\frac{X+Y}{2}+\left|\frac{Y-X}{2}\right|.$$

Similarly, we can deduce that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d(y_0-s)\left(\int_0^1 D_y[t,(1-\mu)y_0+\mu s]\,d\mu\right)dsdt\right| \tag{3.19}$$
$$\leq\frac{\left[(d-c)\left\|D_y(.,y_0)\right\|_1+\left\|D_y\right\|_1\right]}{2(b-a)(d-c)}\left[\frac{1}{2}(d-c)+\left|y_0-\frac{c+d}{2}\right|\right]$$

and

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d(x_0-t)(y_0-s)\right. \tag{3.20}$$
$$\left.\times\left(\int_0^1\int_0^1 D_{xy}[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s]\,d\mu d\lambda\right)dsdt\right|$$
$$\leq\frac{\left[S_1(x_0)+S_2(x_0,y_0)+S_3(y_0)+\left\|D_{xy}\right\|_1\right]}{4(b-a)(d-c)}$$
$$\times\left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y_0-\frac{c+d}{2}\right|\right].$$

Utilizing (3.18), (3.19) and (3.20), substituting into (3.3), thus the Theorem 3 is completely proved. ∎

Again the best result can be given at $x_0=\frac{a+b}{2}$ and $y_0=\frac{c+d}{2}$, as shown in the following corollary

Corollary 3. *Under the above assumptions, we have*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right| \tag{3.21}$$
$$\leq\frac{\left[(b-a)\left\|D_x\left(\frac{a+b}{2},.\right)\right\|_1+\left\|D_x\right\|_1\right]}{4(d-c)}$$
$$+\frac{\left[(d-c)\left\|D_y\left(.,\frac{c+d}{2}\right)\right\|_1+\left\|D_y\right\|_1\right]}{4(b-a)}$$
$$+\frac{\left[S_1(\frac{a+b}{2})+S_2(\frac{a+b}{2},\frac{c+d}{2})+S_3(\frac{c+d}{2})+\left\|D_{xy}\right\|_1\right]}{16}.$$

where

$$S_1(\frac{a+b}{2}):=(b-a)\left\|D_{xy}\left(\frac{a+b}{2},.\right)\right\|_1,$$
$$S_2(\frac{a+b}{2},\frac{c+d}{2}):=(b-a)(d-c)|D_{xy}\left(\frac{a+b}{2},\frac{c+d}{2}\right)|,$$

and

$$S_3(\frac{c+d}{2}) := (d-c)\left\|D_{xy}\left(.,\frac{c+d}{2}\right)\right\|_1 .$$

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