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AN OSTROWSKI TYPE INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. An Ostrowski type integral inequality for convex functions and applications for quadrature rules and integral means are given. A refinement and a counterpart result for Hermite-Hadamard inequalities are obtained and some inequalities for pdf's and (HH) -divergence measure are also mentioned.

1. INTRODUCTION

The following result is known in the literature as Ostrowski's inequality [1].

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

$$(1.2) \quad f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{b-a} \int_a^b p(x, t) f'(t) dt, \quad x \in [a, b],$$

where

$$p(x, t) := \begin{cases} t - a & \text{if } a \leq t \leq x \\ t - b & \text{if } x < t \leq b \end{cases}$$

which holds for absolutely continuous functions $f : [a, b] \rightarrow \mathbb{R}$.

The following Ostrowski type result holds (see [2], [3] and [4]).

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Theorem 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous on $[a, b]$. Then, for all $x \in [a, b]$, we have:

$$(1.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \frac{1}{(p+1)^{\frac{1}{p}}} \left[\left(\frac{x-a}{b-a} \right)^{p+1} + \left(\frac{b-x}{b-a} \right)^{p+1} \right]^{\frac{1}{p}} (b-a)^{\frac{1}{p}} \|f'\|_q & \text{if } f' \in L_q[a, b], \\ & \frac{1}{p} + \frac{1}{q} = 1, p > 1; \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \|f'\|_1 & \end{cases}$$

where $\|\cdot\|_r$ ($r \in [1, \infty)$) are the usual Lebesgue norms on $L_r[a, b]$, i.e.,

$$\|g\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r dt \right)^{\frac{1}{r}}, \quad r \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from Fink's result in [5] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [6]):

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be of r -Hölder type, i.e.,

$$(1.4) \quad |f(x) - f(y)| \leq H |x - y|^r, \quad \text{for all } x, y \in [a, b],$$

where $r \in (0, 1]$ and $H > 0$ are fixed. Then for all $x \in [a, b]$ we have the inequality:

$$(1.5) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{H}{r+1} \left[\left(\frac{b-x}{b-a} \right)^{r+1} + \left(\frac{x-a}{b-a} \right)^{r+1} \right] (b-a)^r.$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [7])

$$(1.6) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) L.$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the continuity condition of the function, and assumes that it is of bounded variation, then the following result may be stated (see [8]).

Theorem 4. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b(f)$ its total variation. Then

$$(1.7) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b(f)$$

for all $x \in [a, b]$.

The constant $\frac{1}{2}$ is the best possible.

If we assume more about f , i.e., f is monotonically increasing, then the inequality (1.7) may be improved in the following manner [9] (see also [10]).

Theorem 5. Let $f : [a, b] \rightarrow \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a, b]$, we have the inequality:

$$(1.8) \quad \begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b \operatorname{sgn}(t-x) f(t) dt \right\} \\ & \leq \frac{1}{b-a} \{ (x-a) [f(x) - f(a)] + (b-x) [f(b) - f(x)] \} \\ & \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)]. \end{aligned}$$

All the inequalities in (1.8) are sharp and the constant $\frac{1}{2}$ is the best possible.

In this paper we establish an Ostrowski type inequality for convex functions. Applications for quadrature rules, for integral means, for probability distribution functions, and for HH -divergences in Information Theory are also considered.

2. THE RESULTS

The following theorem providing a lower bound for the Ostrowski difference $\int_a^b f(t) dt - (b-a) f(x)$ holds.

Theorem 6. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Then for any $x \in (a, b)$ we have the inequality:

$$(2.1) \quad \frac{1}{2} \left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \leq \int_a^b f(t) dt - (b-a) f(x).$$

The constant $\frac{1}{2}$ in the left hand side of (2.1) is sharp in the sense that it cannot be replaced by a larger constant.

Proof. It is easy to see that for any locally absolutely continuous function $f : (a, b) \rightarrow \mathbb{R}$, we have the identity

$$(2.2) \quad \int_a^x (t-a) f'(t) dt + \int_x^b (t-b) f'(t) dt = f(x) - \int_a^b f(t) dt,$$

for any $x \in (a, b)$ where f' is the derivative of f which exists a.e. on (a, b) .

Since f is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any $x \in (a, b)$, we have the inequalities

$$(2.3) \quad f'(t) \leq f'_-(x) \text{ for a.e. } t \in [a, x]$$

and

$$(2.4) \quad f'(t) \geq f'_+(x) \text{ for a.e. } t \in [x, b].$$

If we multiply (2.3) by $t - a \geq 0$, $t \in [a, x]$, and integrate on $[a, x]$, we get

$$(2.5) \quad \int_a^x (t - a) f'(t) dt \leq \frac{1}{2} (x - a)^2 f'_-(x)$$

and if we multiply (2.4) by $b - t \geq 0$, $t \in [x, b]$, and integrate on $[x, b]$, we also have

$$(2.6) \quad \int_x^b (b - t) f'(t) dt \geq \frac{1}{2} (b - x)^2 f'_+(x).$$

Finally, if we subtract (2.6) from (2.5) and use the representation (2.2) we deduce the desired inequality (2.1).

Now, assume that (2.1) holds with a constant $C > 0$ instead of $\frac{1}{2}$, i.e.,

$$(2.7) \quad C \left[(b - x)^2 f'_+(x) - (x - a)^2 f'_-(x) \right] \leq \int_a^b f(t) dt - (b - a) f(x).$$

Consider the convex function $f_0(t) := k \left| t - \frac{a+b}{2} \right|$, $k > 0$, $t \in [a, b]$. Then

$$f'_{0+} \left(\frac{a+b}{2} \right) = k, \quad f'_{0-} \left(\frac{a+b}{2} \right) = -k, \quad f_0 \left(\frac{a+b}{2} \right) = 0$$

and

$$\int_a^b f_0(t) dt = \frac{1}{4} k (b - a)^2.$$

If in (2.7) we choose f_0 as above and $x = \frac{a+b}{2}$, then we get

$$C \left[\frac{1}{4} (b - a)^2 k + \frac{1}{4} (b - a)^2 k \right] \leq \frac{1}{4} k (b - a)^2,$$

which gives $C \leq \frac{1}{2}$, and the sharpness of the constant is proved. ■

Now, recall that the following inequality, which is well known in the literature as the *Hermite-Hadamard inequality* for convex functions, holds:

$$(HH) \quad f \left(\frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}.$$

The following corollary which improves the first Hermite-Hadamard inequality (HH) holds.

Corollary 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Then*

$$(2.8) \quad \begin{aligned} 0 &\leq \frac{1}{8} \left[f'_+ \left(\frac{a+b}{2} \right) - f'_- \left(\frac{a+b}{2} \right) \right] (b - a) \\ &\leq \frac{1}{b-a} \int_a^b f(t) dt - f \left(\frac{a+b}{2} \right). \end{aligned}$$

The constant $\frac{1}{8}$ is sharp.

The proof is obvious by the above theorem. The sharpness of the constant is obtained for $f_0(t) := k \left| t - \frac{a+b}{2} \right|$, $t \in [a, b]$, $k > 0$.

When x is a point of differentiability, we may state the following corollary as well.

Corollary 2. *Let f be as in Theorem 6. If $x \in (a, b)$ is a point of differentiability for f , then*

$$(2.9) \quad \left(\frac{a+b}{2} - x \right) f'(x) \leq \frac{1}{b-a} \int_a^b f(t) dt - f(x).$$

Remark 1. *If $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is convex on I and if we choose $x \in \overset{\circ}{I}$ ($\overset{\circ}{I}$ is the interior of I), $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ is such that $a, b \in I$, then from (2.1) we may write*

$$(2.10) \quad 0 \leq \frac{1}{8} h^2 [f'_+(x) - f'_-(x)] \leq \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t) dt - hf(x),$$

and the constant $\frac{1}{8}$ is sharp in (2.10).

The following result providing an upper bound for the Ostrowski difference $\int_a^b f(t) dt - (b-a)f(x)$ also holds.

Theorem 7. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Then for any $x \in [a, b]$, we have the inequality:*

$$(2.11) \quad \int_a^b f(t) dt - (b-a)f(x) \leq \frac{1}{2} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right].$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. If either $f'_+(a) = -\infty$ or $f'_-(b) = +\infty$, then the inequality (2.11) evidently holds true.

Assume that $f'_+(a)$ and $f'_-(b)$ are finite.

Since f is convex on $[a, b]$, we have

$$(2.12) \quad f'(t) \geq f'_+(a) \text{ for a.e. } t \in [a, x]$$

and

$$(2.13) \quad f'(t) \leq f'_-(b) \text{ for a.e. } t \in [x, b].$$

If we multiply (2.12) by $t-a \geq 0$, $t \in [a, x]$, and integrate on $[a, x]$, then we deduce

$$(2.14) \quad \int_a^x (t-a) f'(t) dt \geq \frac{1}{2} (x-a)^2 f'_+(a)$$

and if we multiply (2.13) by $b-t \geq 0$, $t \in [x, b]$, and integrate on $[x, b]$, then we also have

$$(2.15) \quad \int_x^b (b-t) f'(t) dt \leq \frac{1}{2} (b-x)^2 f'_-(b).$$

Finally, if we subtract (2.14) from (2.15) and use the representation (2.2), we deduce the desired inequality (2.11).

Now, assume that (2.11) holds with a constant $D > 0$ instead of $\frac{1}{2}$, i.e.,

$$(2.16) \quad \int_a^b f(t) dt - (b-a)f(x) \leq D \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right].$$

If we consider the convex function $f_0 : [a, b] \rightarrow \mathbb{R}$, $f_0(t) = k|t - \frac{a+b}{2}|$, then we have $f'_-(b) = k$, $f'_+(a) = -k$ and by (2.16) we deduce for $x = \frac{a+b}{2}$ that

$$\frac{1}{4}k(b-a)^2 \leq D \left[\frac{1}{4}k(b-a)^2 + \frac{1}{4}k(b-a)^2 \right],$$

giving $D \geq \frac{1}{2}$, and the sharpness of the constant is proved. ■

The following corollary related to the Hermite-Hadamard inequality is interesting as well.

Corollary 3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be convex on $[a, b]$. Then*

$$(2.17) \quad 0 \leq \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8} [f'_-(b) - f'_+(a)] (b-a)$$

and the constant $\frac{1}{8}$ is sharp.

Remark 2. Denote $B := f'_-(b)$, $A := f'_+(a)$ and assume that $B \neq A$, i.e., f is not constant on (a, b) . Then

$$\begin{aligned} & (b-x)^2 B - (x-a)^2 A \\ &= (B-A) \left[x - \left(\frac{bB-aA}{B-A} \right) \right]^2 - \frac{AB}{B-A} (b-a)^2 \end{aligned}$$

and by (2.11) we get

$$(2.18) \quad \begin{aligned} & \int_a^b f(t) dt - (b-a) f(x) \\ & \leq \frac{1}{2} (B-A) \left\{ \left[x - \left(\frac{bB-aA}{B-A} \right) \right]^2 - \frac{AB}{(B-A)^2} (b-a)^2 \right\} \end{aligned}$$

for any $x \in [a, b]$.

If $A \geq 0$ then $x_0 = \frac{bB-aA}{B-A} \in [a, b]$ and by (2.18) we get, choosing $x = \frac{bB-aA}{B-A}$, that

$$(2.19) \quad 0 \leq \frac{1}{2} \frac{AB}{B-A} (b-a) \leq f\left(\frac{bB-aA}{B-A}\right) - \frac{1}{b-a} \int_a^b f(t) dt,$$

which is an interesting inequality in itself.

Remark 3. If $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is convex on I and if we choose $x \in I$, $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ is such that $a, b \in I$, then from (2.11) we deduce:

$$(2.20) \quad 0 \leq \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t) dt - hf(x) \leq \frac{1}{8} h^2 \left[f'_-\left(x + \frac{h}{2}\right) - f'_+\left(x - \frac{h}{2}\right) \right],$$

and the constant $\frac{1}{8}$ is sharp.

3. THE COMPOSITE CASE

Consider the division $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ and denote $h_i := x_{i+1} - x_i$, $i = \overline{0, n-1}$. If $\xi_i \in [x_i, x_{i+1}]$ ($i = \overline{0, n-1}$) are intermediate points, then we will denote by

$$(3.1) \quad R_n(f; I_n, \xi) := \sum_{i=0}^{n-1} h_i f(\xi_i)$$

the Riemann sum associated to f , I_n and ξ .

The following theorem providing upper and lower bounds for the remainder in approximating the integral $\int_a^b f(t) dt$ of a convex function f in terms of a general Riemann sum holds.

Theorem 8. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function and I_n and ξ be as above. Then we have:*

$$(3.2) \quad \int_a^b f(t) dt = R_n(f; I_n, \xi) + W_n(f; I_n, \xi),$$

where $R_n(f; I_n, \xi)$ is the Riemann sum defined by (3.1) and the remainder $W_n(f; I_n, \xi)$ satisfies the estimate:

$$(3.3) \quad \begin{aligned} & \frac{1}{2} \left[\sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_+(\xi_i) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_-(\xi_i) \right] \\ & \leq W_n(f; I_n, \xi) \\ & \leq \frac{1}{2} \left[(b - \xi_{n-1})^2 f'_-(b) + \sum_{i=1}^{n-1} \left[(x_i - \xi_{i-1})^2 f'_-(x_i) \right. \right. \\ & \quad \left. \left. - (\xi_i - x_i)^2 f'_+(x_i) \right] - (\xi_0 - a)^2 f'_+(a) \right]. \end{aligned}$$

Proof. If we write the inequalities (2.1) and (2.11) on the interval $[x_i, x_{i+1}]$ and for the intermediate points $\xi_i \in [x_i, x_{i+1}]$, then we have

$$\begin{aligned} & \frac{1}{2} \left[(x_{i+1} - \xi_i)^2 f'_+(\xi_i) - (\xi_i - x_i)^2 f'_-(\xi_i) \right] \\ & \leq \int_{x_i}^{x_{i+1}} f(t) dt - h_i f(\xi_i) \\ & \leq \frac{1}{2} \left[(x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) - (\xi_i - x_i)^2 f'_+(x_i) \right]. \end{aligned}$$

Summing the above inequalities over i from 0 to $n-1$, we deduce

$$(3.4) \quad \begin{aligned} & \frac{1}{2} \sum_{i=0}^{n-1} \left[(x_{i+1} - \xi_i)^2 f'_+(\xi_i) - (\xi_i - x_i)^2 f'_-(\xi_i) \right] \\ & \leq \int_a^b f(t) dt - R_n(f; I_n, \xi) \\ & \leq \frac{1}{2} \left[\sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) \right]. \end{aligned}$$

However,

$$\begin{aligned} \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) &= (b - \xi_{n-1})^2 f'_-(b) + \sum_{i=0}^{n-2} \left[(x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) \right] \\ &= (b - \xi_{n-1})^2 f'_-(b) + \sum_{i=1}^{n-1} \left[(x_i - \xi_{i-1})^2 f'_-(x_i) \right] \end{aligned}$$

and

$$\sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) = \sum_{i=1}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) + (\xi_0 - a)^2 f'_+(a)$$

and then, by (3.4), we deduce the desired estimate (3.3). ■

The following corollary may be useful in practical applications.

Corollary 4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable convex function on (a, b) . Then we have the representation (3.2) and the remainder $W_n(f; I_n, \xi)$ satisfies the estimate:*

$$\begin{aligned} (3.5) \quad & \sum_{i=0}^{n-1} \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right) h_i f'(\xi_i) \\ & \leq W_n(f; I_n, \xi_i) \\ & \leq \frac{1}{2} \left[(b - \xi_{n-1})^2 f'_-(b) - (\xi_0 - a)^2 f'_+(a) \right] \\ & \quad + \sum_{i=1}^{n-1} \left(x_i - \frac{\xi_i + \xi_{i-1}}{2} \right) (\xi_i - \xi_{i-1}) f'(x_i). \end{aligned}$$

We may also consider the mid-point quadrature rule:

$$(3.6) \quad M_n(f, I_n) := \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right).$$

Using Corollaries 1 and 2, we may state the following result as well.

Corollary 5. *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is a convex function on $[a, b]$ and I_n is a division as above. Then we have the representation:*

$$(3.7) \quad \int_a^b f(x) dx = M_n(f, I_n) + S_n(f, I_n),$$

where $M_n(f, I_n)$ is the mid-point quadrature rule given in (3.6) and the remainder $S_n(f, I_n)$ satisfies the estimates:

$$\begin{aligned} (3.8) \quad 0 & \leq \frac{1}{8} \sum_{i=0}^{n-1} \left[f'_+\left(\frac{x_i + x_{i+1}}{2}\right) - f'_-\left(\frac{x_i + x_{i+1}}{2}\right) \right] h_i^2 \\ & \leq S_n(f, I_n) \leq \frac{1}{8} \sum_{i=0}^{n-1} [f'_-(x_{i+1}) - f'_+(x_i)] h_i^2. \end{aligned}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

4. INEQUALITIES FOR INTEGRAL MEANS

We may prove the following result in comparing two integral means.

Theorem 9. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function and $c, d \in [a, b]$ with $c < d$. Then we have the inequalities

$$\begin{aligned}
 (4.1) \quad & \frac{a+b}{2} \cdot \frac{f(d) - f(c)}{d-c} - \frac{df(d) - cf(c)}{d-c} + \frac{1}{d-c} \int_c^d f(x) dx \\
 & \leq \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(x) dx \\
 & \leq \frac{f'_-(b) \left[(b-d)^2 + (b-d)(b-c) + (b-c)^2 \right]}{6(b-a)} \\
 & \quad - \frac{f'_+(a) \left[(d-a)^2 + (d-a)(c-a) + (c-a)^2 \right]}{6(b-a)}.
 \end{aligned}$$

Proof. Since f is convex, then for a.e. $x \in [a, b]$, we have (by (2.9)) that

$$(4.2) \quad \left(\frac{a+b}{2} - x \right) f'(x) \leq \frac{1}{b-a} \int_a^b f(t) dt - f(x).$$

Integrating (5.2) on $[c, d]$ we deduce

$$(4.3) \quad \frac{1}{d-c} \int_c^d \left(\frac{a+b}{2} - x \right) f'(x) dx \leq \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(x) dx.$$

Since

$$\begin{aligned}
 & \frac{1}{d-c} \int_c^d \left(\frac{a+b}{2} - x \right) f'(x) dx \\
 & = \frac{1}{d-c} \left[\left(\frac{a+b}{2} - d \right) f(d) - \left(\frac{a+b}{2} - c \right) f(c) + \int_c^d f(x) dx \right]
 \end{aligned}$$

then by (4.3) we deduce the first part of (4.1).

Using (2.11), we may write for any $x \in [a, b]$ that

$$(4.4) \quad \frac{1}{b-a} \int_a^b f(t) dt - f(x) \leq \frac{1}{2(b-a)} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right].$$

Integrating (4.4) on $[c, d]$, we deduce

$$\begin{aligned}
 (4.5) \quad & \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(x) dx \\
 & \leq \frac{1}{2(b-a)} \left[f'_-(b) \frac{1}{d-c} \int_c^d (b-x)^2 dx - f'_+(a) \frac{1}{d-c} \int_c^d (x-a)^2 dx \right].
 \end{aligned}$$

Since

$$\frac{1}{d-c} \int_c^d (b-x)^2 dx = \frac{(b-d)^2 + (b-d)(b-c) + (b-c)^2}{3}$$

and

$$\frac{1}{d-c} \int_c^d (x-a)^2 dx = \frac{(d-a)^2 + (d-a)(c-a) + (c-a)^2}{3},$$

then by (4.5) we deduce the second part of (4.1). ■

Remark 4. If we choose $f(x) = x^p$, $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ or $f(x) = \frac{1}{x}$ or even $f(x) = -\ln x$, $x \in [a, b] \subset (0, \infty)$, in the above inequalities, then a great number of interesting results for p -logarithmic, logarithmic and identric means may be obtained. We leave this as an exercise to the interested reader.

5. APPLICATIONS FOR P.D.F.S

Let X be a random variable with the *probability density function* $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$ and with *cumulative distribution function* $F(x) = \Pr(X \leq x)$.

The following theorem holds.

Theorem 10. If $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$ is monotonically increasing on $[a, b]$, then we have the inequality:

$$\begin{aligned} (5.1) \quad & \frac{1}{2} \left[(b-x)^2 f_+(x) - (x-a)^2 f_-(x) \right] \\ & \leq b - E(X) - (b-a) F(x) \\ & \leq \frac{1}{2} \left[(b-x)^2 f_-(b) - (x-a)^2 f_+(a) \right] \end{aligned}$$

for any $x \in (a, b)$, where $f_-(\alpha)$ means the left limit in α while $f_+(\alpha)$ means the right limit in α and $E(X)$ is the expectation of X .

The constant $\frac{1}{2}$ is sharp in both inequalities.

The second inequality also holds for $x = a$ or $x = b$.

Proof. Follows by Theorem 6 and 7 applied for the convex cdf function $F(x) = \int_a^x f(t) dt$, $x \in [a, b]$ and taking into account that

$$\int_a^b F(x) dx = b - E(X).$$

■

Finally, we may state the following corollary in estimating the probability $\Pr(X \leq \frac{a+b}{2})$.

Corollary 6. With the above assumptions, we have

$$\begin{aligned} (5.2) \quad & b - E(X) - \frac{1}{8} (b-a)^2 [f_-(b) - f_+(a)] \\ & \leq \Pr\left(X \leq \frac{a+b}{2}\right) \\ & \leq b - E(X) - \frac{1}{8} (b-a)^2 \left[f_+\left(\frac{a+b}{2}\right) - f_-\left(\frac{a+b}{2}\right) \right]. \end{aligned}$$

6. APPLICATIONS FOR HH -DIVERGENCE

Assume that a set χ and the σ -finite measure μ are given. Consider the set of all probability densities on μ to be

$$(6.1) \quad \Omega := \left\{ p | p : \Omega \rightarrow \mathbb{R}, p(x) \geq 0, \int_{\chi} p(x) d\mu(x) = 1 \right\}.$$

Csiszár's f -divergence is defined as follows [11]

$$(6.2) \quad D_f(p, q) := \int_{\chi} p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \quad p, q \in \Omega,$$

where f is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived.

In [12], Shioya and Da-te introduced the generalised Lin-Wong f -divergence $D_f(p, \frac{1}{2}p + \frac{1}{2}q)$ and the Hermite-Hadamard (HH) divergence

$$(6.3) \quad D_{HH}^f(p, q) := \int_{\chi} \frac{p^2(x)}{q(x) - p(x)} \left(\int_1^{\frac{q(x)}{p(x)}} f(t) dt \right) d\mu(x), \quad p, q \in \Omega,$$

and, by the use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality

$$(6.4) \quad D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right) \leq D_{HH}^f(p, q) \leq \frac{1}{2}D_f(p, q),$$

provided that f is convex and normalised, i.e., $f(1) = 0$.

The following result in estimating the difference

$$D_{HH}^f(p, q) - D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right)$$

holds.

Theorem 11. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a convex function and $p, q \in \Omega$. Then we have the inequality:*

$$(6.5) \quad \begin{aligned} 0 &\leq \frac{1}{8} \left[D_{f'_+ \cdot |\cdot|^{\frac{+1}{2}}} (p, q) - D_{f'_- \cdot |\cdot|^{\frac{+1}{2}}} (p, q) \right] \\ &\leq D_{HH}^f(p, q) - D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right) \\ &\leq \frac{1}{8} D_{f'_- \cdot (-1)} (p, q). \end{aligned}$$

Proof. Using the double inequality

$$\begin{aligned} 0 &\leq \frac{1}{8} \left[f'_+ \left(\frac{a+b}{2} \right) - f'_- \left(\frac{a+b}{2} \right) \right] |b-a| \\ &\leq \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{8} [f_-(b) - f'_+(a)] (b-a) \end{aligned}$$

for the choices $a = 1$, $b = \frac{q(x)}{p(x)}$, $x \in \chi$, multiplying with $p(x) \geq 0$ and integrating over x on χ we get

$$\begin{aligned} 0 &\leq \frac{1}{8} \int_{\chi} \left[f'_+ \left(\frac{p(x) + q(x)}{2p(x)} \right) - f'_- \left(\frac{p(x) + q(x)}{2p(x)} \right) \right] |q(x) - p(x)| d\mu(x) \\ &\leq D_{HH}^f(p, q) - D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right) \\ &\leq \frac{1}{8} \int_{\chi} \left[f'_- \left(\frac{q(x)}{p(x)} \right) - f'_+(1) \right] (q(x) - p(x)) d\mu(x), \end{aligned}$$

which is clearly equivalent to (6.5). ■

Corollary 7. *With the above assumptions and if f is differentiable on $(0, \infty)$, then*

$$(6.6) \quad 0 \leq D_{HH}^f(p, q) - D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right) \leq \frac{1}{8}D_{f' \cdot (-1)}(p, q).$$

REFERENCES

- [1] A. OSTROWSKI, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 226-227.
- [2] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_1 -norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.
- [3] S.S. DRAGOMIR and S. WANG, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.
- [4] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_p -norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.*, **40**(3) (1998), 245-304.
- [5] A.M. FINK, Bounds on the deviation of a function from its averages, *Czech. Math. J.*, **42**(117) (1992), 289-310.
- [6] S.S. DRAGOMIR, P. CERONE, J. ROUMELIOTIS and S. WANG, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Roumanie*, **42**(90)(4) (1992), 301-314.
- [7] S.S. DRAGOMIR, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. and Math. with Appl.*, **38** (1999), 33-37.
- [8] S.S. DRAGOMIR, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.*, **4**(1) (2001), 33-40.
- [9] S.S. DRAGOMIR, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM*, **3**(1) (1999), 127-135.
- [10] P. CERONE and S.S. DRAGOMIR, Midpoint type rules from an inequalities point of view, in *Analytic-Computational Methods in Applied Mathematics*, G.A. Anastassiou (Ed), CRC Press, New York, 2000, 135-200.
- [11] I. CSISZÁR, Information-type measures of difference of probability distributions and indirect observations, *Studia Math. Hungarica*, **2** (1967), 299-318.
- [12] H. SHIOYA and T. DA-TE, A generalisation of Lin divergence and the derivation of a new information divergence, *Elec. and Comm. in Japan*, **78**(7), (1995), 37-40.

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