An Ostrowski Type Inequality for Convex Functions

This is the Published version of the following publication


The publisher's official version can be found at

Note that access to this version may require subscription.

Downloaded from VU Research Repository  https://vuir.vu.edu.au/17689/
AN OSTROWSKI TYPE INEQUALITY FOR CONVEX
FUNCTIONS

S.S. DRAGOMIR

Abstract. An Ostrowski type integral inequality for convex functions and
applications for quadrature rules and integral means are given. A refinement
and a counterpart result for Hermite-Hadamard inequalities are obtained and
some inequalities for pdf’s and (HH) – divergence measure are also mentioned.

1. Introduction

The following result is known in the literature as Ostrowski’s inequality [1].

Theorem 1. Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with the
property that \( |f'(t)| \leq M \) for all \( t \in (a, b) \). Then

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) M
\]

for all \( x \in [a, b] \).

The constant \( \frac{1}{4} \) is the best possible in the sense that it cannot be replaced by a
smaller constant.

A simple proof of this fact can be done by using the identity:

\[
f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + \frac{1}{b-a} \int_a^b p(x, t) f'(t) \, dt, \quad x \in [a, b],
\]

where

\[
p(x, t) := \begin{cases} 
  t - a & \text{if } a \leq t \leq x \\
  t - b & \text{if } x < t \leq b
\end{cases}
\]

which holds for absolutely continuous functions \( f : [a, b] \to \mathbb{R} \).

The following Ostrowski type result holds (see [2], [3] and [4]).
Theorem 2. Let \( f : [a, b] \to \mathbb{R} \) be absolutely continuous on \([a, b]\). Then, for all \( x \in [a, b] \), we have:

\[
|f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt| \leq \begin{cases} 
\frac{1}{4} + \left( \frac{x - a}{b - a} \right)^2 (b - a) \|f'\|_\infty & \text{if } f' \in L_\infty [a, b]; \\
\frac{1}{(p+1)^{\frac{1}{p}}} \left( \left( \frac{x - a}{b - a} \right)^{p+1} + \left( \frac{b - x}{b - a} \right)^{p+1} \right)^{\frac{1}{p}} (b - a)^{\frac{1}{p}} \|f'\|_q & \text{if } f' \in L_q [a, b], \quad \frac{1}{p} + \frac{1}{q} = 1, p > 1; \\
\left[ \frac{1}{2} + \left| \frac{x - a}{b - a} \right| \right] \|f'\|_1 & \end{cases}
\]

where \( \|\cdot\|_r \ (r \in [1, \infty]) \) are the usual Lebesgue norms on \( L_r [a, b] \), i.e.,

\[\|g\|_\infty := \text{ess sup}_{t \in [a,b]} |g(t)|\]

and

\[\|g\|_r := \left( \int_a^b |g(t)|^r \, dt \right)^{\frac{1}{r}}, \ r \in [1, \infty).\]

The constants \( \frac{1}{4} \), \( \frac{1}{(p+1)^{\frac{1}{p}}} \) and \( \frac{1}{2} \) respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from Fink’s result in [5] on choosing \( n = 1 \) and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that \( f \) is Hölder continuous, then one may state the result (see [6]):

Theorem 3. Let \( f : [a, b] \to \mathbb{R} \) be of \( r \)-Hölder type, i.e.,

\[
|f(x) - f(y)| \leq H |x - y|^r, \ \text{for all} \ x, y \in [a, b],
\]

where \( r \in (0, 1] \) and \( H > 0 \) are fixed. Then for all \( x \in [a, b] \) we have the inequality:

\[
|f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt| \leq \frac{H}{r+1} \left[ \left( \frac{b - x}{b - a} \right)^{r+1} + \left( \frac{x - a}{b - a} \right)^{r+1} \right] (b - a)^r.
\]

The constant \( \frac{1}{r+1} \) is also sharp in the above sense.

Note that if \( r = 1 \), i.e., \( f \) is Lipschitz continuous, then we get the following version of Ostrowski’s inequality for Lipschitzian functions (with \( L \) instead of \( H \)) (see [7])

\[
|f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt| \leq \frac{1}{4} + \left( \frac{x - a+b}{b - a} \right)^2 (b - a) \ L.
\]

Here the constant \( \frac{1}{4} \) is also best.

Moreover, if one drops the continuity condition of the function, and assumes that it is of bounded variation, then the following result may be stated (see [8]).
**Theorem 4.** Assume that \( f : [a, b] \to \mathbb{R} \) is of bounded variation and denote by \( \int_a^b (f) \) its total variation. Then

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{2} + \frac{|x-a|}{b-a} \right] \int_a^b (f)
\]

for all \( x \in [a, b] \).

The constant \( \frac{1}{2} \) is the best possible.

If we assume more about \( f \), i.e., \( f \) is monotonically increasing, then the inequality (1.7) may be improved in the following manner [9] (see also [10]).

**Theorem 5.** Let \( f : [a, b] \to \mathbb{R} \) be monotonic nondecreasing. Then for all \( x \in [a, b] \), we have the inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{b-a} \left\{ |2x - (a+b)| f(x) + \int_a^b \text{sgn}(t-x) f(t) \, dt \right\} \leq \frac{1}{b-a} \left\{ (x-a) \left[ f(x) - f(a) \right] + (b-x) \left[ f(b) - f(x) \right] \right\} \leq \frac{1}{2} \left[ \frac{x-a}{b-a} \right] \left[ f(b) - f(a) \right].
\]

All the inequalities in (1.8) are sharp and the constant \( \frac{1}{2} \) is the best possible.

In this paper we establish an Ostrowski type inequality for convex functions. Applications for quadrature rules, for integral means, for probability distribution functions, and for \( HH \)–divergences in Information Theory are also considered.

2. The Results

The following theorem providing a lower bound for the Ostrowski difference \( \int_a^b f(t) \, dt - (b-a) f(x) \) holds.

**Theorem 6.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\). Then for any \( x \in (a, b) \) we have the inequality:

\[
\frac{1}{2} \left[ (b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \leq \int_a^b f(t) \, dt - (b-a) f(x).
\]

The constant \( \frac{1}{2} \) in the left hand side of (2.1) is sharp in the sense that it cannot be replaced by a larger constant.

**Proof.** It is easy to see that for any locally absolutely continuous function \( f : (a, b) \to \mathbb{R} \), we have the identity

\[
\int_a^x (t-a) f'(t) \, dt + \int_x^b (t-b) f'(t) \, dt = f(x) - \int_a^b f(t) \, dt,
\]

for any \( x \in (a, b) \) where \( f' \) is the derivative of \( f \) which exists a.e. on \((a, b)\).
Since \( f \) is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any \( x \in (a,b) \), we have the inequalities

\[
(2.3) \quad f'(t) \leq f'_-(x) \text{ for a.e. } t \in [a,x]
\]

and

\[
(2.4) \quad f'(t) \geq f'_+(x) \text{ for a.e. } t \in [x,b].
\]

If we multiply (2.3) by \( t - a \geq 0 \), \( t \in [a,x] \), and integrate on \([a,x]\), we get

\[
(2.5) \quad \int_a^x (t - a) f'(t) \, dt \leq \frac{1}{2} (x - a)^2 f'_-(x)
\]

and if we multiply (2.4) by \( b - t \geq 0 \), \( t \in [x,b] \), and integrate on \([x,b]\), we also have

\[
(2.6) \quad \int_x^b (b - t) f'(t) \, dt \geq \frac{1}{2} (b - x)^2 f'_+(x).
\]

Finally, if we subtract (2.6) from (2.5) and use the representation (2.2) we deduce the desired inequality (2.1).

Now, assume that (2.1) holds with a constant \( C > 0 \) instead of \( \frac{1}{2} \), i.e.,

\[
(2.7) \quad C \left[ (b - x)^2 f'_+(x) - (x - a)^2 f'_-(x) \right] \leq \int_a^b f(t) \, dt - (b - a) f(x).
\]

Consider the convex function \( f_0(t) := k \left| t - \frac{a+b}{2} \right|, \, k > 0, \, t \in [a,b] \). Then

\[
f'_0+ \left( \frac{a+b}{2} \right) = k, \quad f'_0- \left( \frac{a+b}{2} \right) = -k, \quad f_0 \left( \frac{a+b}{2} \right) = 0
\]

and

\[
\int_a^b f_0(t) \, dt = \frac{1}{4} k (b - a)^2.
\]

If in (2.7) we choose \( f_0 \) as above and \( x = \frac{a+b}{2} \), then we get

\[
C \left[ \frac{1}{4} (b - a)^2 k + \frac{1}{4} (b - a)^2 k \right] \leq \frac{1}{4} k (b - a)^2,
\]

which gives \( C \leq \frac{1}{2} \), and the sharpness of the constant is proved.

Now, recall that the following inequality, which is well known in the literature as the **Hermite-Hadamard inequality** for convex functions, holds:

\[
(fH) \quad f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2}.
\]

The following corollary which improves the first Hermite-Hadamard inequality (HH) holds.

**Corollary 1.** Let \( f : [a,b] \to \mathbb{R} \) be a convex function on \([a,b]\). Then

\[
(2.8) \quad 0 \leq \frac{1}{8} \left[ f'_+ \left( \frac{a+b}{2} \right) - f'_- \left( \frac{a+b}{2} \right) \right] (b-a)
\]

\[
\leq \frac{1}{b-a} \int_a^b f(t) \, dt - f \left( \frac{a+b}{2} \right).
\]

The constant \( \frac{1}{8} \) is sharp.
Finally, if we subtract (2.14) from (2.15) and use the representation (2.2), we deduce

**Proof.** If either

\[ f(x) \in (a,b) \]

and the constant \( \frac{1}{2} \) is sharp in (2.10).

The following result providing an upper bound for the Ostrowski difference

\[ \int_a^b f(t) \, dt - (b-a) f(x) \]

also holds.

**Theorem 7.** Let \( f : [a,b] \to \mathbb{R} \) be a convex function on \([a,b]\). Then for any \( x \in [a,b] \), we have the inequality:

\[ \int_a^b f(t) \, dt - (b-a) f(x) \leq \frac{1}{2} \left[ (b-x)^2 f'_+ (b) - (x-a)^2 f'_+ (a) \right]. \tag{2.11} \]

The constant \( \frac{1}{2} \) is sharp in the sense that it cannot be replaced by a smaller constant.

**Proof.** If either \( f'_+ (a) = -\infty \) or \( f'_- (b) = +\infty \), then the inequality (2.11) evidently holds true.

Assume that \( f'_+ (a) \) and \( f'_- (b) \) are finite.

Since \( f \) is convex on \([a,b]\), we have

\[ f'(t) \geq f'_+ (a) \text{ for a.e. } t \in [a,x] \tag{2.12} \]

and

\[ f'(t) \leq f'_- (b) \text{ for a.e. } t \in [x,b]. \tag{2.13} \]

If we multiply (2.12) by \( t-a \geq 0, t \in [a,x] \), and integrate on \([a,x]\), then we deduce

\[ \int_a^x (t-a) f'(t) \, dt \geq \frac{1}{2} (x-a)^2 f'_+ (a) \tag{2.14} \]

and if we multiply (2.13) by \( b-t \geq 0, t \in [x,b] \), and integrate on \([x,b]\), then we also have

\[ \int_x^b (b-t) f'(t) \, dt \leq \frac{1}{2} (b-x)^2 f'_- (b) \tag{2.15} \]

Finally, if we subtract (2.14) from (2.15) and use the representation (2.2), we deduce the desired inequality (2.11).

Now, assume that (2.11) holds with a constant \( D > 0 \) instead of \( \frac{1}{2} \), i.e.,

\[ \int_a^b f(t) \, dt - (b-a) f(x) \leq D \left[ (b-x)^2 f'_- (b) - (x-a)^2 f'_+ (a) \right]. \tag{2.16} \]
If we consider the convex function $f_0 : [a, b] \to \mathbb{R}$, $f_0(t) = k |t - \frac{a+b}{2}|$, then we have $f'_-(b) = k$, $f'_+(a) = -k$ and by (2.16) we deduce for $x = \frac{a+b}{2}$ that
\[
\frac{1}{4}k(b-a)^2 \leq D \left[ \frac{1}{4}k(b-a)^2 + \frac{1}{4}k(b-a)^2 \right],
\]
giving $D \geq \frac{1}{2}$, and the sharpness of the constant is proved. \(\Box\)

The following corollary related to the Hermite-Hadamard inequality is interesting as well.

**Corollary 3.** Let $f : [a, b] \to \mathbb{R}$ be convex on $[a, b]$. Then
\[
0 \leq \frac{1}{b-a} \int_a^b f(t) \, dt - f \left( \frac{b+a}{2} \right) \leq \frac{1}{8} \left[ f'_-(b) - f'_+(a) \right] (b-a)
\]
and the constant $\frac{1}{8}$ is sharp.

**Remark 2.** Denote $B := f'_-(b)$, $A := f'_+(a)$ and assume that $B \neq A$, i.e., $f$ is not constant on $(a, b)$. Then
\[
(b-x)^2 B - (x-a)^2 A = (B-A) \left[ x - \left( \frac{bB-aA}{B-A} \right) \right]^2 - \frac{AB}{B-A} (b-a)^2
\]
and by (2.17) we get
\[
\int_a^b f(t) \, dt - (b-a) f(x) \leq \frac{1}{2} (B-A) \left\{ \left[ x - \left( \frac{bB-aA}{B-A} \right) \right]^2 - \frac{AB}{(B-A)^2} (b-a)^2 \right\}
\]
for any $x \in [a, b]$.

If $A \geq 0$ then $x_0 = \frac{bB-aA}{B-A} \in [a,b]$ and by (2.18) we get, choosing $x = \frac{bB-aA}{B-A}$, that
\[
0 \leq \frac{1}{2} \frac{AB}{B-A} (b-a) \leq f \left( \frac{bB-aA}{B-A} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt,
\]
which is an interesting inequality in itself.

**Remark 3.** If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on $I$ and if we choose $x \in I$, $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ is such that $a, b \in I$, then from (2.11) we deduce:
\[
0 \leq \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t) \, dt - hf(x) \leq \frac{1}{8} h^2 \left[ f'_-(x + \frac{h}{2}) - f'_+(x - \frac{h}{2}) \right],
\]
and the constant $\frac{1}{8}$ is sharp.

### 3. The Composite Case

Consider the division $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and denote $h_i := x_{i+1} - x_i$, $i = 0, n-1$. If $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, n-1$) are intermediate points, then we will denote by
\[
R_n (f; I_n, \xi) := \sum_{i=0}^{n-1} h_i f(\xi_i)
\]
satisfies the estimate:

approximating the integral

where

Then we have:

Theorem 8. Let \( f : [a, b] \to \mathbb{R} \) be a convex function and \( I_n \) and \( \xi \) be as above. Then we have:

\[
\int_a^b f(t) \, dt = R_n(f; I_n, \xi) + W_n(f; I_n, \xi),
\]

where \( R_n(f; I_n, \xi) \) is the Riemann sum defined by (3.1) and the remainder \( W_n(f; I_n, \xi) \) satisfies the estimate:

\[
\frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_+(\xi_i) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_-(\xi_i)
\]

\[
\leq W_n(f; I_n, \xi)
\]

\[
\leq \frac{1}{2} \left[ (b - \xi_{n-1})^2 f'_-(b) + \sum_{i=1}^{n-1} \left( (x_i - \xi_{i-1})^2 f'_-(x_i) - (\xi_i - x_i)^2 f'_+(x_i) \right) \right].
\]

Proof. If we write the inequalities (2.1) and (2.11) on the interval \([x_i, x_{i+1}]\), then we have

\[
\frac{1}{2} \left[ (x_{i+1} - \xi_i)^2 f'_+(\xi_i) - (\xi_i - x_i)^2 f'_-(\xi_i) \right] \leq \int_{x_i}^{x_{i+1}} f(t) \, dt - h_i f(\xi_i)
\]

\[
\leq \frac{1}{2} \left[ (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) - (\xi_i - x_i)^2 f'_+(x_i) \right].
\]

Summing the above inequalities over \( i \) from 0 to \( n - 1 \), we deduce

\[
\frac{1}{2} \sum_{i=0}^{n-1} \left[ (x_{i+1} - \xi_i)^2 f'_+(\xi_i) - (\xi_i - x_i)^2 f'_-(\xi_i) \right] \leq \int_a^b f(t) \, dt - R_n(f; I_n, \xi)
\]

\[
\leq \frac{1}{2} \left[ \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) \right].
\]

However,

\[
\sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) = (b - \xi_{n-1})^2 f'_-(b) + \sum_{i=0}^{n-2} \left[ (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) \right]
\]

\[
= (b - \xi_{n-1})^2 f'_-(b) + \sum_{i=1}^{n-1} \left[ (x_i - \xi_{i-1})^2 f'_-(x_i) \right]
\]
and
\[ \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+ (x_i) = \sum_{i=1}^{n-1} (\xi_i - x_i)^2 f'_+ (x_i) + (\xi_0 - a)^2 f'_+ (a) \]
and then, by (3.4), we deduce the desired estimate (3.3).

The following corollary may be useful in practical applications.

**Corollary 4.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable convex function on \((a, b)\). Then we have the representation (3.2) and the remainder \( W_n (f; I_n, \xi) \) satisfies the estimate:

\[ \sum_{i=0}^{n-1} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right) h_i f' (\xi_i) \leq W_n (f; I_n, \xi) \leq \frac{1}{2} \left[ (b - \xi_{n-1})^2 f'_- (b) - (\xi_0 - a)^2 f'_+ (a) \right] \]
\[ + \sum_{i=1}^{n-1} \left( x_i - \frac{\xi_i + \xi_{i-1}}{2} \right) (\xi_i - \xi_{i-1}) f' (x_i). \]

We may also consider the mid-point quadrature rule:

\[ M_n (f, I_n) := \sum_{i=0}^{n-1} h_i f \left( \frac{x_i + x_{i+1}}{2} \right). \]

Using Corollaries 1 and 2, we may state the following result as well.

**Corollary 5.** Assume that \( f : [a, b] \to \mathbb{R} \) is a convex function on \([a, b]\) and \( I_n \) is a division as above. Then we have the representation:

\[ \int_a^b f (x) \, dx = M_n (f, I_n) + S_n (f, I_n), \]

where \( M_n (f, I_n) \) is the mid-point quadrature rule given in (3.6) and the remainder \( S_n (f, I_n) \) satisfies the estimates:

\[ 0 \leq \frac{1}{8} \sum_{i=0}^{n-1} \left[ f'_+ \left( \frac{x_i + x_{i+1}}{2} \right) - f'_- \left( \frac{x_i + x_{i+1}}{2} \right) \right] h_i^2 \]
\[ \leq S_n (f, I_n) \leq \frac{1}{8} \sum_{i=0}^{n-1} \left[ f'_- (x_{i+1}) - f'_+ (x_i) \right] h_i^2. \]

The constant \( \frac{1}{8} \) is sharp in both inequalities.

4. Inequalities for Integral Means

We may prove the following result in comparing two integral means.
Theorem 9. Let \( f : [a, b] \to \mathbb{R} \) be a convex function and \( c, d \in [a, b] \) with \( c < d \). Then we have the inequalities

\[
\frac{a+b}{2} \cdot \frac{f(d) - f(c)}{d-c} - \frac{df(d) - cf(c)}{d-c} + \frac{1}{d-c} \int_c^d f(x) \, dx \leq \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{d-c} \int_c^d f(x) \, dx
\]

\[
\leq \frac{f'_-(b) \left( (d-a)^2 + (b-d)(b-c) + (b-c)^2 \right)}{6(b-a)} - \frac{f'_+(a) \left( (d-a)^2 + (d-a)(c-a) + (c-a)^2 \right)}{6(b-a)}.
\]

Proof. Since \( f \) is convex, then for a.e. \( x \in [a, b] \), we have (by (2.9)) that

\[
\left( \frac{a+b}{2} - x \right) f'(x) \leq \frac{1}{b-a} \int_a^b f(t) \, dt - f(x).
\]

Integrating (5.2) on \([c, d]\) we deduce

\[
\frac{1}{d-c} \int_c^d \left( \frac{a+b}{2} - x \right) f'(x) \, dx \leq \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{d-c} \int_c^d f(x) \, dx.
\]

Since

\[
\frac{1}{d-c} \int_c^d \left( \frac{a+b}{2} - x \right) f'(x) \, dx = \frac{1}{d-c} \left( \left( \frac{a+b}{2} - d \right) f(d) - \left( \frac{a+b}{2} - c \right) f(c) + \int_c^d f(x) \, dx \right)
\]

then by (4.3) we deduce the first part of (4.1).

Using (2.11), we may write for any \( x \in [a, b] \) that

\[
\frac{1}{b-a} \int_a^b f(t) \, dt - f(x) \leq \frac{1}{2(b-a)} \left( (b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right).
\]

Integrating (4.4) on \([c, d]\), we deduce

\[
\frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{d-c} \int_c^d f(x) \, dx
\]

\[
\leq \frac{1}{2(b-a)} \left[ f'_-(b) \frac{1}{d-c} \int_c^d (b-x)^2 \, dx - f'_+(a) \frac{1}{d-c} \int_c^d (x-a)^2 \, dx \right].
\]

Since

\[
\frac{1}{d-c} \int_c^d (b-x)^2 \, dx = \frac{(b-d)^2 + (b-d)(b-c) + (b-c)^2}{3}
\]

and

\[
\frac{1}{d-c} \int_c^d (x-a)^2 \, dx = \frac{(d-a)^2 + (d-a)(c-a) + (c-a)^2}{3},
\]

then by (4.5) we deduce the second part of (4.1).
Remark 4. If we choose $f(x) = x^p$, $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ or $f(x) = \frac{1}{x}$ or even $f(x) = -\ln x$, $x \in [a, b] \subset (0, \infty)$, in the above inequalities, then a great number of interesting results for $p$-logarithmic, logarithmic and identric means may be obtained. We leave this as an exercise to the interested reader.

5. Applications for P.D.F.s

Let $X$ be a random variable with the probability density function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}^+$ and with cumulative distribution function $F(x) = \Pr(X \leq x)$.

The following theorem holds.

Theorem 10. If $f : [a, b] \subset \mathbb{R} \to \mathbb{R}^+$ is monotonically increasing on $[a, b]$, then we have the inequality:
\[
\frac{1}{2} \left[ (b-x)^2 f_+(x) - (x-a)^2 f_-(x) \right] \leq b - E(X) - (b-a) F(x) \leq \frac{1}{2} \left[ (b-x)^2 f_-(b) - (x-a)^2 f_+(a) \right]
\]
for any $x \in (a, b)$, where $f_-(\alpha)$ means the left limit in $\alpha$ while $f_+(\alpha)$ means the right limit in $\alpha$ and $E(X)$ is the expectation of $X$.

The constant $\frac{1}{2}$ is sharp in both inequalities.

The second inequality also holds for $x = a$ or $x = b$.

Proof. Follows by Theorem 6 and 7 applied for the convex cdf function $F(x) = \int_a^x f(t) \, dt$, $x \in [a, b]$ and taking into account that
\[
\int_a^b F(x) \, dx = b - E(X).
\]

Finally, we may state the following corollary in estimating the probability $\Pr\left(X \leq \frac{a+b}{2}\right)$.

Corollary 6. With the above assumptions, we have
\[
\begin{align*}
&b - E(X) - \frac{1}{8} (b-a)^2 [f_-(b) - f_+(a)] \\
&\quad \leq \Pr\left(X \leq \frac{a+b}{2}\right) \\
&\quad \leq b - E(X) - \frac{1}{8} (b-a)^2 \left[ f_+\left(\frac{a+b}{2}\right) - f_-\left(\frac{a+b}{2}\right) \right].
\end{align*}
\]

6. Applications for HH-Divergence

Assume that a set $\chi$ and the $\sigma$-finite measure $\mu$ are given. Consider the set of all probability densities on $\mu$ to be
\[
\Omega := \left\{ p | p : \Omega \to \mathbb{R}, p(x) \geq 0, \int_{\chi} p(x) \, d\mu(x) = 1 \right\}.
\]

Csiszár’s $f$-divergence is defined as follows [11]
\[
D_f(p, q) := \int_{\chi} p(x) f\left[ \frac{q(x)}{p(x)} \right] \, d\mu(x), \; p, q \in \Omega,
\]
where $f$ is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived.

In [12], Shioya and Da-te introduced the generalised Lin-Wong $f$–divergence $D_f(p, \frac{1}{2}p + \frac{1}{2}q)$ and the Hermite-Hadamard (HH) divergence

$$D_{HH}^f(p, q) := \int_{\chi} \frac{p^2(x)}{q(x) - p(x)} \left( \int_1^{q(x)/p(x)} f(t) \, dt \right) \, d\mu(x), \quad p, q \in \Omega,$$

and, by the use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality

$$D_{HH}^f(p, q) \leq D_f(p, \frac{1}{2}p + \frac{1}{2}q) \leq \frac{1}{2} D_f(p, q),$$

provided that $f$ is convex and normalised, i.e., $f(1) = 0$.

The following result in estimating the difference

$$D_{HH}^f(p, q) - D_f(p, \frac{1}{2}p + \frac{1}{2}q)$$

holds.

**Theorem 11.** Let $f : [0, \infty) \to \mathbb{R}$ be a convex function and $p, q \in \Omega$. Then we have the inequality:

$$0 \leq \frac{1}{8} \left[ f'_+ \left( \frac{a + b}{2} \right) - f'_- \left( \frac{a + b}{2} \right) \right] |b - a|$$

$$\leq \frac{1}{8} \left[ f'_+ \left( \frac{q(x) + \frac{q(x)}{p(x)}}{2p(x)} \right) - f'_- \left( \frac{p(x) + \frac{q(x)}{p(x)}}{2p(x)} \right) \right] |q(x) - p(x)| \, d\mu(x)$$

$$\leq \frac{1}{8} \left[ f'_+ \left( \frac{q(x)}{p(x)} \right) - f'_+ (1) \right] \langle q(x) - p(x) \rangle \, d\mu(x),$$

which is clearly equivalent to (6.5).

Proof. Using the double inequality

$$0 \leq \frac{1}{8} \left[ f'_+ \left( \frac{a + b}{2} \right) - f'_- \left( \frac{a + b}{2} \right) \right] |b - a|$$

$$\leq \frac{1}{8} \left[ f'_+ \left( \frac{a+b}{2} \right) - f'_- \left( \frac{a+b}{2} \right) \right] \left| \int_a^b f(t) \, dt - f \left( \frac{a+b}{2} \right) \right|$$

$$\leq \frac{1}{8} \left[ f'_- (b) - f'_+ (a) \right] (b - a)$$

for the choices $a = 1$, $b = \frac{q(x)}{p(x)}$, $x \in \chi$, multiplying with $p(x) \geq 0$ and integrating over $x$ on $\chi$ we get

$$0 \leq \frac{1}{8} \int_{\chi} \left[ f'_+ \left( \frac{p(x) + q(x)}{2p(x)} \right) - f'_- \left( \frac{p(x) + q(x)}{2p(x)} \right) \right] |q(x) - p(x)| \, d\mu(x)$$

$$\leq \frac{1}{8} \int_{\chi} \left[ f'_+ \left( \frac{q(x)}{p(x)} \right) - f'_+ (1) \right] \langle q(x) - p(x) \rangle \, d\mu(x),$$
Corollary 7. With the above assumptions and if $f$ is differentiable on $(0, \infty)$, then

$$0 \leq D_{HH}^f (p, q) - D_f \left( p, \frac{1}{2} p + \frac{1}{2} q \right) \leq \frac{1}{8} D_{f'}(\cdot - 1) (p, q).$$

REFERENCES


