An Ostrowski Type Inequality for Convex Functions

AN OSTROWSKI TYPE INEQUALITY FOR CONVEX FUNCTIONS

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Abstract. An Ostrowski type integral inequality for convex functions and applications for quadrature rules and integral means are given. A refinement and a counterpart result for Hermite-Hadamard inequalities are obtained and some inequalities for pdf's and (HH) −divergence measure are also mentioned.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [1].

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \leq M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)M \tag{1.1}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt, \quad x \in [a,b], \tag{1.2}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad a \leq t \leq x \\ t-b & \text{if} \quad x < t \leq b \end{cases}$$

which holds for absolutely continuous functions $f : [a,b] \to \mathbb{R}$.
The following Ostrowski type result holds (see [2], [3] and [4]).

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Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x \in [a,b]$, we have:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.3}$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q & \text{if } f' \in L_q[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1; \end{cases}$$

where $\|\cdot\|_r$ $(r \in [1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}},\ r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from Fink's result in [5] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [6]):

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x) - f(y)| \leq H|x-y|^r,\ \text{for all } x,y \in [a,b], \tag{1.4}$$

where $r \in (0,1]$ and $H > 0$ are fixed. Then for all $x \in [a,b]$ we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{1.5}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [7])

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{1.6}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the continuity condition of the function, and assumes that it is of bounded variation, then the following result may be stated (see [8]).

Theorem 4. *Assume that* $f:[a,b]\rightarrow\mathbb{R}$ *is of bounded variation and denote by* $\bigvee_a^b(f)$ *its total variation. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.7}$$

for all $x\in[a,b]$.
The constant $\frac{1}{2}$ *is the best possible.*

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.7) may be improved in the following manner [9] (see also [10]).

Theorem 5. *Let* $f:[a,b]\rightarrow\mathbb{R}$ *be monotonic nondecreasing. Then for all* $x\in[a,b]$, *we have the inequality:*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \leq\ &\frac{1}{b-a}\left\{[2x-(a+b)]\,f(x)+\int_a^b sgn(t-x)\,f(t)\,dt\right\} \\ \leq\ &\frac{1}{b-a}\left\{(x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]\right\} \\ \leq\ &\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)]\,.\end{aligned} \tag{1.8}$$

All the inequalities in (1.8) are sharp and the constant $\frac{1}{2}$ *is the best possible.*

In this paper we establish an Ostrowski type inequality for convex functions. Applications for quadrature rules, for integral means, for probability distribution functions, and for $HH-$divergences in Information Theory are also considered.

2. The Results

The following theorem providing a lower bound for the Ostrowski difference $\int_a^b f(t)\,dt-(b-a)\,f(x)$ holds.

Theorem 6. *Let* $f:[a,b]\rightarrow\mathbb{R}$ *be a convex function on* $[a,b]$. *Then for any* $x\in(a,b)$ *we have the inequality:*

$$\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right]\leq\int_a^b f(t)\,dt-(b-a)\,f(x)\,. \tag{2.1}$$

The constant $\frac{1}{2}$ *in the left hand side of (2.1) is sharp in the sense that it cannot be replaced by a larger constant.*

Proof. It is easy to see that for any locally absolutely continuous function $f:(a,b)\rightarrow\mathbb{R}$, we have the identity

$$\int_a^x (t-a)\,f'(t)\,dt+\int_x^b (t-b)\,f'(t)\,dt=f(x)-\int_a^b f(t)\,dt, \tag{2.2}$$

for any $x\in(a,b)$ where f' is the derivative of f which exists a.e. on (a,b).

Since f is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any $x\in(a,b)$, we have the inequalities

$$f'(t)\leq f'_{-}(x)\ \text{ for a.e. }\ t\in[a,x] \tag{2.3}$$

and

$$f'(t)\geq f'_{+}(x)\ \text{ for a.e. }\ t\in[x,b]\,. \tag{2.4}$$

If we multiply (2.3) by $t-a\geq 0$, $t\in[a,x]$, and integrate on $[a,x]$, we get

$$\int_a^x (t-a)\,f'(t)\,dt\leq\frac{1}{2}(x-a)^2 f'_{-}(x) \tag{2.5}$$

and if we multiply (2.4) by $b-t\geq 0$, $t\in[x,b]$, and integrate on $[x,b]$, we also have

$$\int_x^b (b-t)\,f'(t)\,dt\geq\frac{1}{2}(b-x)^2 f'_{+}(x)\,. \tag{2.6}$$

Finally, if we subtract (2.6) from (2.5) and use the representation (2.2) we deduce the desired inequality (2.1).

Now, assume that (2.1) holds with a constant $C>0$ instead of $\frac{1}{2}$, i.e.,

$$C\left[(b-x)^2 f'_{+}(x)-(x-a)^2 f'_{-}(x)\right]\leq\int_a^b f(t)\,dt-(b-a)\,f(x)\,. \tag{2.7}$$

Consider the convex function $f_0(t):=k\left|t-\frac{a+b}{2}\right|$, $k>0$, $t\in[a,b]$. Then

$$f'_{0+}\left(\frac{a+b}{2}\right)=k,\quad f'_{0-}\left(\frac{a+b}{2}\right)=-k,\quad f_0\left(\frac{a+b}{2}\right)=0$$

and

$$\int_a^b f_0(t)\,dt=\frac{1}{4}k\,(b-a)^2\,.$$

If in (2.7) we choose f_0 as above and $x=\frac{a+b}{2}$, then we get

$$C\left[\frac{1}{4}(b-a)^2 k+\frac{1}{4}(b-a)^2 k\right]\leq\frac{1}{4}k\,(b-a)^2\,,$$

which gives $C\leq\frac{1}{2}$, and the sharpness of the constant is proved. ∎

Now, recall that the following inequality, which is well known in the literature as the *Hermite-Hadamard inequality* for convex functions, holds:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}. \tag{HH}$$

The following corollary which improves the first Hermite-Hadamard inequality (HH) holds.

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \\ &\leq \frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right). \end{aligned} \tag{2.8}$$

The constant $\frac{1}{8}$ is sharp.

The proof is obvious by the above theorem. The sharpness of the constant is obtained for $f_0(t) := k\left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$, $k > 0$.

When x is a point of differentiability, we may state the following corollary as well.

Corollary 2. *Let f be as in Theorem 6. If $x \in (a,b)$ is a point of differentiability for f, then*

$$\left(\frac{a+b}{2} - x\right) f'(x) \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f(x). \tag{2.9}$$

Remark 1. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I and if we choose $x \in \mathring{I}$ ($\mathring{I}$ is the interior of I), $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ is such that $a, b \in I$, then from (2.1) we may write*

$$0 \leq \frac{1}{8}h^2\left[f'_+(x) - f'_-(x)\right] \leq \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)\,dt - hf(x), \tag{2.10}$$

and the constant $\frac{1}{8}$ is sharp in (2.10).

The following result providing an upper bound for the Ostrowski difference $\int_a^b f(t)\,dt - (b-a)f(x)$ also holds.

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in [a,b]$, we have the inequality:*

$$\int_a^b f(t)\,dt - (b-a)f(x) \leq \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right]. \tag{2.11}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. If either $f'_+(a) = -\infty$ or $f'_-(b) = +\infty$, then the inequality (2.11) evidently holds true.

Assume that $f'_+(a)$ and $f'_-(b)$ are finite.

Since f is convex on $[a,b]$, we have

$$f'(t) \geq f'_+(a) \text{ for a.e. } t \in [a,x] \tag{2.12}$$

and

$$f'(t) \leq f'_-(b) \text{ for a.e. } t \in [x,b]. \tag{2.13}$$

If we multiply (2.12) by $t - a \geq 0$, $t \in [a,x]$, and integrate on $[a,x]$, then we deduce

$$\int_a^x (t-a) f'(t)\,dt \geq \frac{1}{2}(x-a)^2 f'_+(a) \tag{2.14}$$

and if we multiply (2.13) by $b - t \geq 0$, $t \in [x,b]$, and integrate on $[x,b]$, then we also have

$$\int_x^b (b-t) f'(t)\,dt \leq \frac{1}{2}(b-x)^2 f'_-(b). \tag{2.15}$$

Finally, if we subtract (2.14) from (2.15) and use the representation (2.2), we deduce the desired inequality (2.11).

Now, assume that (2.11) holds with a constant $D > 0$ instead of $\frac{1}{2}$, i.e.,

$$\int_a^b f(t)\,dt - (b-a)f(x) \leq D\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right]. \tag{2.16}$$

If we consider the convex function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = k\left|t - \frac{a+b}{2}\right|$, then we have $f'_-(b) = k$, $f'_+(a) = -k$ and by (2.16) we deduce for $x = \frac{a+b}{2}$ that

$$\frac{1}{4}k(b-a)^2 \le D\left[\frac{1}{4}k(b-a)^2 + \frac{1}{4}k(b-a)^2\right],$$

giving $D \ge \frac{1}{2}$, and the sharpness of the constant is proved. ∎

The following corollary related to the Hermite-Hadamard inequality is interesting as well.

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ be convex on $[a,b]$. Then*

$$0 \le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \le \frac{1}{8}\left[f'_-(b) - f'_+(a)\right](b-a) \tag{2.17}$$

and the constant $\frac{1}{8}$ is sharp.

Remark 2. *Denote $B := f'_-(b)$, $A := f'_+(a)$ and assume that $B \ne A$, i.e., f is not constant on (a,b). Then*

$$\begin{aligned}&(b-x)^2 B - (x-a)^2 A \\ = \;& (B-A)\left[x - \left(\frac{bB-aA}{B-A}\right)\right]^2 - \frac{AB}{B-A}(b-a)^2\end{aligned}$$

and by (2.11) we get

$$\begin{aligned}&\int_a^b f(t)\,dt - (b-a)f(x) \\ \le\;& \frac{1}{2}(B-A)\left\{\left[x - \left(\frac{bB-aA}{B-A}\right)\right]^2 - \frac{AB}{(B-A)^2}(b-a)^2\right\}\end{aligned} \tag{2.18}$$

for any $x \in [a,b]$.

If $A \ge 0$ then $x_0 = \frac{bB-aA}{B-A} \in [a,b]$ and by (2.18) we get, choosing $x = \frac{bB-aA}{B-A}$, that

$$0 \le \frac{1}{2}\frac{AB}{B-A}(b-a) \le f\left(\frac{bB-aA}{B-A}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt, \tag{2.19}$$

which is an interesting inequality in itself.

Remark 3. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I and if we choose $x \in \mathring{I}$, $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ is such that $a,b \in I$, then from (2.11) we deduce:*

$$0 \le \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)\,dt - hf(x) \le \frac{1}{8}h^2\left[f'_-\left(x+\frac{h}{2}\right) - f'_+\left(x-\frac{h}{2}\right)\right], \tag{2.20}$$

and the constant $\frac{1}{8}$ is sharp.

3. The Composite Case

Consider the division $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and denote $h_i := x_{i+1} - x_i$, $i = \overline{0, n-1}$. If $\xi_i \in [x_i, x_{i+1}]$ $\left(i = \overline{0, n-1}\right)$ are intermediate points, then we will denote by

$$R_n(f; I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} h_i f(\xi_i) \tag{3.1}$$

the Riemann sum associated to f, I_n and $\boldsymbol{\xi}$.

The following theorem providing upper and lower bounds for the remainder in approximating the integral $\int_a^b f(t)\,dt$ of a convex function f in terms of a general Riemann sum holds.

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and I_n and $\boldsymbol{\xi}$ be as above. Then we have:*

$$\int_a^b f(t)\,dt = R_n(f;I_n,\boldsymbol{\xi}) + W_n(f;I_n,\boldsymbol{\xi}), \tag{3.2}$$

where $R_n(f;I_n,\boldsymbol{\xi})$ is the Riemann sum defined by (3.1) and the remainder $W_n(f;I_n,\boldsymbol{\xi})$ satisfies the estimate:

$$\begin{aligned}
&\frac{1}{2}\left[\sum_{i=0}^{n-1}(x_{i+1}-\xi_i)^2 f'_+(\xi_i) - \sum_{i=0}^{n-1}(\xi_i-x_i)^2 f'_-(\xi_i)\right] \\
\le\ & W_n(f;I_n,\boldsymbol{\xi}) \\
\le\ & \frac{1}{2}\Bigg[\left(b-\xi_{n-1}\right)^2 f'_-(b) + \sum_{i=1}^{n-1}\Big[\left(x_i-\xi_{i-1}\right)^2 f'_-(x_i) \\
&- (\xi_i-x_i)^2 f'_+(x_i)\Big] - (\xi_0-a)^2 f'_+(a)\Bigg].
\end{aligned} \tag{3.3}$$

Proof. If we write the inequalities (2.1) and (2.11) on the interval $[x_i,x_{i+1}]$ and for the intermediate points $\xi_i\in[x_i,x_{i+1}]$, then we have

$$\begin{aligned}
&\frac{1}{2}\left[(x_{i+1}-\xi_i)^2 f'_+(\xi_i) - (\xi_i-x_i)^2 f'_-(\xi_i)\right] \\
\le\ & \int_{x_i}^{x_{i+1}} f(t)\,dt - h_i f(\xi_i) \\
\le\ & \frac{1}{2}\left[(x_{i+1}-\xi_i)^2 f'_-(x_{i+1}) - (\xi_i-x_i)^2 f'_+(x_i)\right].
\end{aligned}$$

Summing the above inequalities over i from 0 to $n-1$, we deduce

$$\begin{aligned}
&\frac{1}{2}\sum_{i=0}^{n-1}\left[(x_{i+1}-\xi_i)^2 f'_+(\xi_i) - (\xi_i-x_i)^2 f'_-(\xi_i)\right] \\
\le\ & \int_a^b f(t)\,dt - R_n(f;I_n,\boldsymbol{\xi}) \\
\le\ & \frac{1}{2}\left[\sum_{i=0}^{n-1}(x_{i+1}-\xi_i)^2 f'_-(x_{i+1}) - \sum_{i=0}^{n-1}(\xi_i-x_i)^2 f'_+(x_i)\right].
\end{aligned} \tag{3.4}$$

However,

$$\begin{aligned}
\sum_{i=0}^{n-1}(x_{i+1}-\xi_i)^2 f'_-(x_{i+1}) &= \left(b-\xi_{n-1}\right)^2 f'_-(b) + \sum_{i=0}^{n-2}\left[(x_{i+1}-\xi_i)^2 f'_-(x_{i+1})\right] \\
&= \left(b-\xi_{n-1}\right)^2 f'_-(b) + \sum_{i=1}^{n-1}\left[\left(x_i-\xi_{i-1}\right)^2 f'_-(x_i)\right]
\end{aligned}$$

and

$$\sum_{i=0}^{n-1}(\xi_i-x_i)^2 f'_+(x_i)=\sum_{i=1}^{n-1}(\xi_i-x_i)^2 f'_+(x_i)+(\xi_0-a)^2 f'_+(a)$$

and then, by (3.4), we deduce the desired estimate (3.3). ∎

The following corollary may be useful in practical applications.

Corollary 4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable convex function on (a,b). Then we have the representation (3.2) and the remainder $W_n(f;I_n,\boldsymbol{\xi})$ satisfies the estimate:*

$$\begin{aligned}(3.5)\qquad & \sum_{i=0}^{n-1}\left(\frac{x_i+x_{i+1}}{2}-\xi_i\right)h_i f'(\xi_i)\\ \leq\ & W_n(f;I_n,\xi_i)\\ \leq\ & \frac{1}{2}\left[\left(b-\xi_{n-1}\right)^2 f'_-(b)-(\xi_0-a)^2 f'_+(a)\right]\\ & +\sum_{i=1}^{n-1}\left(x_i-\frac{\xi_i+\xi_{i-1}}{2}\right)(\xi_i-\xi_{i-1})f'(x_i).\end{aligned}$$

We may also consider the mid-point quadrature rule:

$$M_n(f,I_n):=\sum_{i=0}^{n-1}h_i f\left(\frac{x_i+x_{i+1}}{2}\right). \tag{3.6}$$

Using Corollaries 1 and 2, we may state the following result as well.

Corollary 5. *Assume that $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$ and I_n is a division as above. Then we have the representation:*

$$\int_a^b f(x)\,dx=M_n(f,I_n)+S_n(f,I_n), \tag{3.7}$$

where $M_n(f,I_n)$ is the mid-point quadrature rule given in (3.6) and the remainder $S_n(f,I_n)$ satisfies the estimates:

$$\begin{aligned}(3.8)\qquad 0\ \leq\ & \frac{1}{8}\sum_{i=0}^{n-1}\left[f'_+\left(\frac{x_i+x_{i+1}}{2}\right)-f'_-\left(\frac{x_i+x_{i+1}}{2}\right)\right]h_i^2\\ \leq\ & S_n(f,I_n)\leq\frac{1}{8}\sum_{i=0}^{n-1}\left[f'_-(x_{i+1})-f'_+(x_i)\right]h_i^2.\end{aligned}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

4. Inequalities for Integral Means

We may prove the following result in comparing two integral means.

Theorem 9. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $c,d\in[a,b]$ with $c<d$. Then we have the inequalities*

$$\begin{aligned}
&\frac{a+b}{2}\cdot\frac{f(d)-f(c)}{d-c}-\frac{df(d)-cf(c)}{d-c}+\frac{1}{d-c}\int_c^d f(x)\,dx \\
\leq\ & \frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(x)\,dx \\
\leq\ & \frac{f'_-(b)\left[(b-d)^2+(b-d)(b-c)+(b-c)^2\right]}{6(b-a)} \\
&-\frac{f'_+(a)\left[(d-a)^2+(d-a)(c-a)+(c-a)^2\right]}{6(b-a)}.
\end{aligned}\tag{4.1}$$

Proof. Since f is convex, then for $a.e.$ $x\in[a,b]$, we have (by (2.9)) that

$$\left(\frac{a+b}{2}-x\right)f'(x)\leq\frac{1}{b-a}\int_a^b f(t)\,dt-f(x).\tag{4.2}$$

Integrating (5.2) on $[c,d]$ we deduce

$$\frac{1}{d-c}\int_c^d\left(\frac{a+b}{2}-x\right)f'(x)\,dx\leq\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(x)\,dx.\tag{4.3}$$

Since

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d\left(\frac{a+b}{2}-x\right)f'(x)\,dx \\
=\ &\frac{1}{d-c}\left[\left(\frac{a+b}{2}-d\right)f(d)-\left(\frac{a+b}{2}-c\right)f(c)+\int_c^d f(x)\,dx\right]
\end{aligned}$$

then by (4.3) we deduce the first part of (4.1).

Using (2.11), we may write for any $x\in[a,b]$ that

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)\leq\frac{1}{2(b-a)}\left[(b-x)^2f'_-(b)-(x-a)^2f'_+(a)\right].\tag{4.4}$$

Integrating (4.4) on $[c,d]$, we deduce

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(x)\,dx \\
\leq\ &\frac{1}{2(b-a)}\left[f'_-(b)\frac{1}{d-c}\int_c^d(b-x)^2\,dx-f'_+(a)\frac{1}{d-c}\int_c^d(x-a)^2\,dx\right].
\end{aligned}\tag{4.5}$$

Since

$$\frac{1}{d-c}\int_c^d(b-x)^2\,dx=\frac{(b-d)^2+(b-d)(b-c)+(b-c)^2}{3}$$

and

$$\frac{1}{d-c}\int_c^d(x-a)^2\,dx=\frac{(d-a)^2+(d-a)(c-a)+(c-a)^2}{3},$$

then by (4.5) we deduce the second part of (4.1). ∎

Remark 4. *If we choose* $f(x) = x^p,\ p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ *or* $f(x) = \frac{1}{x}$ *or even* $f(x) = -\ln x,\ x \in [a, b] \subset (0, \infty)$, *in the above inequalities, then a great number of interesting results for* $p-$*logarithmic, logarithmic and identric means may be obtained. We leave this as an exercise to the interested reader.*

5. Applications for P.D.F.s

Let X be a random variable with the *probability density function* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ and with *cumulative distribution function* $F(x) = \Pr(X \leq x)$.

The following theorem holds.

Theorem 10. *If* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ *is monotonically increasing on* $[a, b]$, *then we have the inequality:*

$$\begin{aligned} &\frac{1}{2}\left[(b-x)^2 f_+(x) - (x-a)^2 f_-(x)\right] \\ \leq\ & b - E(X) - (b-a) F(x) \\ \leq\ & \frac{1}{2}\left[(b-x)^2 f_-(b) - (x-a)^2 f_+(a)\right] \end{aligned} \tag{5.1}$$

for any $x \in (a, b)$, *where* $f_-(\alpha)$ *means the left limit in* α *while* $f_+(\alpha)$ *means the right limit in* α *and* $E(X)$ *is the expectation of* X.
The constant $\frac{1}{2}$ *is sharp in both inequalities.*
The second inequality also holds for $x = a$ *of* $x = b$.

Proof. Follows by Theorem 6 and 7 applied for the convex cdf function $F(x) = \int_a^x f(t)\,dt,\ x \in [a, b]$ and taking into account that

$$\int_a^b F(x)\,dx = b - E(X).$$

∎

Finally, we may state the following corollary in estimating the probability $\Pr\left(X \leq \frac{a+b}{2}\right)$.

Corollary 6. *With the above assumptions, we have*

$$\begin{aligned} & b - E(X) - \frac{1}{8}(b-a)^2 \left[f_-(b) - f_+(a)\right] \\ \leq\ & \Pr\left(X \leq \frac{a+b}{2}\right) \\ \leq\ & b - E(X) - \frac{1}{8}(b-a)^2 \left[f_+\left(\frac{a+b}{2}\right) - f_-\left(\frac{a+b}{2}\right)\right]. \end{aligned} \tag{5.2}$$

6. Applications for $HH-$Divergence

Assume that a set χ and the $\sigma-$finite measure μ are given. Consider the set of all probability densities on μ to be

$$\Omega := \left\{p | p : \Omega \to \mathbb{R},\ p(x) \geq 0,\ \int_\chi p(x)\,d\mu(x) = 1\right\}. \tag{6.1}$$

Csiszár's $f-$divergence is defined as follows [11]

$$D_f(p, q) := \int_\chi p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x),\ p, q \in \Omega, \tag{6.2}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$. By appropriately defining this convex function, various divergences are derived.

In [12], Shioya and Da-te introduced the generalised Lin-Wong $f-$divergence $D_f\left(p,\frac{1}{2}p+\frac{1}{2}q\right)$ and the Hermite-Hadamard (HH) divergence

$$D_{HH}^{f}(p,q):=\int_{\chi}\frac{p^{2}(x)}{q(x)-p(x)}\left(\int_{1}^{\frac{q(x)}{p(x)}}f(t)\,dt\right)d\mu(x)\,,\ p,q\in\Omega, \tag{6.3}$$

and, by the use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality

$$D_f\left(p,\frac{1}{2}p+\frac{1}{2}q\right)\leq D_{HH}^{f}(p,q)\leq\frac{1}{2}D_f(p,q)\,, \tag{6.4}$$

provided that f is convex and normalised, i.e., $f(1)=0$.

The following result in estimating the difference

$$D_{HH}^{f}(p,q)-D_f\left(p,\frac{1}{2}p+\frac{1}{2}q\right)$$

holds.

Theorem 11. *Let $f:[0,\infty)\to\mathbb{R}$ be a convex function and $p,q\in\Omega$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[D_{f'_{+}\cdot\left|\frac{\cdot+1}{2}\right|}(p,q)-D_{f'_{-}\cdot\left|\frac{\cdot+1}{2}\right|}(p,q)\right] \\ &\leq D_{HH}^{f}(p,q)-D_f\left(p,\frac{1}{2}p+\frac{1}{2}q\right) \\ &\leq \frac{1}{8}D_{f'_{-}\cdot(\cdot-1)}(p,q)\,. \end{aligned} \tag{6.5}$$

Proof. Using the double inequality

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right]|b-a| \\ &\leq \frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{8}\left[f_{-}(b)-f'_{+}(a)\right](b-a) \end{aligned}$$

for the choices $a=1$, $b=\frac{q(x)}{p(x)}$, $x\in\chi$, multiplying with $p(x)\geq 0$ and integrating over x on χ we get

$$\begin{aligned} 0 &\leq \frac{1}{8}\int_{\chi}\left[f'_{+}\left(\frac{p(x)+q(x)}{2p(x)}\right)-f'_{-}\left(\frac{p(x)+q(x)}{2p(x)}\right)\right]|q(x)-p(x)|\,d\mu(x) \\ &\leq D_{HH}^{f}(p,q)-D_f\left(p,\frac{1}{2}p+\frac{1}{2}q\right) \\ &\leq \frac{1}{8}\int_{\chi}\left[f'_{-}\left(\frac{q(x)}{p(x)}\right)-f'_{+}(1)\right](q(x)-p(x))\,d\mu(x)\,, \end{aligned}$$

which is clearly equivalent to (6.5). ∎

Corollary 7. *With the above assumptions and if f is differentiable on $(0,\infty)$, then*

$$0 \le D_{HH}^{f}(p,q) - D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right) \le \frac{1}{8} D_{f'\cdot(\cdot-1)}(p,q). \tag{6.6}$$

References

[1] A. OSTROWSKI, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

[2] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[3] S.S. DRAGOMIR and S. WANG, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[4] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[5] A.M. FINK, Bounds on the deviation of a function from its averages, *Czech. Math. J.,* **42**(117) (1992), 289-310.

[6] S.S. DRAGOMIR, P. CERONE, J. ROUMELIOTIS and S. WANG, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Roumanie,* **42**(90)(4) (1992), 301-314.

[7] S.S. DRAGOMIR, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. and Math. with Appl.,* **38** (1999), 33-37.

[8] S.S. DRAGOMIR, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[9] S.S. DRAGOMIR, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[10] P. CERONE and S.S. DRAGOMIR, Midpoint type rules from an inequalities point of view, in *Analytic-Computational Methods in Applied Mathematics*, G.A. Anastassiou (Ed), CRC Press, New York, 2000, 135-200.

[11] I. CSISZÁR, Information-type measures of difference of probability distributions and indirect observations, *Studia Math. Hungarica,* **2** (1967), 299-318.

[12] H. SHIOYA and T. DA-TE, A generalisation of Lin divergence and the derivation of a new information divergence, *Elec. and Comm. in Japan,* **78**(7), (1995), 37-40.

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