Another Grüss Type Inequality for Sequences of Vectors in Normed Linear Spaces and Applications

ANOTHER GRÜSS TYPE INEQUALITY FOR SEQUENCES OF VECTORS IN NORMED LINEAR SPACES AND APPLICATIONS

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ABSTRACT. A discrete inequality of Grüss type in normed linear sapces and applications for the Fourier transform, Mellin transform of sequences, for polynomials with coefficients in normed spaces and for vector valued Lipschitzian mappings, are given.

1. INTRODUCTION

In 1935, G. Grüss [9] proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

where $f,\ g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi,\ \gamma\leq g(x)\leq\Gamma \tag{1.2}$$

for each $x\in[a,b]$, where ϕ,Φ,γ,Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the recent book [11] and the papers [1]-[8] and [10].

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [11, Chapter X] established the following discrete version of Grüss' inequality:

Theorem 1. *Let $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ be two $n-$tuples of real numbers such that $r\leq a_i\leq R$ and $s\leq b_i\leq S$ for $i=1,...,n$. Then one has*

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i\right|\leq\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s), \tag{1.3}$$

where $[x]$ denotes the integer part of x, $x\in\mathbb{R}$.

A weighted version of the discrete Grüss inequality was proved by J. E. Pečarić in 1979 [11, Chapter X]:

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Theorem 2. *Let a and b be two monotonic $n-$tuples and p a positive one. Then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i\right| \leq |a_n-a_1|\,|b_n-b_1|\max_{1\leq k\leq n-1}\left[\frac{P_k\bar{P}_{k+1}}{P_n^2}\right], \tag{1.4}$$

where $P_n:=\sum_{i=1}^{n}p_i$, and $\bar{P}_{k+1}=P_n-P_{k+1}$.

In 1981, A. Lupaş, [11, Chapter X] proved some similar results for the first difference of a as follows.

Theorem 3. *Let a,b be two monotonic $n-$tuples in the same sense and p a positive n-tuple. Then*

$$\begin{aligned}
&\min_{1\leq i\leq n-1}|\Delta a_i|\min_{1\leq i\leq n-1}|\Delta b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right] \\
&\leq \frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i \\
&\leq \max_{1\leq i\leq n-1}|\Delta a_i|\max_{1\leq i\leq n-1}|\Delta b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right],
\end{aligned} \tag{1.5}$$

where $\Delta a_i:=a_{i+1}-a_i$ is the forward first difference. If there exist the numbers $\bar{a},\bar{a}_1,r,r_1$ $(rr_1>0)$ such that $a_k=\bar{a}+kr$ and $b_k=\bar{a}_1+kr_1$, then equality holds in (1.5).

In the recent paper [6], the authors obtained the following related result

Theorem 4. *Let $(X,\|\cdot\|)$ be a normed linear space over $\mathbb{K}$, $\mathbb{K}=\mathbb{R},\mathbb{C}$, $x_i\in X$, $\alpha_i\in\mathbb{K}$ and $p_i\geq 0$ $(i=1,...,n)$ such that $\sum_{i=1}^{n}p_i=1$.*
Then we have the inequality

$$\left\|\sum_{i=1}^{n}p_i\alpha_ix_i-\sum_{i=1}^{n}p_i\alpha_i\sum_{i=1}^{n}p_ix_i\right\| \leq \max_{1\leq j\leq n-1}|\Delta\alpha_j|\max_{1\leq j\leq n-1}\|\Delta x_j\|\left[\sum_{i=1}^{n}i^2p_i-\left(\sum_{i=1}^{n}ip_i\right)^2\right]. \tag{1.6}$$

The inequality (1.6) *is sharp in the sense that the constant $c=1$ in the right hand side cannot be replaced by a smaller one.*

In this paper we point out another inequality of Grüss type and apply it in approximating the discrete Fourier transform, the Mellin transform of sequences, for polynomials with coefficients in normed linear spaces and for vector valued Lipschitzian mappings.

2. An Inequality of Grüss Type

The following result holds.

Theorem 5. *Let* $(X, \|\cdot\|)$ *be a normed linear space over* $\mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $x_i \in X$, $\alpha_i \in \mathbb{K}$ *and* $p_i \geq 0$ $(i = 1, ..., n)$ $(n \geq 2)$ *such that* $\sum_{i=1}^n p_i = 1$. *Then we have the inequality*

$$\left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i \right\| \leq \sum_{1 \leq j < i \leq n} (i-j)\, p_i p_j \left(\sum_{k=1}^{n-1} |\Delta \alpha_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}}, \tag{2.1}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $c = 1$ *in the right hand side of* (2.1) *is sharp in the sense that it cannot be replaced by a smaller one.*

Proof. It is well known that the following identity holds in normed linear spaces [6]

$$\begin{aligned} \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i &= \frac{1}{2} \sum_{i,j=1}^n p_i p_j (\alpha_i - \alpha_j)(x_i - x_j) \\ &= \sum_{1 \leq j < i \leq n} p_i p_j (\alpha_i - \alpha_j)(x_i - x_j). \end{aligned}$$

We observe that for $i > j$, we can write

$$\alpha_i - \alpha_j = \sum_{k=j}^{i-1} \Delta \alpha_k, \; x_i - x_j = \sum_{k=j}^{i-1} \Delta x_k$$

and by the generalized triangle inequality we get

$$\begin{aligned} &\left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i \right\| \\ &\leq \sum_{1 \leq j < i \leq n} p_i p_j \left| \sum_{k=j}^{i-1} \Delta \alpha_k \right| \left\| \sum_{k=j}^{i-1} \Delta x_k \right\| \\ &\leq \sum_{1 \leq j < i \leq n} p_i p_j \sum_{k=j}^{i-1} |\Delta \alpha_k| \sum_{k=j}^{i-1} \|\Delta x_k\| =: A. \end{aligned} \tag{2.2}$$

Using Hölder's discrete inequality, we can state that

$$\sum_{k=j}^{i-1} |\Delta \alpha_k| \leq (i-j)^{\frac{1}{q}} \left(\sum_{k=j}^{i-1} |\Delta \alpha_k|^p \right)^{\frac{1}{p}}$$

and

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq (i-j)^{\frac{1}{p}} \left(\sum_{k=j}^{i-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and then

$$A \leq \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \left(\sum_{k=j}^{i-1} |\Delta \alpha_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=j}^{i-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}}. \tag{2.3}$$

As
$$\sum_{k=j}^{i-1}|\Delta\alpha_k|^p \le \sum_{k=1}^{n-1}|\Delta\alpha_k|^p$$
and
$$\sum_{k=j}^{i-1}\|\Delta x_k\|^q \le \sum_{k=1}^{n-1}\|\Delta x_k\|^q,$$
for all $1 \le j < i \le n$, then by (2.3) and (2.2), we get the desired inequality (2.1).

To prove the sharpness of the constant, let us assume that (2.1) holds with a constant $c > 0$. That is,

$$\begin{aligned}(2.4)\qquad &\left\|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right\| \\ &\le c\sum_{1\le j<i\le n}(i-j)\,p_ip_j\left(\sum_{k=1}^{n-1}|\Delta\alpha_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Choose $n = 2$. Then
$$\begin{aligned}\left\|\sum_{i=1}^{2} p_i\alpha_i x_i - \sum_{i=1}^{2} p_i\alpha_i \sum_{i=1}^{2} p_i x_i\right\| &= \left\|\frac{1}{2}\sum_{i,j=1}^{2} p_ip_j\,(\alpha_i-\alpha_j)\,(x_i-x_j)\right\| \\ &= \left\|\sum_{1\le j<i\le 2} p_ip_j\,(\alpha_i-\alpha_j)\,(x_i-x_j)\right\| \\ &= p_1p_2\,|\alpha_1-\alpha_2|\,\|x_1-x_2\|\end{aligned}$$
and
$$\begin{aligned}&\sum_{1\le j<i\le 2}(i-j)\,p_ip_j\left(\sum_{k=1}^{1}|\Delta\alpha_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{1}\|\Delta x_k\|^q\right)^{\frac{1}{q}} \\ &= p_1p_2\,|\alpha_1-\alpha_2|\,\|x_1-x_2\|.\end{aligned}$$
Therefore, from (2.4), we obtain
$$p_1p_2\,|\alpha_1-\alpha_2|\,\|x_1-x_2\| \le cp_1p_2\,|\alpha_1-\alpha_2|\,\|x_1-x_2\|$$
for all $\alpha_1 \ne \alpha_2$, $x_1 \ne x_2$, and then $c \ge 1$, which proves the sharpness of the constant. ∎

Remark 1. *A coarser upper bound, which can be more useful may be obtained by applying Cauchy-Schwartz's inequality:*
$$\sum_{1\le j<i\le n}(i-j)\,p_ip_j \le \left(\sum_{1\le j<i\le n}p_ip_j\right)^{\frac{1}{2}}\left(\sum_{1\le j<i\le n}p_ip_j\,(i-j)^2\right)^{\frac{1}{2}}$$

and taking into account that

$$\begin{aligned}\sum_{1\leq j<i\leq n} p_i p_j &= \frac{1}{2}\left(\sum_{i,j=1}^{n} p_i p_j - \sum_{i=j}^{n} p_i p_j\right) \\ &= \frac{1}{2}\left(1-\sum_{i=1}^{n} p_i^2\right) \\ &= \frac{1}{2}\sum_{i=1}^{n} p_i\left(1-p_i\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{1\leq j<i\leq n} p_i p_j (i-j)^2 &= \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j (i-j)^2 \\ &= \left[\sum_{i=1}^{n} p_i \sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right] \\ &= \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right].\end{aligned}$$

Thus, from (2.1), *we can state the inequality*

$$\begin{aligned}&\left\|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right\| \\ &\leq \frac{\sqrt{2}}{2}\left[\sum_{i=1}^{n} p_i\left(1-p_i\right)\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right]^{\frac{1}{2}} \\ &\quad\times\left(\sum_{k=1}^{n-1}\left|\Delta\alpha_k\right|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|^q\right)^{\frac{1}{q}}.\end{aligned} \tag{2.5}$$

The following corollary holds.

Corollary 1. *With the above assumptions, we have*

$$\left\|\frac{1}{n}\sum_{i=1}^{n}\alpha_i x_i - \frac{1}{n}\sum_{i=1}^{n}\alpha_i\cdot\frac{1}{n}\sum_{i=1}^{n} x_i\right\| \leq \frac{n^2-1}{6n}\left(\sum_{k=1}^{n-1}\left|\Delta\alpha_k\right|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|^q\right)^{\frac{1}{q}}. \tag{2.6}$$

The constant $\frac{1}{6}$ *is the best possible.*

Proof. The proof follows by (2.1), putting $p_i = \frac{1}{n}$ and taking into account that

$$\begin{aligned}
& \sum_{1\leq j<i\leq n} (i-j) \\
&= \sum_{1\leq j\leq 2} (2-j) + \sum_{1\leq j\leq 3} (3-j) + ... + \sum_{1\leq j\leq n} (n-j) \\
&= 2\cdot 2 - (1+2) + 3\cdot 3 - (1+2+3) + ... + n\cdot n - (1+2+...+n) \\
&= 1^2 + 2^2 + ... + n^2 - 1 - (1+2) - (1+2+3) - ... - (1+2+...+n) \\
&= \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{1}{2}\left(\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k\right) = \frac{n(n^2-1)}{6},
\end{aligned}$$

and the corollary is thus proved. ∎

Remark 2. *If in* (2.1) *and* (2.6) *we assume that* $p = q = 2$, *then we get the inequalities:*

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right\| \qquad (2.7)\\
&\leq \sum_{1\leq i<j\leq n} (i-j) p_i p_j \left(\sum_{k=1}^{n-1} |\Delta\alpha_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n-1} \|\Delta\alpha_k\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \frac{1}{n}\sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n}\sum_{i=1}^{n} \alpha_i \cdot \frac{1}{n}\sum_{i=1}^{n} x_i \right\| \qquad (2.8)\\
&\leq \frac{n^2-1}{6n} \left(\sum_{k=1}^{n-1} |\Delta\alpha_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

respectively.

3. Applications for the Discrete Fourier Transform

Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, and $\bar{x} = (x_1, ..., x_n)$ be a sequence of vectors in X.

For a given $w \in \mathbb{R}$, define the *discrete Fourier transform*

$$\mathcal{F}_w(\bar{x})(m) := \sum_{k=1}^{n} x_k \exp(2wimk), \ m = 1, ..., n. \qquad (3.1)$$

The following approximation result for the Fourier transform (3.1) holds.

Theorem 6. *Let* $(X, \|\cdot\|)$ *and* $\bar{x} \in X^n$ *be as above. Then we have the inequality*

$$\begin{aligned}
&\left\| \mathcal{F}_w(\bar{x})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp[(n+1)mi] \times \frac{1}{n}\sum_{k=1}^{n} x_k \right\| \qquad (3.2)\\
&\leq \frac{n+1}{3}(n-1)^{1+\frac{1}{p}} |\sin(wm)| \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $m \in \{1, ..., n\}$, $w \neq \frac{k}{n}\pi$, $k \in \mathbb{Z}$.

Proof. Using the inequality (2.6), we can state that

$$\left\|\sum_{k=1}^{n}a_kx_k-\sum_{k=1}^{n}a_k\cdot\frac{1}{n}\sum_{k=1}^{n}x_k\right\| \leq \frac{n^2-1}{6}\left(\sum_{k=1}^{n-1}|\Delta a_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^q\right)^{\frac{1}{q}} \tag{3.3}$$

for all $a_k\in\mathbb{K},\ x_k\in X,\ k=1,...,n.$
Now, choose in (3.3), $a_k=\exp(2wimk)$ to obtain

$$\left\|\mathcal{F}_w(\bar{x})(m)-\sum_{k=1}^{n}\exp(2wimk)\times\frac{1}{n}\sum_{k=1}^{n}x_k\right\| \leq \frac{n^2-1}{6}\left(\sum_{k=1}^{n-1}|\exp(2wim(k+1))-\exp(2wimk)|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^q\right)^{\frac{1}{q}}, \tag{3.4}$$

for all $m\in\{1,...,n\}$.
However,

$$\begin{aligned}\sum_{k=1}^{n}\exp(2wimk) &= \exp(2wim)\times\left[\frac{\exp(2wimn)-1}{\exp(2wim)-1}\right]\\
&= \exp(2wim)\times\left[\frac{\cos(2wmn)+i\sin(2wmn)-1}{\cos(2wm)+i\sin(2wm)-1}\right]\\
&= \exp(2wim)\times\left[\frac{-2\sin^2(wmn)+2i\sin(wmn)\cos(wmn)}{-2\sin^2(wm)+2i\sin(wm)\cos(wm)}\right]\\
&= \exp(2wim)\times\frac{\sin(wmn)}{\sin(wm)}\left[\frac{\sin(wmn)-i\cos(wmn)}{\sin(wm)-i\cos(wm)}\right]\\
&= \exp(2wim)\times\frac{\sin(wmn)}{\sin(wm)}\left[\frac{\cos(wmn)+i\sin(wmn)}{\cos(wm)+i\sin(wm)}\right]\\
&= \frac{\sin(wmn)}{\sin(wm)}\times\exp(2wim)\left[\frac{\exp(iwmn)}{\exp(iwm)}\right]\\
&= \frac{\sin(wmn)}{\sin(wm)}\times\exp[2wim+iwmn-iwm]\\
&= \frac{\sin(wmn)}{\sin(wm)}\times\exp[(n+1)mi].\end{aligned}$$

We observe that

$$\begin{aligned}&\exp(2wim(k+1))-\exp(2wimk)\\
=\;&\cos(2wm(k+1))+i\sin(2wm(k+1))-\cos(2wmk)-i\sin(2wmk)\\
=\;&\cos(2wm(k+1))-\cos(2wmk)+i[\sin(2wm(k+1))-\sin(2wmk)]\end{aligned}$$

$$\begin{aligned}
&= -2\sin\left[\frac{2wm(k+1)+2wmk}{2}\right]\sin\left[\frac{2wm(k+1)-2wmk}{2}\right] \\
&\quad +i2\cos\left[\frac{2wm(k+1)+2wmk}{2}\right]\sin\left[\frac{2wm(k+1)-2wmk}{2}\right] \\
&= -2\sin((2k+1)wm)\sin(wm)+2i\cos((2k+1)wm)\sin(wm) \\
&= 2i\sin(wm)\left[\cos\left[(2k+1)mw\right]+i\sin\left[(2k+1)mw\right]\right] \\
&= 2i\sin(wm)\exp\left[(2k+1)mwi\right],
\end{aligned}$$

and then

$$|\exp(2wim(k+1))-\exp(2wimk)| = 2|\sin(wm)|$$

Then, by (3.3), we deduce the desired inequality (3.2). ∎

The following particular case can be useful in practice.

Corollary 2. *Under the above assumptions, we have*

$$\left\|\mathcal{F}_w(\bar{x})(m)-\frac{\sin(wmn)}{\sin(wm)}\exp\left[(n+1)mi\right]\times\frac{1}{n}\sum_{k=1}^{n}x_k\right\| \le \frac{(n+1)(n-1)^{\frac{3}{2}}}{3}|\sin(wn)|\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^2\right)^{\frac{1}{2}}, \tag{3.5}$$

for all $m\in\{1,...,n\}$, $w\neq\frac{k}{n}\pi$, $k\in\mathbb{Z}$.

4. Applications for the Discrete Mellin Transform

Let $(X,\|\cdot\|)$ be a normed linear space over $\mathbb{K}$, $\mathbb{K}=\mathbb{C},\mathbb{R}$, and $\bar{x}=(x_1,...,x_n)$ be a sequence of vectors in X.

Define the *Mellin transform*

$$\mathcal{M}(\bar{x})(m) := \sum_{k=1}^{n}k^{m-1}x_k,\ m=1,...,n; \tag{4.1}$$

of the sequence $\bar{x}\in X^n$.

The following approximation result for the Mellin transform (4.1) holds.

Theorem 7. *Let* X *and* $\bar{x}$ *be as above. Then we have the inequality*

$$\left\|\mathcal{M}(\bar{x})(m)-S_{m-1}(n)\times\frac{1}{n}\sum_{k=1}^{n}x_k\right\| \le \frac{n^2-1}{6}\left(\sum_{k=1}^{n-1}\left[(k+1)^{m-1}-k^{m-1}\right]^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^q\right)^{\frac{1}{q}}, \tag{4.2}$$

where $S_m(n)$, $m\in\mathbb{R}$, $n\in\mathbb{N}$ *is the* $m-$*powers sum of the first* n *natural numbers, i.e.,*

$$S_m(n) = \sum_{k=1}^{n}k^m \tag{4.3}$$

and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the inequality (3.3), we can state that

$$\left\| \sum_{k=1}^{n} k^{m-1} x_k - \sum_{k=1}^{n} k^{m-1} \times \frac{1}{n} \sum_{k=1}^{n} x_k \right\|$$
$$\leq \frac{n^2-1}{6} \left(\sum_{k=1}^{n-1} \left[(k+1)^{m-1} - k^{m-1} \right]^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}}$$

and the inequality (4.2) is obtained. ∎

Consider the following particular values of the Mellin transform

$$\mu_1(\bar{x}) := \sum_{k=1}^{n} k x_k$$

and

$$\mu_2(\bar{x}) := \sum_{k=1}^{n} k^2 x_k.$$

The following corollary holds.

Corollary 3. *Let X and $\bar{x}$ be as specified above. Then we have the inequalities:*

$$\left\| \mu_1(\bar{x}) - \frac{n+1}{2} \sum_{k=1}^{n} x_k \right\| \leq \frac{\left(n^2-1\right)(n-1)^{\frac{1}{p}}}{6} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \tag{4.4}$$

and

$$\left\| \mu_2(\bar{x}) - \frac{(n+1)(2n+1)}{6} \sum_{k=1}^{n} x_k \right\| \tag{4.5}$$
$$\leq \frac{n^2-1}{6} \left[\sum_{k=1}^{n-1} (2k-1)^p \right]^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 3. *If we assume that $p = (p_1, ..., p_n)$ is a probability distribution, i.e., $p_k \geq 0$ $(k = 1, ..., n)$ and $\sum_{k=1}^{n} p_k = 1$, then, by (4.4) and (4.5), we get the inequalities*

$$\left| \sum_{k=1}^{n} k p_k - \frac{n+1}{2} \right| \leq \frac{\left(n^2-1\right)(n-1)^{\frac{1}{p}}}{6} \left(\sum_{k=1}^{n-1} |p_{k+1} - p_k|^q \right)^{\frac{1}{q}} \tag{4.6}$$

and

$$\left| \sum_{k=1}^{n} k^2 p_k - \frac{(n+1)(2n+1)}{6} \right| \tag{4.7}$$
$$\leq \frac{n^2-1}{6} \left[\sum_{k=1}^{n-1} (2k-1)^p \right]^{\frac{1}{p}} \left(\sum_{k=1}^{n} |p_{k+1} - p_k|^q \right)^{\frac{1}{q}},$$

which can be applied for the estimation of the 1 and 2-moments of a guessing mapping as in the paper [4]. We omit the details.

5. Applications for Polynomials

Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, and $\bar{c} = (c_0, ..., c_n)$ be a sequence of vectors in X.

Define the polynomial $P : \mathbb{C} \to X$ with the coefficients $\bar{c}$ by

$$P(z) = c_0 + zc_1 + ... + z^n c_n, \ z \in \mathbb{C}, \ c_n \neq 0.$$

The following approximation result for the polynomial P holds.

Theorem 8. *Let $X, \bar{c}$ and P be as above. Then we have the inequality:*

$$\left\| P(z) - \frac{z^{n+1} - 1}{z - 1} \times \frac{c_0 + ... + c_n}{n+1} \right\| \leq \frac{n(n+1)}{6} |z - 1| \left[\frac{|z|^{pn} - 1}{|z|^p - 1} \right]^{\frac{1}{p}} \left(\sum_{k=0}^{n-1} \|\Delta c_k\|^q \right)^{\frac{1}{q}} \tag{5.1}$$

for all $z \in \mathbb{C}$, $|z| \neq 1$ and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the inequality (3.3), we can state that

$$\begin{aligned} & \left\| \sum_{k=0}^{n} z^k c_k - \sum_{k=0}^{n} z^k \times \frac{1}{n+1} \sum_{k=0}^{n} c_k \right\| \\ \leq & \frac{n(n+1)}{6} \left(\sum_{k=0}^{n-1} \left| z^{k+1} - z^k \right|^p \right)^{\frac{1}{p}} \left(\sum_{k=0}^{n-1} \|\Delta c_k\|^q \right)^{\frac{1}{q}} \\ = & \frac{n(n+1)}{6} \left(\sum_{k=0}^{n-1} |z|^{pk} |z-1|^p \right)^{\frac{1}{p}} \left(\sum_{k=0}^{n-1} \|\Delta c_k\|^q \right)^{\frac{1}{q}} \\ = & \frac{n(n+1)}{6} |z-1| \left(\sum_{k=0}^{n-1} |z|^{pk} \right)^{\frac{1}{p}} \left(\sum_{k=0}^{n-1} \|\Delta c_k\|^q \right)^{\frac{1}{q}} \\ = & \frac{n(n+1)}{6} |z-1| \left(\frac{|z|^{pn} - 1}{|z|^p - 1} \right)^{\frac{1}{p}} \left(\sum_{k=0}^{n-1} \|\Delta c_k\|^q \right)^{\frac{1}{q}} \end{aligned} \tag{5.2}$$

and, as

$$\sum_{k=0}^{n} z^k = \frac{z^{n+1} - 1}{z - 1}, \ z \neq 1,$$

we obtain the desired result (5.1). ∎

The following result for the complex roots of the unity also holds:

Theorem 9. *Let $z_k := \cos\left(\frac{k\pi}{n+1}\right) + i \sin\left(\frac{k\pi}{n+1}\right)$, $k \in \{0, ..., n\}$ be the complex $(n+1)-$roots of the unity. Then we have the inequality*

$$\|P(z_k)\| \leq \frac{n^{1+\frac{1}{p}}(n+1)}{3} \sin\left[\frac{k\pi}{2(n+1)}\right] \left(\sum_{k=0}^{n-1} \|\Delta c_k\|^q \right)^{\frac{1}{q}} \tag{5.3}$$

for all $k \in \{1, ..., n\}$.

Proof. We have, from (5.2), that

$$\left\| P(z_k) - \frac{z_k^{n+1}-1}{z_k-1} \times \frac{1}{n+1}\sum_{k=0}^{n} c_k \right\| \leq \frac{n(n+1)}{6}|z_k-1| \left(\sum_{k=1}^{n-1}|z_k|^{pk}\right)^{\frac{1}{p}} \left(\sum_{k=0}^{n-1}\|\Delta c_k\|^q\right)^{\frac{1}{q}}. \tag{5.4}$$

Now, as $z_k^{n+1}=1$, $|z_k|=1$, we get, by (5.4), that

$$\|P(z_k)\| \leq \frac{n^{1+\frac{1}{p}}(n+1)}{6}|z_k-1| \left(\sum_{k=0}^{n-1}\|\Delta c_k\|^q\right)^{\frac{1}{q}}. \tag{5.5}$$

We observe that

$$\begin{aligned} z_k - 1 &= \cos\left(\frac{k\pi}{n+1}\right) + i\sin\left(\frac{k\pi}{n+1}\right) - 1 \\ &= -2\sin^2\left(\frac{k\pi}{2(n+1)}\right) + 2i\sin\left[\frac{k\pi}{2(n+1)}\right]\cos\left[\frac{k\pi}{2(n+1)}\right] \\ &= 2i\sin\left(\frac{k\pi}{2(n+1)}\right)\left\{\cos\left[\frac{k\pi}{2(n+1)}\right] + i\sin\left[\frac{k\pi}{2(n+1)}\right]\right\}, \end{aligned}$$

and then

$$|z_k-1| = 2\left|\sin\left[\frac{k\pi}{2(n+1)}\right]\right|, \text{ for all } k \in \{1,...,n\},$$

and the theorem is proved. ∎

6. Applications for Lipschitzian Mappings

Let $(X, \|\cdot\|)$ be as above and $F : X \to Y$ a mapping defined on the normed linear space X with values in the normed linear space Y which satisfies the *Lipschitz condition*

$$|F(x) - F(y)| \leq L\|x-y\| \text{ for all } x, y \in X, \tag{6.1}$$

where $|\cdot|$ denotes the norm of Y.

The following theorem holds.

Theorem 10. *Let $F : X \to Y$ be as above and $x_i \in X$, $p_i \geq 0$ $(i = 1, ..., n)$ and $\sum_{i=1}^n p_i = 1$. Then we have the inequality:*

$$\left|\sum_{i=1}^n p_i F(x_i) - F\left(\sum_{i=1}^n p_i x_i\right)\right| \leq L\sum_{i,j=1}^n p_i p_j |i-j|^{\frac{1}{q}} \left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{1}{p}}, \tag{6.2}$$

where $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. As F is Lipschitzian, we have (6.1) for all $x, y \in X$. Choose $x = \sum_{i=1}^n p_i x_i$ and $y = x_j$ $(j = 1, ..., n)$ to get

$$\left|F\left(\sum_{i=1}^n p_i x_i\right) - F(x_j)\right| \leq L\left\|\sum_{i=1}^n p_i x_i - x_j\right\| \tag{6.3}$$

for all $j \in \{1, ..., n\}$.

If we multiply (6.3) by $p_j \geq 0$ and sum over j from 1 to n, we obtain

$$\sum_{j=1}^{n} p_j \left| F\left(\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| \leq L \sum_{j=1}^{n} p_j \left\| \sum_{i=1}^{n} p_i x_i - x_j \right\|. \tag{6.4}$$

Using the generalized triangle inequality, we have

$$\sum_{j=1}^{n} p_j \left| F\left(\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| \geq \left| F\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{j=1}^{n} p_j F(x_j)\right|. \tag{6.5}$$

By the generalized triangle inequality in normed linear space X we also have

$$\begin{aligned} \sum_{j=1}^{n} p_j \left\| \sum_{i=1}^{n} p_i x_i - x_j \right\| &= \sum_{j=1}^{n} p_j \left\| \sum_{i=1}^{n} p_i (x_i - x_j) \right\| \\ &\leq \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \\ &= 2 \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\| := C. \end{aligned} \tag{6.6}$$

As in Theorem 5, we have, for $i < j$ that

$$\|x_i - x_j\| = \left\| \sum_{k=i}^{j-1} \Delta x_k \right\| \leq \sum_{k=i}^{j-1} \|\Delta x_k\|$$

and then, by Hölder's inequality for sums, we have

$$\begin{aligned} C &\leq 2 \sum_{1 \leq i < j \leq n} p_i p_j \sum_{k=i}^{j-1} \|\Delta x_k\| \\ &\leq 2 \sum_{1 \leq i < j \leq n} p_i p_j (j-i)^{\frac{1}{q}} \left(\sum_{k=i}^{j-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \\ &\leq 2 \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \sum_{1 \leq i < j \leq n} p_i p_j (j-i)^{\frac{1}{p}} \\ &= \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \sum_{i,j=1}^{n} p_i p_j |j-i|^{\frac{1}{p}} \end{aligned} \tag{6.7}$$

and the inequality (6.2) is proved. ∎

The following corollary is a natural consequence of the above results.

Corollary 4. *Let X be a normed linear space and $x_i \in X$ $(i = 1, ..., n)$, $p_i \geq 0$ $(i = 1, ..., n)$ with $\sum_{i=1}^{n} p_i = 1$. Then we have*

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\| - \left\| \sum_{i=1}^{n} p_i x_i \right\| \leq \sum_{i,j=1}^{n} p_i p_j |i-j|^{\frac{1}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}}, \tag{6.8}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

References

[1] S.S. Dragomir, Grüss inequality in inner product spaces, *The Australian Math Soc. Gazette*, **26** (1999), No. 2, 66-70.

[2] S.S. Dragomir, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74-82.

[3] S.S. Dragomir, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, **31**(1) (2000), 43-47.

[4] S.S. Dragomir, Some discrete inequalities of Grüss type and applications in guessing theory, *Honam Math. J.*, **21**(1) (1999), 145-156.

[5] S.S. Dragomir, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.,* **31**(4) (2000), 397-415.

[6] S.S. Dragomir and G.L. Booth, On a Grüss-Lupaş type inequality and its applications for the estimation of p-moments of guessing mappings, *Mathematical Communications,* **5** (2000), 117-126.

[7] S.S. Dragomir and I. Fedotov, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4)(1998), 286-292.

[8] S.S. Dragomir and J.E. Pečarić, On some inequalities for the moments of guessing mapping, *submitted.*

[9] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.

[10] A.M. Fink, A treatise on Grüss' inequality, *Analytic and Geometric Inequalities,* 93-113, Math. Appl. 478, Kluwer Academic Publ., 1999.

[11] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

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